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Optimum advertising pulsation strategies: A dynamic programming approach

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OPTIMUM ADVERTISING PULSATION STRATEGIES:

A DYNAMIC PROGRAMMING APPROACH

by

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ABSTRACT

This study, using the dynamic programming approach, has addressed the problem of optimally allocating a fixed advertising budget of a monopolistic firm over a planning horizon comprised of n equal periods to maximize two popular measures of advertising performance: (1) profits related to the advertising effort (discount factor $r = 0$), and (2) present value of profits related to the advertising effort (discount factor $r > 0$).

Two dynamic programming models that use the modified Vidale-Wolfe model to represent sales response to advertising are formulated with respect to whether the time value of money is considered. For a planning horizon comprised of four equal time periods, computing routines are developed to solve two sample problems with respect to the dynamic programming models. Sensitivity analyses are performed to assess the impacts of a change in some key model parameters upon the behavior patterns of the optimum dynamic programming advertising policy and the associated total return.

Four alternative types of traditional advertising pulsation policies are modeled for the purpose of comparing their performance with the optimum advertising policy determined by dynamic programming. For a planning horizon comprised of four equal time periods, computing routines are also developed to generate total returns under these traditional advertising pulsation policies. Computational results show that the performance under the optimal advertising policy determined by dynamic programming,

as expected, is at least as good as the maximum performance among the four traditional advertising pulsation policies.

The plausibility of the modified Vidale-Wolfe model is empirically examined using the well-known Lydia Pinkham vegetable compound annual data covering the period from 1907 to 1960. Model parameters have been estimated using the Gauss-Newton algorithm related to nonlinear regression. The model selected is one corrected for first-order autoregressive residuals. The empirical results indicate that the model parameters are statistically significant and of the expected signs. More important, it is found that the advertising response function is concave.

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CHAPTER 1

INTRODUCTION

Advertising is a key factor in a firm's marketing efforts, and significant amounts of resources are usually committed to it. For example, Procter & Gamble Company's yearly advertising expenditure reached a level of 3.4 billion U.S. dollars in 1997, and during the period from 1991 to 1997 the company spent approximately one dollar in advertising for every 10 dollars of net sales (Proctor & Gamble Annual Reports 1991-97.) At the national level, the average advertising expenditure per year in the United States was approximately 93 billion dollars in the 1980s, and it rose to 139 billion dollars in the first six years of the 1990s. In the year 1996 alone, more than 173 billion dollars were spent on advertising in this country (Statistical Abstracts of the United States 1993-97.) Accordingly, the determination of an optimal adverting policy with respect to a certain performance measure over time is of central importance to professionals as well as academicians. While numerous previous studies have explored sales response to advertising, two questions of particular significance stand only partially answered. The first is concerned with what is the best way of allocating advertising funds over a number of equal consecutive time periods so that a certain performance measure is optimized? The second question asks if the optimal advertising policy differs from the best policy within the cyclic class of advertising pulsation policies frequently discussed in the literature, and if so, how?

The advertising pulsation class includes the following four main alternative policies shown in Figure 1.1.

1. Blitz Policy (BP): This is a one-pulse policy in which the firm concentrates all advertising efforts in a single period.
2. Advertising Pulsing Policy (APP) : This is a policy in which the firm alternates between high and zero levels of advertising.
3. Advertising Pulsing/Maintenance Policy (APMP): This is a policy in which the firm alternates between high and low levels of advertising.
4. Uniform Advertising Policy (UAP): According to this policy, the firm advertises at some constant level.

The average sales revenue or mean awareness related to the above advertising pulsation policies have often been compared with each other under the assumption that initial sales rate or awareness is zero as in the case of new products (e.g., Mahajan and Muller, 1986), infinite planning horizon (e.g., Park and Hahn, 1991), or a zero discount rate (e.g. Hahn an Hyun, 1991). The above simplifying assumptions have resulted in the development of tractable models and the production of powerful results at the expense of ignoring important aspects of reality that often exist in the business environment. In addition, the best policy within the above narrow set of pulsation policies may not necessarily be the optimal policy within a broader class of advertising pulsation policies.

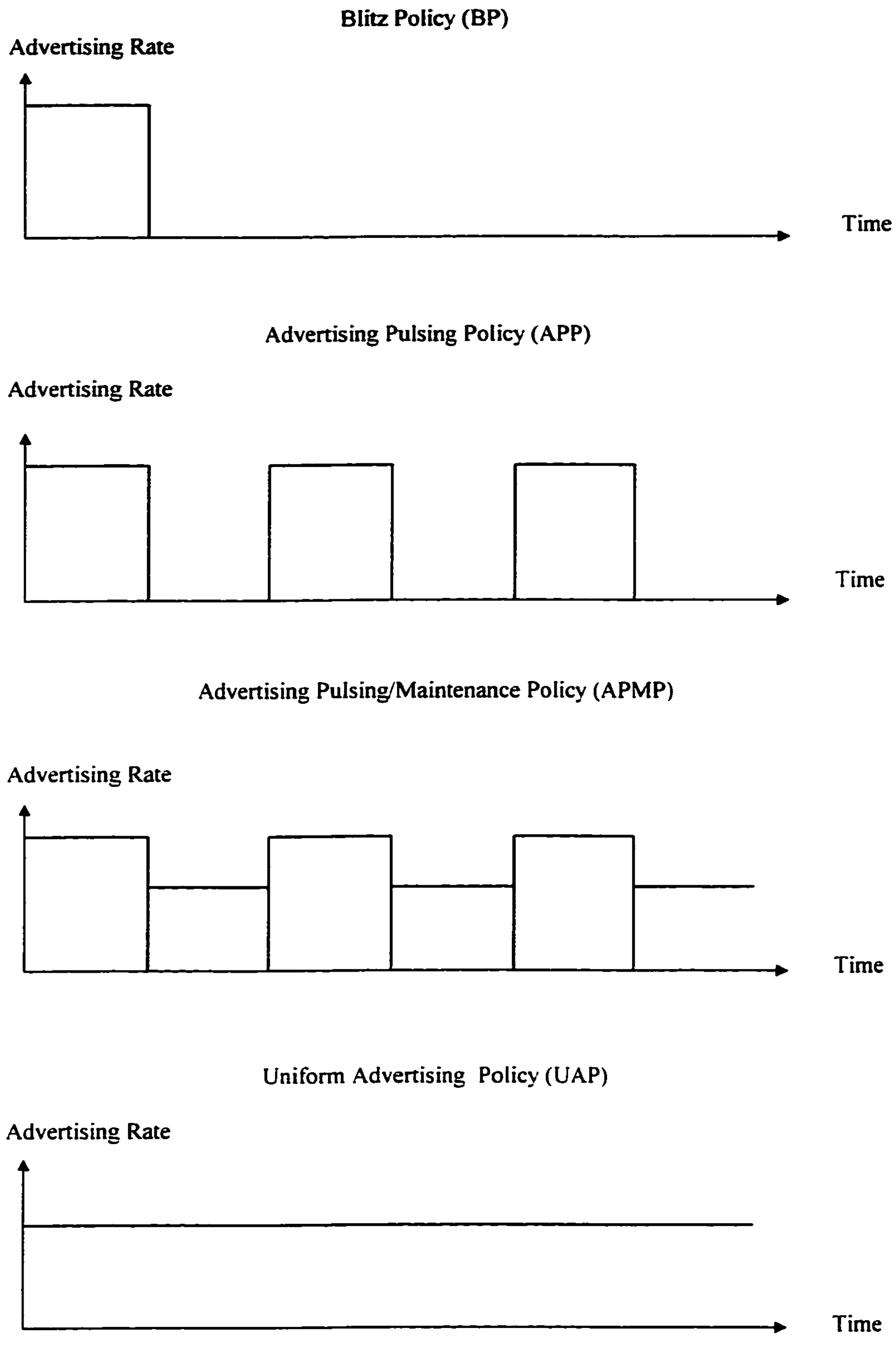

Figure 1.1

Alternative Advertising Pulsation Policies

Statement of the Problem

In this study, it is assumed that the advertiser sells a single product in a monopolistic market and that advertising is the major element of the firm's marketing efforts. The monopoly assumption may well represent one or more of the following situations: (i) the firm is granted a patent, (ii) the product is highly differentiated, and (iii) the firm has a dominant market share and faces competition from a fringe of many small suppliers, each too small to noticeably influence the market dynamics (Mesak, 1992). The problem that will be addressed in this dissertation can be briefly stated as follows:

"An advertising budget, I, of a firm in a monopolistic market is to be allocated over n equal periods over a planning horizon of length L. What is the optimal allocation of the advertising appropriations over time to maximize either one of the following two popular performance measures:

1. Profits related to the advertising effort (discount factor $r = 0$), and
2. Present value of profits related to advertising (discount factor $r > 0$)?"

For each of the above performance measures, both zero and positive initial sales rates are considered in the analysis. The advertising amplitude (advertising rate) is assumed to be constant over a given period in the planning horizon. The advertising rate, however, may differ for different periods. The duration of these equal time periods T and the advertising budget I are assumed to have been determined exogenously. The above problem will be formulated as a dynamic programming problem. Sales response to advertising is assumed to be explained by a modified version of the Vidale-Wolfe (1957) model proposed by Little (1979).

Objectives of the Study

This study has five main objectives. They are (1) the formulation of a dynamic programming (DP) model that would represent the problem stated above, (2) the development of a computer routine to solve numerically the DP model for a given set of parameters, (3) the performing of a sensitivity analysis to assess the impact of changes in certain parameters on the performance measures, (4) the comparison of the performance of the DP optimal policy with the pulsation policies of BP, APP, APMP, and UAP that cost the same, and (5) the conducting of an empirical analysis to assess the plausibility of the assumed dynamic model that relates advertising to sales and to assess the shape of the advertising response function. It is of course expected that the performance related to the DP optimal advertising policy would be at least as good as the maximum performance among the four pulsation policies depicted in Figure 1.1. To achieve the objectives stated above, the solution procedure will make use of a hybrid of analytical and numerical analyses.

Contribution and Applicability

To the best knowledge of the author, the study reported herein is the first attempt in the literature wherein DP is used to solve the finite-horizon advertising pulsation problem wherein both the initial sales and the discount rates are allowed to be different from zero. In addition, the modeling framework is significantly more flexible than the rigid ones already found in the literature. The intended research is thought to be applicable for frequently purchased unseasonal products in the mature stage of their product life cycle for which advertising is the main element of the marketing mix.

Organization of the Dissertation

The remaining chapters are organized as follows: Chapter 2 presents a review of the literature relevant to this study. Chapter 3 incorporates an analysis of traditional pulsation policies. Chapter 4 contains the methodology to be employed in this study: the formulation of the DP model. Chapter 5 contains the solution methodology for solving some practical advertising pulsation problems. Chapter 6 includes a sensitivity analysis related to the impact of changes in the shaping parameter of the advertising response function and/or the value of initial sales on the pattern of the DP optimal advertising policy and its associated return. In addition, the chapter incorporates a comparison between the DP optimal advertising policy return and its traditional advertising pulsation counterparts that cost the same. Chapter 7 includes a discussion of the findings of an empirical analysis conducted to validate the assumed dynamic relationship between advertising and sales together with an assessment of the shape of the advertising response function. Chapter 8 contains a summary of the main results, conclusions and implications for managerial practice and future research. In order to improve readability, derivation of key mathematical formulas, and documentation of detailed results are relegated to separate Appendices.

CHAPTER 2

REVIEW OF RELATED LITERATURE AND POSITIONING OF PROPOSED RESEARCH

Relevant studies have been published with respect to three areas pertinent to this study: (1) studies related to advertising pulsation, (2) studies addressing the Vidale-Wolfe model, and (3) studies related to the applications of dynamic programming in marketing.

Review of Advertising Pulsation Studies

Whether it is best to adopt a cyclic policy of advertising or one of even spending that costs the same has been a fundamental research question in the literature. Several researchers have examined the optimal policy within the advertising pulsation class from various perspectives. Nerlove and Arrow (1962) and Sethi (1973, 1977) argued that a one-time pulse, followed by constant advertising in subsequent periods, constitutes the optimal policy under certain circumstances. Gould (1970) and Jacquemin (1973) illustrated that the optimal policy leads to a unique, stable, steady-state level of constant advertising spending. Sasieni (1971) found that, for a general class of sales response models incorporating a concave advertising response function, a cyclic advertising policy can never be superior, in the long run, to a uniform policy of advertising spending. Mahajan and Muller (1986) and Sasieni (1989) provided normative guidelines as to the number and timing of successive exposures in a given time period in the presence of an S-shaped advertising response function. After formulating the market share response to advertising as a first-order Markov process, Horsky (1977) found that the optimal policy consists of an advertising pulse to reach the optimal market share and constant advertising spending in the subsequent periods. Based on modeling Haley's (1978) wearout phenomenon, Simon (1982) and later Mesak (1992) found that an advertising pulsing policy is optimal under either a constrained or unconstrained advertising budget. Mesak (1985) derived the conditions under which an advertising pulsing policy dominates a uniform advertising policy for both stationary and nonstationary markets. Hahn and Hyun (1991) analyzed the effect of different costs on the optimal advertising policy and found that a pulsing policy is optimal when the ratio of pulsation costs to fixed advertising costs is sufficiently large. Desai and Gupta (1996) employed a discrete-time Markov decision model to obtain optimal control limit policies and concluded that as the high-level advertising cost increases, pulsing becomes optimal. Feinberg (1992) found that a pulsation policy (other than chattering) is optimal if there is a gradual build-up in advertising goodwill in the presence of a convex advertising response function. Mesak and Darrat (1992) compared five alternative advertising policies that belong to the advertising pulsation class using a modified Vidale-Wolfe model (to be discussed shortly) and considered the impact of the shape of the advertising response function on the optimal policy. They found that for a concave or linear advertising response function, a policy of even spending is optimal, whereas for a convex response function, the best advertising policy is one of pulsing.

The above literature review suggests that the shape of the advertising response function plays an important role in determining the optimal advertising policy. To arrive at the optimal policy, researchers have mainly pursued one of the following two alternative methodologies: (1) proposing a few alternative advertising pulsation policies that cost the same and comparing their effectiveness with respect to a certain performance measure (e.g., Mahajan and Muller 1986, Mesak and Darrat 1992) or (2) optimizing a certain measure of performance using optimal control methods (e.g., Sasieni 1971, 1989). It appears that because of the rigidity of media, a certain advertising level must be applied for a certain time period. Therefore, the former approach seems to be more applicable in practice than the latter. The first approach employed in the current literature, however, suffers from a rigidity in its modeling framework and the limited number of advertising pulsation policies investigated. This dissertation will mitigate these shortcomings by allowing the modeling framework to be more flexible and by enlarging the number of alternative advertising pulsation policies considered using dynamic programming. Table 2.1 is self-explanatory and compares the proposed dissertation with the closely related studies of Mahajan and Muller (1986) and Mesak and Darrat (1992) along several dimensions.

Review of the Vidale-Wolfe Model

The Vidale-Wolfe model (1957) is one of the earliest and most intensively analyzed mathematical models of dynamic advertising response (e.g., Mahajan and Muller 1986, Sasieni 1989, Mesak and Darrat 1992). For that model, the instantaneous change in the sales rate is given by a first-order linear differential equation:

Table 2.1

Comparison of Three Inquiries

Factor \ Study	Mahajan and Muller (1986)	Mesak and Darrat (1992)	Proposed Dissertation
Model Employed	Modified Vidale-Wofle model	Modified Vidale-Wofle model	Modified Vidale-Wofle model
Shape of Advertising Response Function Considered	S-Shaped	Concave, linear, and convex	Concave, linear, and convex
Decision Variable	Advertising	Advertising	Advertising
Market Structure	Monopoly	Monopoly	Monopoly
Modeling Framework	Equal periods of alternating high and low advertising rates	Equal periods of alternating high and low advertising rates	Arbitrary levels of advertising rates over equal time periods
Planning Horizon	Finite	Infinite	Finite
Solution Concept	Dominance concept of Game Theory	Dominance concept of Game Theory	Deterministic Dynamic Programming
Performance Measure	Average undiscounted awareness	Average undiscounted sales revenues	Average undiscounted and present value of discounted sales revenues
Initial Conditions	Zero initial awareness	Non-negative initial sales rate	Non-negative initial sales rate

$$dS / dt = (\rho / m)x(m - S) - aS, \tag{2.1}$$

where S = sales rate ($/unit time), x = advertising rate ($/unit time), ρ = response constant, a = decay constant, and m = saturation sales. The advertising response function for the Vidale-Wolfe model is linear given by f(x) = (ρ/m)x. A modified version of the Vidale-Wolfe model has been proposed by Little (1979) for which f(x) takes on a power function of the form

$$f(x) = bx^{\delta}, \tag{2.2}$$

where b = measure of advertising effectiveness (Krishnan and Gupta, 1967), δ = measure of the degree of convexity (concavity) of the advertising response function (Little, 1979). The function (2.2) is concave for $0 < \delta < 1$, linear for $\delta = 1$, and convex for $\delta > 1$. By using the more general form for f(x) instead of (ρ/m)x, the modified version of the Vidale-Wolfe model takes the general form

$$dS / dt = f(x)(m - S) - aS. \tag{2.3}$$

The steady-state sales response S(x) related to a constant level of advertising spending x is derived through setting dS/dt = 0, and solving equation (2.3) for S to obtain

$$S(x) = mf(x) / (a + f(x)). \tag{2.4}$$

It is noted here that the steady-state sales response (2.4) is concave if f(x) is linear or concave (that is $0 < \delta \le 1$) whereas it is S-shaped if f(x) is convex (that is $\delta > 1$). Using (2.4), it can be easily shown that (2.3) may be rewritten as

$$dS / dt = \phi(x)[S(x) - S], \tag{2.5}$$

where,

$$\phi(x) = a + f(x). \tag{2.6}$$

The modified Vidale-Wolfe model has been used extensively in analyzing pulsation policies in monopolistic markets (e.g., Mahajan and Muller 1986; Sasieni 1989) and thoroughly analyzed in the marketing literature (e.g., Feichtinger *et al.* 1994). Mesak and Darrat (1992) provided empirical support for the modified Vidale-Wolfe model and offered a procedure based on OLS for assessing the shape of an advertising response function. Using the well-known Lydia Pinkham annual data, this dissertation will employ a nonlinear regression procedure to estimate and identify the shape of the advertising response function in the modified Vidale-Wolfe model.

Review of Dynamic Programming Applications in Marketing

Dynamic programming (DP) is a mathematical approach designed primarily to improve computational efficiency by decomposing a large problem into smaller, and hence computationally simpler, subproblems. Dynamic programming typically solves the problem in *stages*, with each stage involving a few decision variables and usually one state variable normally defined to reflect the status of the constraints that bind all the stages together. The name *dynamic programming* probably evolved because of its use with applications involving decision-making over time. However, other situations in which time is not a factor are also solved by DP. For this reason a more apt name may be *multistage programming*, since the procedure typically determines the solution in stages (Taha, 1992). Notable studies that have used DP in solving problems related to different areas in marketing are briefly reviewed below.

Marketing-Production Joint Decision Making

Thomas (1974) formulated a stochastic DP model to minimize the expected discounted cost of an inventory control system over a planning horizon of n periods. The decision variables for each period (stage) were the unit price and the quantity of the product to be produced. The state variable was the inventory level at the beginning of the period.

Lodish (1980) used a stochastic DP model to maximize the present value of profits over a multiperiod planning horizon. For each period (stage), the decision variables were the price to be charged and the units of the product to be added to the inventory during the period. The single-state variable stood for the inventory level at the beginning of the period.

Stokes *et al.* (1997) developed a scholastic DP model which captures the existence of a value-added, serial-stage production process with intra- and interyear dynamics of multiple nursery crops. The objective was to maximize the expected value of after-tax cash flows associated with the sale of two different categories of products (one- and three-gallon container-grown nursery crops.) The decision variable at either one of the two stages (Fall and Spring) represented the amount of one-gallon production to be marketed. A unique feature of this two-stage DP model was that the state variables varied by stages. The state variables used to characterize the system were acres of non-salable one-gallon production, acres of salable one-gallon production, acres of salable three-gallon production, carryover business loss, and Spring net income. For the Fall stage, the first three and the fifth state variables defined the status of the system, whereas for the Spring stage, the system was characterized by the first four stage variables mentioned above.

Market Segmentation

Blattberg *et al.* (1978) formulated a mathematical programming model of households' purchasing processes to identify household characteristics that should affect deal proneness. Key factors influencing the household's purchasing decisions were identified as transaction costs, holding costs, stockout costs, and price. Household characteristics were then related to these cost parameters to identify households likely to be deal prone. The problem was solved using a probabilistic DP approach for which the household aims at minimizing the expected product costs over a finite time horizon. In each time period (stage), the decision variables were the quantities of the product purchased from different stores. The state variable was the inventory on hand at the beginning of the period.

Pricing

Robinson and Lakhani (1975) proposed a deterministic DP model for maximizing the present value of profits of a new product produced and sold by a monopolistic firm over a planning horizon of 20 periods. For each period (stage), the decision variable represented the price to be set, whereas the state variable was the cumulative sales volume at the beginning of the period.

Ladany (1996) applied a deterministic DP model to maximize the daily profits of a hotel. Each market segment for which a certain price per room prevails was treated as a stage. For each stage, the decision variable was referred to as the number of rooms to be assigned to the segment. The single state variable considered was the number of rooms available for assignment.

Distribution

Zufryden (1986) employed a deterministic DP model to allocate a certain available integer shelf-space units among a set of products in a supermarket with the objective of maximizing net profits. Within the DP formulation, each product was considered as a stage. For each stage, the decision variable was the space to be allocated to the product. The related state variable was the amount of space available for allocation.

Boronico and Bland (1996) used a stochastic DP model to explore the issue of procuring adequate stocks of seasonal food products. More specifically, their study focused on a distribution system which typifies operations for a major food producer where the major retail outlets must determine optimal order quantities for products received from vendors, subject to uncertainty in the distribution channel. Demand was assumed to be known while the receipt quantity from the supplier was probabilistic. The overall objective was to minimize the total expected delivery and holding costs over a multiperiod planning horizon. The decision variable for each stage (period) was defined as the lot size ordered. The state variable was the equilibrium quantity of the product at which the quantity received from vendors equals that demanded by customers.

Salesforce

A mathematical model was developed by Beswick (1977), for allocating selling efforts and setting sales force size, which explicitly takes into account interactions with territorial design, forecasting, and performance evaluation. The objective was defined as maximizing the total profits of the firm. The problem was cast into a deterministic DP formulation where all the control units (individual customers) were treated as a sequence of interrelated stages. The decision variable at each stage was referred to as the selling time to be allocated to the corresponding control unit, whereas the single state variable considered represented the selling time available for allocation.

Gaucherand *et al.* (1995) modeled the situation where the productivity of members of a salesforce was evaluated in each period over a finite time horizon. Those members with a performance measure (accumulated expected sales) lower than a threshold value were replaced by new members. The firm's objective was to maximize the average productivity by choosing an optimal threshold value for each period of evaluation. A stochastic DP model was developed where each period was defined to be a stage. At each stage, the decision variable was the threshold value, while the state variable was referred to as the accumulated sales level achieved by the salesperson.

Consumer behavior

Gönül and Srinivasan (1996), from the perspective of a household, developed a stochastic DP model with the objective of minimizing the expected expenditures over a finite multiperiod time horizon. For each period (stage), the decision variable was binary: to buy or not to buy. The state vector at each stage was composed of the inventory level and the coupons available from preceding stages.

Advertising

Little and Lodish (1966) introduced a mathematical programming model for media selection which takes into account market segmentation, sales potential, and forgetting patterns of the audience. The objective of maximizing the total sales over the planning horizon was subject to a set of constraints, where the exposure value constraints contained probabilistic components. The problem was cast into a DP formulation, where each stage was referred to as a particular medium. The decision variable at each stage represented the number of advertising insertions, and the state variable considered stood for the budget available for allocation.

Zufryden (1974) employed DP in optimizing the reach of local radio advertising. A mathematical programming model was put forward where the objective was to minimize the uncovered audience proportion subject to a budget constraint. The model was translated into a deterministic DP model where each decision stage corresponded to a radio station. At each stage, the decision variable stood for the number of spots to be inserted in the corresponding station, and the state variable was defined as the budget available for allocation.

A nonlinear integer programming model was developed by Zufryden (1975) to explore the impact of the dual objectives of maximizing media reach and frequency in relation to a problem of media selection. The problem was cast into a deterministic DP formulation where each stage corresponded to a radio station. The decision variable at each stage was referred to as the number of spots to be inserted in the corresponding station, and the state vector at each stage contained two elements: the budget available at the end of the current stage and the frequency resulting from the current decision.

As discussed above, both stochastic and deterministic DP models have been applied to solve a variety of decision problems in marketing. It is observed that deterministic dynamic programming formulations in the current literature mainly contain a single state variable. However, to the author's knowledge, the use of dynamic programming to solve the advertising pulsation problem has not yet been addressed in the literature. This dissertation applies a two-state deterministic dynamic programming approach to solve the advertising pulsation problem. This approach will be illustrated in more details in Chapter 4. Analysis of traditional advertising pulsation policies is discussed next.

CHAPTER 3

ANALYSIS OF TRADITIONAL ADVERTISING PULSATION POLICIES

In this chapter, four traditional alternative advertising policies that belong to the advertising pulsation class are analyzed using the modified version of the Vidale-Wolfe model introduced in Chapter 2. These are the BP, APMP, APP, and UAP policies depicted schematically in Figure 1.1. First, we discuss the response of sales to rectangular advertising pulses. Performance measures of both BP and APMP are then analytically derived in two cases: (1) the time value of money is not considered ($r = 0$) and (2) the time value of money is taken into account ($r > 0$), where r stands for the discount factor. Finally, two advertising policy parameters are defined and discussed for the characterization of APMP, APP, and UAP.

Sales Response to Advertising

The finite planning horizon under consideration consists of n equal time periods and the length of each period equals T (see Figure 3.1.) Beginning from the starting point of the planning horizon, the n periods are successively denoted as period i ($i = 1, 2, ..., n$). Since the firm is not going out of business by the end of the nth period, the infinite period immediately following the planning horizon must also be considered to assess the

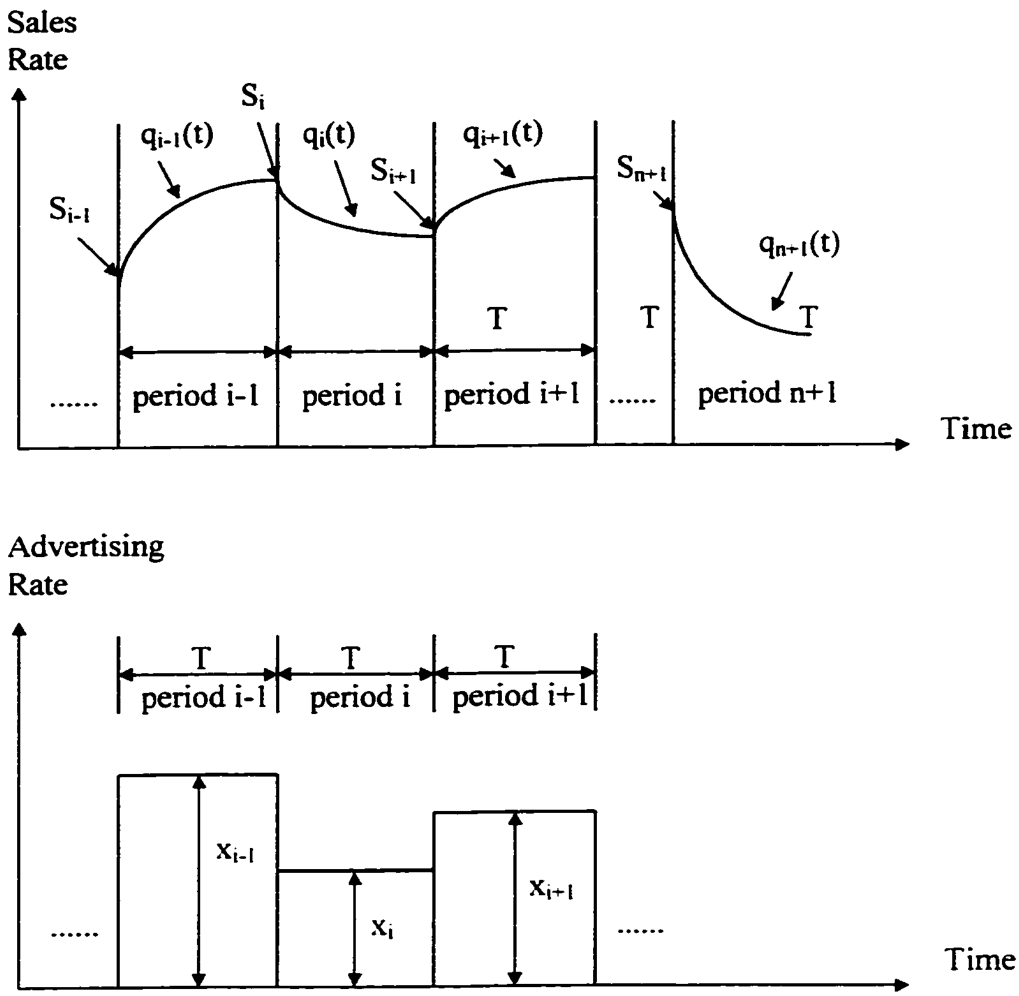

Figure 3.1

Sales Response to Advertising

effect of advertising spending in previous periods. For convenience of discussion, this infinite period is denoted as period n+1. For comparison purposes only, it is assumed that the firm does not advertise after time L = nT. That is, the sales rate level at time nT decays indefinitely according to equations (2.2) and (2.3) with f(x) = 0 corresponding to x = 0 (for further discussion on end effects, see Little and Lodish 1969.)

At first, the following variables are defined:

S_i = the sales rate at the beginning of period i (i = 1, 2, ..., n+1);

I = the total advertising budget if r = 0, or the present value of the total advertising budget if r > 0.

Now the sales rate curve $q_i(t)$ in Figure 3.1 over period i (i = 1, 2, ..., n) in which advertising funds are assumed to be evenly spent at rate x_i is considered. Upon solving the differential equation of the modified version of the Vidale-Wolfe model (Equation (2.5)), the sales rate curve over this time period takes the following form:

$$q_i(t) = S_i e^{-\phi(x_i)(t-(i-1)T)} + S(x_i)(1 - e^{-\phi(x_i)(t-(i-1)T)}),$$

$$(i-1)T \leq t \leq iT, \tag{3.1}$$

where,

S_i = the sales rate at the beginning of period i;

x_i = the rate of advertising spending over period i (If the time value of money is considered, x_i stands for the advertising rate measured in current dollars);

$S(x_i)$ = the steady state sales rate defined by (2.4);

$\phi(x_i)$ is defined by (2.6).

Referring to Figure 1.1, since it is assumed that the firm does not advertise in period n+1, the sales rate decreases exponentially as time elapses, as a result of solving (2.5) when x = 0. The sales rate curve for this case takes on the form shown below:

$$q_{n+1}(t) = S_{n+1}e^{-a(t-nT)}, \quad t \geq nT. \tag{3.2}$$

Equation (3.2) may also be derived from (3.1) by replacing S_i with S_{n+1} and substituting $S(x_{n+1} = 0) = 0$. It is worth mentioning that for a set of alternative advertising policies that cost the same, regardless of whether they are BP, APMP, APP, or UAP, maximizing sales revenue (or its present value) is equivalent to maximizing profit (or its present value), given that the ratio of cost (other than advertising expenditure) to sales revenue is constant over time and independent of these policies (See Mesak 1992 for a detailed discussion.)

Blitz Policy (BP)

It has been mentioned in Chapter 1 that the firm, by adopting a blitz policy, concentrates its advertising efforts only in a single time period over the planning horizon. Without loss in generality, assume that the single advertising pulse coincides with period i where i $\in\{1, 2, ..., n\}$, as shown in Figure 3.2. Governed by (3.1) and (3.2), the sales rate curves depicted in Figure 3.2 take the following forms:

$$q_1(t) = S_1e^{-at}, \quad 0 \leq t \leq (i-1)T; \tag{3.3}$$

$$q_2(t) = S_ie^{-\phi(x)(t-(i-1)T)} + S(x)(1 - e^{-\phi(x)(t-(i-1)T)}),$$

$$(i\text{-}1)T \leq t \leq iT; \tag{3.4}$$

$$q_3(t) = S_{i+1}e^{-a(t-iT)}, \quad iT \leq t < \infty; \tag{3.5}$$

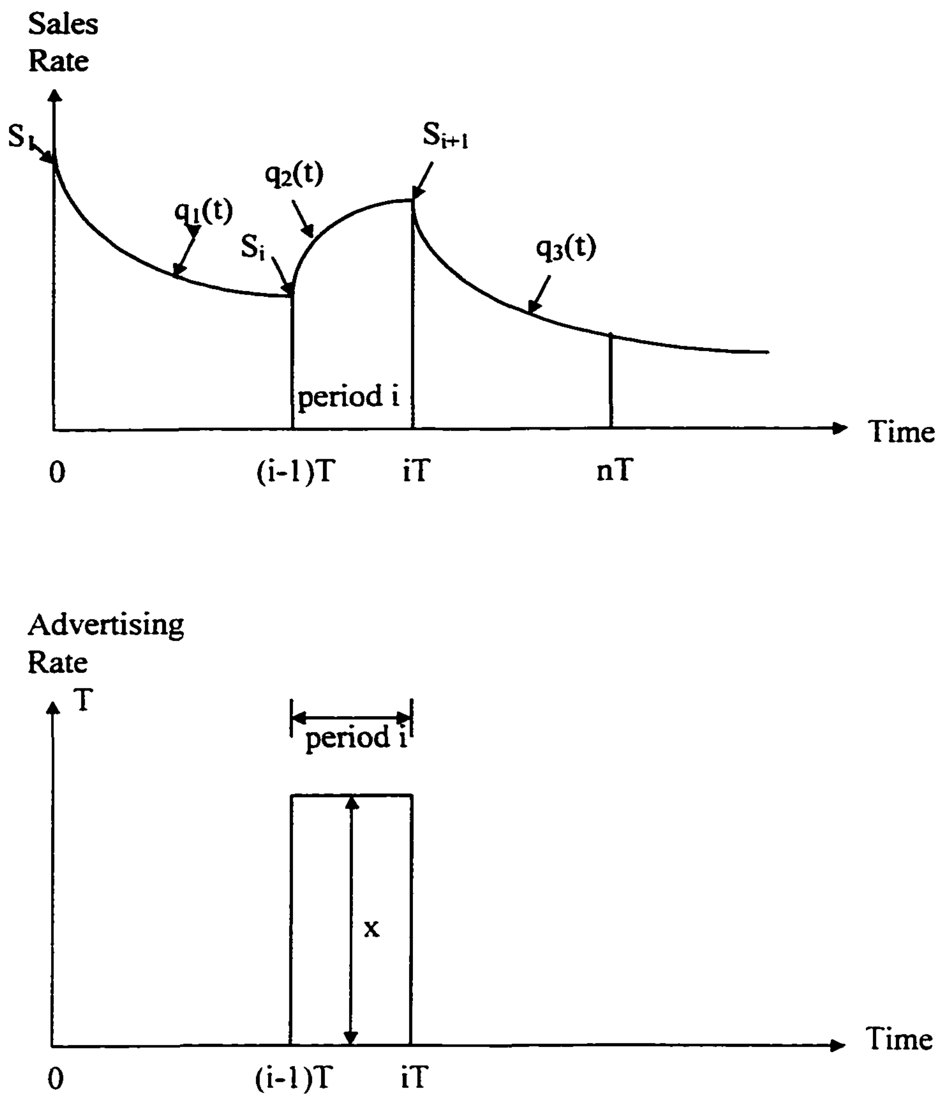

Figure 3.2

Sales Response under BP

where, S_1 is given, $S_i = q_1((i-1)T)$, $S_{i+1}=q_2(iT)$, and x is the rate of advertising spending (measured in current dollars if $r > 0$) over period i.

The performance measure of an advertising policy when the time value of money is not considered is different from that when the time value is taken into account. The two cases are separately addressed in the ensuing discussions.

Case 1: r = 0

In this case, the time value of money is not considered, and the performance of advertising is measured by the sales revenue over the planning horizon and the succeeding infinite period given by

$$R = \int_0^{(i-1)T} q_1(t)dt + \int_0^{T} q_2(t)dt + \int_0^{\infty} q_3(t)dt. \tag{3.6}$$

Notice that in the above formulations a change in the time variable has been employed, so that time is set equal to zero at the beginning of each time period (from here on, this method of changing the time variable will be used unless stated otherwise.) Substituting $q_1(t)$, $q_2(t)$ and $q_3(t)$ from (3.3)—(3.5) produces, after carrying out the integrations.

$$R = \frac{S_1}{a}(1 - e^{-(i-1)aT}) + \frac{S_i - S(x)}{\phi(x)}(1 - e^{-\phi(x)T}) + S(x)T + \frac{S_{i+1}}{a}, \tag{3.7}$$

where the advertising spending over period i equals the advertising budget available at the beginning of the planning horizon, or xT = I, as the advertising funds are exhaustively committed in this single period.

Case 2: r > 0

When the time value of money is taken into consideration, the performance of advertising is measured by the present value of sales revenue over the planning horizon and the succeeding infinite period. In this case, R is given by

$$R = \int_0^{(i-1)T} q_1(t)e^{-rt}dt + e^{-(i-1)rT}\int_0^{T} q_2(t)e^{-rt}dt + e^{-irT}\int_0^{\infty} q_3(t)e^{-rt}dt. \qquad (3.8)$$

Substituting for $q_i(t)$ (i = 1, 2, 3) from (3.3) — (3.5) and carrying out the integrations yield

$$R = \frac{S_1}{a+r}(1-e^{-(i-1)(a+r)T}) + e^{-(i-1)rT}[\frac{S_i - S(x)}{\phi(x)+r}(1-e^{-(\phi(x)+r)T})$$

$$+\frac{S(x)}{r}(1-e^{-rT})] + e^{-irT}\frac{S_{i+1}}{a+r}. \qquad (3.9)$$

As shown in Appendix A, the relationship between the current and the present values of advertising spending over period i (note that the Blitz policy requires all the advertising efforts to be concentrated within a single time period only) is portrayed by

$$x = \frac{r}{e^{-(i-1)rT}(1-e^{-rT})}I, \qquad (3.10)$$

where x is the advertising rate measured in current value, whereas I stands for the present value of the advertising budget available for allocation at time t = 0 (note that this budget is exhaustively spent over period i.)

Advertising Pulsing/Maintenance Policy (APMP)

APMP is an advertising policy in which the firm alternates between two different levels of advertising spending as shown in Figure 3.3. As in the discussion of BP, the two cases where r = 0 and r > 0 are also addressed with respect to APMP. The n-period planning horizon may be composed of an even or odd number of equal time periods. These two situations are considered in both cases as well.

Case 1: r = 0

The time value of money is not considered in this case. and the performance of advertising efforts is measured by the sales revenue generated over the planning horizon and the ensuing infinite time period. For illustrative purposes, let us consider the following terms:

x_1 = the rate of advertising spending over period i given that i is an odd integer;

x_2 = the rate of advertising spending over period i given that i is an even integer;

where i = 1, 2, ..., n. Derived from the solution of (2.5), the sales rate curve over period i is given by

$$q_i(t) = S_i e^{-\phi(x_1)t} + S(x_1)(1 - e^{-\phi(x_1)t}),$$

$$0 \leq t \leq T \text{ if i is odd;} \tag{3.11}$$

$$q_i(t) = S_i e^{-\phi(x_2)t} + S(x_2)(1 - e^{-\phi(x_2)t}),$$

$$0 \leq t \leq T \text{ if i is even;} \tag{3.12}$$

$$i = 1, 2, \ldots, n;$$

$$q_{n+1}(t) = S_{n+1} e^{-at}, \quad t \geq 0; \tag{3.13}$$

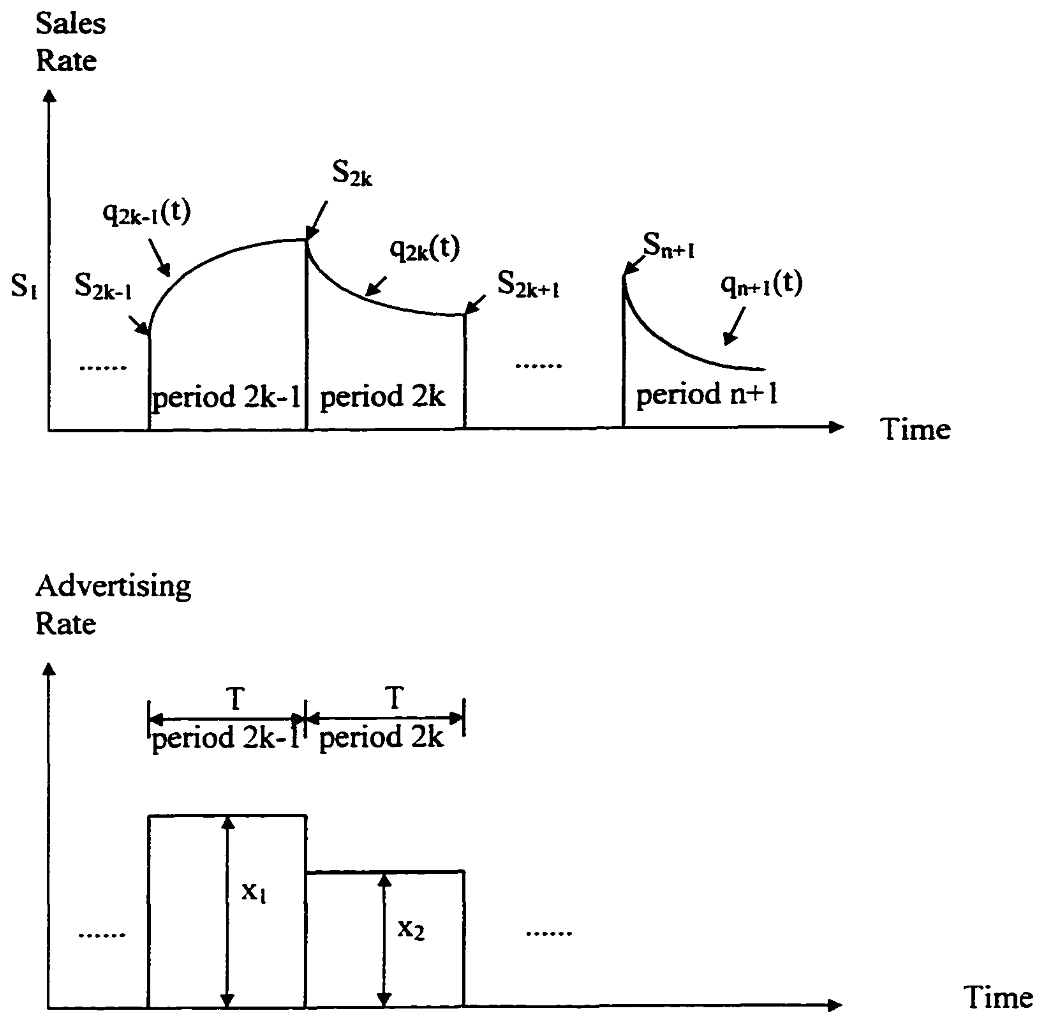

Figure 3.3

Sales Response under APMP

where S_1 is given and $S_i = q_{i-1}(T)$ for i = 2, 3, ..., n+1. It is noted that the sales rate at the beginning of each period (except period 1) is determined by the rate of advertising spending in the preceding period together with its beginning sales rate.

The following two situations are considered:

Situation A: 2m = n. The planning horizon is composed of an even number of equal time periods in this situation (note that m is a positive integer.) The sales revenue over the planning horizon and the ensuing infinite time period is determined by

$$R = \sum_{k=1}^{m} [\int_0^T q_{2k-1}(t)dt + \int_0^T q_{2k}(t)dt] + \int_0^\infty q_{n+1}(t)dt. \tag{3.14}$$

Substituting $q_{2k-1}(t)$, $q_{2k}(t)$, and $q_{n+1}(t)$ from (3.11) — (3.13) and carrying out the integrations, it can be shown that (3.14) may be rewritten as

$$R = \sum_{k=1}^{m} \{\frac{S_{2k-1} - S(x_1)}{\phi(x_1)}(1 - e^{-\phi(x_1)T}) + \frac{S_{2k} - S(x_2)}{\phi(x_2)}(1 - e^{-\phi(x_2)T})$$

$$+ [S(x_1) + S(x_2)]T\} + \frac{S_{n+1}}{a}. \tag{3.15}$$

The advertising expenditures over the entire planning horizon altogether are constrained by the equation $m(x_1 + x_2)T = I$ where I is the advertising budget available for allocation at the beginning of the planning horizon.

Situation B: 2m+1 = n. In this situation, the planning horizon comprises an odd number of equal time periods and the total sales revenue now is given by

$$R = \sum_{k=1}^{m} [\int_0^T q_{2k-1}(t)dt + \int_0^T q_{2k}(t)dt] + \int_0^T q_{2m+1}(t)dt + \int_0^\infty q_{n+1}(t)dt, \tag{3.16}$$

where all the integrations are the same as those included in (3.15) except

$$\int_0^T q_{2m+1}(t)dt = \frac{S_{2m+1} - S(x_1)}{\phi(x_1)}(1 - e^{-\phi(x_1)T}) + S(x_1)T. \tag{3.17}$$

The advertising expenditures over the entire planning horizon are constrained by

$[(m+1)x_1 + mx_2]\,T = I.$

Case 2: r > 0

Since the time value of money is now taken into consideration, the performance of advertising is measured by the present value of the sales revenue generated over periods 1 through n+1. For the purpose of illustration, let us consider the following terms:

y_1 = the present value of advertising spending over period i given that i is odd;

y_2 = the present value of advertising spending over period i given that i is even;

x_i = the rate of advertising spending measured in current dollars over period i for

$i = 1, 2, \ldots, n.$

The relationship between the advertising rate in current value and the present value of adverting spending over each period of the planning horizon is depicted by

$$x_i = \frac{r}{e^{-(i-1)rT}(1 - e^{-rT})} y_1, \text{ given that i is odd;} \tag{3.18}$$

$$x_i = \frac{r}{e^{-(i-1)rT}(1 - e^{-rT})} y_2, \text{ given that i is even;} \tag{3.19}$$

$$i = 1, 2, \ldots, n.$$

The two alternative levels of advertising spending inherent in APMP here are stated in terms of their present values. The present value of the total sales revenue is given by

$$R = \sum_{i=1}^{n} e^{-(i-1)rT} \int_0^T q_i(t) e^{-rt} dt + e^{-nrT} \int_0^{\infty} q_{n+1}(t) e^{-rt} dt, \tag{3.20}$$

where the sales rate curves $q_i(t)$ ($i = 1, 2, ..., n$) are governed by (3.1) and $q_{n+1}(t)$ by (3.2). Substituting for $q_i(t)$ ($i = 1, 2, ..., n+1$) and carrying out the integrations in (3.20) yield

$$R = \sum_{i=1}^{n} e^{-(i-1)rT} \left[\frac{S_i - S(x_i)}{\phi(x_i) + r} (1 - e^{-(\phi(x_i)+r)T}) + \frac{S(x_i)}{r} (1 - e^{-rT}) \right] + e^{-nrT} \frac{S_{n+1}}{a + r}, \tag{3.21}$$

where the advertising rate stated in current dollars, x_i, depending on whether i is odd or even, is determined by (3.18) or (3.19).

In the situation where the planning horizon consists of an even number of equal time periods ($2m = n$), the present values of advertising spending over the entire horizon are restricted by the budget constraint $m(y_1 + y_2) = I$, which indirectly confines the current value of the advertising rate x_i ($i = 1, 2, ..., n$) through (3.18) and (3.19). When the planning horizon is composed of an odd number of equal time periods ($2m+1 = n$), the present values of advertising spending as a whole are confined by $(m+1)y_1 + my_2 = I$, which, along with (3.18) and (3.19), restricts the sequence of x_i ($i = 1, 2, ..., n$).

Advertising Policy Parameters

Under APMP, the firm alternates between high and low levels of advertising spending over the planning horizon, and two different patterns of this policy can be identified: (1) the high level of advertising starts first, and (2) the low level of advertising starts first. For illustrative purposes, these two policy patterns are denoted as APMP-I and APMP-II respectively. It will be shown shortly that both APMP-I and

APMP-II are closely related to APP and UAP. Mesak and Darrat (1992) introduced the concept of policy sets and treated APMP, APP, and UAP each as such a set. In their study, each policy set is characterized by a certain value (or a range of values) of a policy parameter. Following their approach, two advertising policy parameters are defined next, both of which account for APMP, APP, and UAP.

For convenience of exposition, let us restate the notations considered previously:

x_1 = the rate of advertising spending over period i given that i is odd;

x_2 = the rate of advertising spending over period i given that i is even;

y_1 = the present value of advertising spending over period i given that i is odd;

y_2 = the present value of advertising spending over period i given that i is even.

Definition. The advertising policy parameter of APMP-I, λ_1, is a numerical value such that

1. $\lambda_1 \in [0,1]$;
2. $D_1 = (2-\lambda_1)D$ and $D_2 = \lambda_1 D$, where D_i (i =1, 2) stands for x_i given r = 0 and y_i given r > 0 and D is a common factor greater than zero. D stands for the mean advertising rate over the planning horizon for r = 0, or the average present value of advertising expenditures in a period of length T over the planning horizon for r > 0.
3. the relevant budget constraint is maintained.

The advertising policy parameter of APMP-II, λ_2, can be similarly defined by letting $D_1 = \lambda_2 D$ and $D_2 = (2-\lambda_2)D$.

The common factor D assumes various specifications under different conditions. It can be shown that given APMP-I,

1. $D = I/(2mT)$, when $2m = n$ and $r = 0$.
2. $D = I/\{[2(m+1)- \lambda_1]T\}$, when $2m+1 = n$ and $r = 0$.
3. $D = I/(2m)$, when $2m = n$ and $r > 0$.
4. $D = I/[2(m+1)- \lambda_1]$, when $2m+1 = n$ and $r > 0$.

It can be similarly verified that, under APMP-II,

1. $D = I/(2mT)$, when $2m = n$ and $r = 0$.
2. $D = I/[(2m+\lambda_2)T]$, when $2m+1 = n$ and $r = 0$.
3. $D = I/(2m)$, when $2m = n$ and $r > 0$.
4. $D = I/(2m+\lambda_2)$, when $2m+1 = n$ and $r > 0$.

The three different advertising policies, APMP, APP, and UAP, are characterized by different values of the policy parameters. More specifically,

1. When $\lambda_1 \in (0,1)$, $D_1 = (2-\lambda_1)D$ and $D_2 = \lambda_1 D$, indicating an APMP-I policy.
2. When $\lambda_1 = 0$, $D_1 = 2D$ and $D_2 = 0$, indicating an APP-I policy.
3. When $\lambda_1 = 1$, $D_1 = D_2 = D$, indicating a UAP policy.
4. When $\lambda_2 \in (0,1)$, $D_1 = \lambda_2 D$ and $D_2 = (2-\lambda_2)D$, indicating an APMP-II policy.
5. When $\lambda_2 = 0$, $D_1 = 0$ and $D_2 = 2D$, indicating an APP-II policy.
6. When $\lambda_2 = 1$, $D_1 = D_2 = D$, indicating a UAP policy.

Having shed light on the performance of traditional advertising pulsation policies, Dynamic Programming (DP) is introduced in the next chapter to solve two specific maximization problems.

CHAPTER 4

FORMULATION OF THE DYNAMIC PROGRAMMING MODELS

The primary objective of this study is to determine the optimal advertising policy over a finite planning horizon within an enlarged advertising pulsation policy class to maximize either the total or the present value of profits for a given budget available at the beginning of the planning horizon. This chapter consists of two major topics: (1) the formulation of the mathematical programming models of two maximization problems, and (2) the introduction of a dynamic programming approach to solve the above formulated problems.

Formulation of the Maximization Problems

Here advertising policies within an enlarged pulsation class are considered to have a finite planning horizon of n equal time periods. The advertising rate is assumed to be constant over each period. Unlike the BP, APMP, APP, and UAP policies examined in Chapter 3, however, the advertising rate is allowed to vary from period to period. Figure 3.1 delineates schematically sales response to an advertising policy within the enlarged pulsation class. Clearly, the traditional advertising pulsation policies shown in Figure 1.1 and discussed in Chapter 3 are special cases of the advertising policy depicted in

Figure 3.1. For convenience of illustration, in this regard, let us restate the sales rate curve over period i for i = 1, 2, ..., n+1 depicted in Figure 3.1 as follows:

$$q_i(t) = S_i e^{-\phi(x_i)t} + S(x_i)(1 - e^{-\phi(x_i)t}); \quad 0 \le t \le T, \quad i = 1,2,\dots,n; \tag{4.1}$$

$$q_{n+1}(t) = S_{n+1} e^{-at}, \quad t > 0; \tag{4.2}$$

where x_i = the advertising rate (measured in current dollars if the time value of money is considered) during period i.

It is worth mentioning at this point that $q_i(t)$ does not only depend on x_i, but also on the advertising rates in the previous time periods. In other words, the advertising rate in a given period influences the sales rate in the same period together with the sales rates in subsequent periods. Accordingly, for an advertising budget I available at time t = 0, the maximization problem for which the time value of money is not considered, MP1, and the maximization problem for which the time value of money is considered, MP2, may be formulated as follows:

MP1: Find $x_1^*, x_2^*, \dots, x_n^*$ to

$$Max \sum_{i=1}^{n} \int_0^T q_i(t)dt + \int_0^\infty q_{n+1}(t)dt$$

s.t.

$$\sum_{i=1}^{n} x_i T = I$$

and $x_i \ge 0$, i = 1, 2, ..., n. (4.3)

MP2: Find $y_1^*, y_2^*, \dots, y_n^*$ to

$$Max \sum_{i=1}^{n} e^{-(i-1)rT} \int_0^T q_i(t)e^{-rt}dt + e^{-nrT} \int_0^\infty q_{n+1}(t)e^{-rt}dt$$

s.t.

$$\sum_{i=1}^{n} y_i = I$$

and $y_i \geq 0$, $i = 1, 2, \ldots, n$. (4.4)

It is noticed in the above formulations that the change in the time variable introduced in Chapter 3 is used: that is, the time variable is set equal to zero at the beginning of each time period. Confirming earlier ideas, it is reiterated here that since all alternative feasible advertising policies cost the same from (4.3) and (4.4), maximizing profit (or its present value) is equivalent to maximizing sales revenue (or its present value), provided that the ratio of cost (other than advertising expenditure) to sales revenue is constant over time and independent of the alternative advertising policies. In addition, it should be noted that in MP2, the decision variables y_i ($i = 1, 2, \ldots, n$) stand for the present value of advertising spending over period i. If the current values of advertising rates over period i ($i = 1, 2, \ldots, n$) are denoted as x_i, then the relationship between y_i and x_i is dictated by

$$y_i = e^{-(i-1)rT} \int_0^T x_i e^{-rt} dt = e^{-(i-1)rT} (1 - e^{-rT}) x_i / r. \tag{4.5}$$

Once the solution to MP2, $y_1^*, y_2^*, \ldots, y_n^*$, are found, the optimum series of the current values of advertising rates, $x_1^*, x_2^*, \ldots, x_n^*$, can be determined through equation (4.5). The optimal advertising policy, therefore, may be stated by either the series of advertising rates measured in current dollars for different periods or the series of associated present-value advertising expenditures for different periods.

The complex nonlinear structure of the objective functions of the mathematical programming models, MP1 and MP2, are tremendously difficult to model and solve using ordinary nonlinear programming methods such as those based on the well-known Karush-Kuhn-Tucker conditions and gradient search, since the equations related to the KKT conditions are difficult, if not impossible, to solve analytically for the decision variables. Thanks to the principle of decomposition inherent in dynamic programming, it appears to provide an effective solution technique that meets the requirements of the maximization problems. As Zufryden (1986) pointed out, one of the advantages of dynamic programming is that it can easily handle arbitrary objective function specifications, as long as they are separable in the decision variables. For solution purposes, each of the mathematical programming problems MP1 and MP2 can be cast in a dynamic programming formulation. The dynamic programming formulation of problem MP1 is discussed first.

The Dynamic Programming Model for MP1

In general, the components of a dynamic programming model are (1) the sequence of decision stages, (2) input state vector, (3) decision vector, (4) transition function, (5) stage return, and (6) recursive relationship. With respect to the optimization problem of MP1 at hand, these components, shown in Figure 4.1, are identified and discussed below.

The Sequence of Decision Stages

The entire planning horizon is divided into a sequence of consecutive decision stages and each time period stands for a stage. The stages are indexed corresponding to the indices of the time periods defined earlier in Chapter 3. The n+1 stages provide a

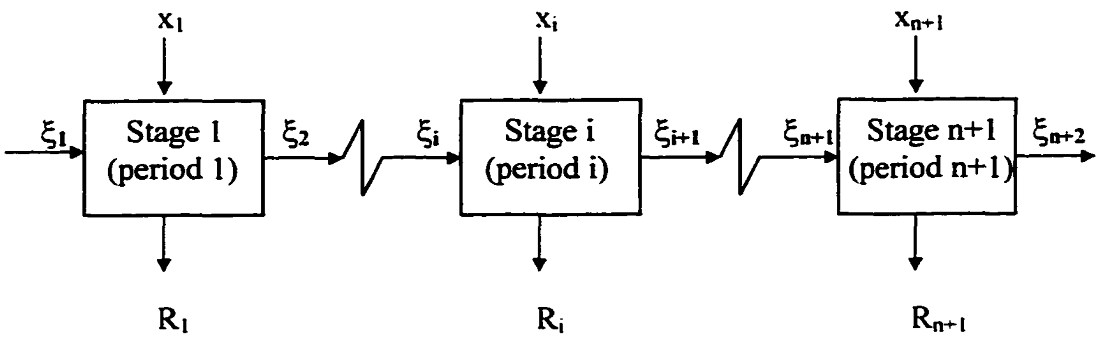

Figure 4.1

The Components of the Dynamic Programming Model for MP1 framework to decompose the problem represented by MP1 into a sequence of smaller and simpler subproblems.

The State Vector (ξ_i)

Each stage has an input state vector as well as an output vector with which it is associated. The input state vector of stage i (i = 1, 2, ..., n+1), ξ_i , contains two elements, the sales rate, S_i, and the advertising budget available, I_i, at the beginning of the stage. Obviously, as shown in Figure 5, each stage's output state vector serves as the input state vector to the next stage. The state vectors contain information about the conditions of the system at various stages, and convey the variation of these conditions from one stage to the next. In particular, ξ_{n+1} stands for the output state vector of the last stage of the planning horizon, which contains the sales rate and zero advertising budget available at the end of the planning horizon (notice that the total advertising budget, I, must be exhausted over the planning horizon.) The input state vector for stage i, ξ_i, is a key factor in determining the return associated with that stage.

The Decision Vector ($\mathbf{x}_i$)

The decision vector of each stage, in general, consists of a number of elements called decision variables and represents the decision alternatives available at the stage. Given the input state vector, each decision alternative will determine a possible value of the stage return (to be discussed shortly) for the particular stage. In our case, the decision vector of each stage contains only one decision variable, i.e., the rate of advertising spending in the stage, and thus, the decision vectors, x_i , (i = 1, 2, ..., n+1) reduce to scalar variables. It is noted that $x_{n+1} = 0$ due to the assumption that no advertising occurs in period n+1. It should be also noted that all the other decision variables are constrained by the budget constraint depicted in (4.3).

The Transition Function

The process in a dynamic programming problem passes from stage to stage. As it does so, it moves through one state vector to the next. As a result of the decision-making at each stage, the transition function describes how the stages of a dynamic programming model are interconnected. The transition function specifies the relationship between the output state vector of a stage to its input state vector and the decision made in the stage. Recalling S_i and I_i to be the sales rate and the advertising budget available for allocation at the beginning of stage i, then the transition function may be expressed as follows:

$$\xi_i = t_i(\xi_{i-1}) \qquad (4.6)$$

where,

$\xi_i = (S_i, I_i)^T$; $\xi_{i-1} = (S_{i-1}, I_{i-1})^T$;

S_1 is given;

$$S_i = S_{i-1}e^{-\phi(x_{i-1})T} + S(x_{i-1})(1 - e^{-\phi(x_{i-1})T}),\ i = 2, 3, \ldots, n+1;\ S_{n+2} = 0. \qquad (4.7)$$

$\phi(x_{i-1}) = a + f(x_{i-1})$, and $S(x_{i-1}) = mf(x_{i-1}) / \phi(x_{i-1})$;

$$I_1 = I\ \text{(given)};\ I_i = I_{i-1} - x_{i-1}T,\ i = 2, 3, \ldots, n;\ I_{n+1} = I_{n+2} = 0. \qquad (4.8)$$

The Stage Return (R_i)

The return for stage i, $R_i(\xi_i, x_i)$, is a function of the input state vector $\xi_i = (S_i, I_i)^T$ and the stage decision x_i. For stage n+1, for example, R_{n+1} is given by

$$R_{n+1} = \int_0^\infty q_{n+1}(t)dt = \int_0^\infty S_{n+1}e^{-at}dt = S_{n+1} / a \tag{4.9}$$

In general for $1 \le i \le n$,

$$R_i(\xi_i, x_i) = \int_0^T q_i(t)dt = \frac{S_i - S(x_i)}{\phi(x_i)}(1 - e^{-\phi(x_i)T}) + S(x_i)T. \tag{4.10}$$

The Recursive Relationship

The solution of a dynamic programming problem having the characteristics mentioned above is based upon Bellman's (1957) principle of optimality.

Principle of Optimality. An optimal policy must have the property that, regardless of the decision made to enter a particular state, the remaining decisions must constitute an optimal policy for leaving that state.

To solve a dynamic programming problem, we begin by first solving a one-stage problem, and then we sequentially add a series of one-stage problems that are solvable until the overall optimum is found. Usually, this solution procedure is based on a *backward induction process*, in which the first stage analyzed is the final stage of the problem, and the solution of the problem proceeds by moving back one stage at a time until all stages in the problem are included. The solution procedure for dynamic programming problems generally begins by finding the optimal policy for each state of the *last stage* of the process.

A final characteristic of dynamic programming problems is the following. The solution proceeds in a fashion that identifies the optimal policy for each state with i stages remaining, given the optimal policy for each state with i -1 stages remaining, using a *recursive relationship.* The recursive relationship for the problem at hand takes the form

$$F_i^*(\xi_i) = \underset{x_i}{Max} \{ F_i(\xi_i, x_i) \} \text{ subject to } x_i T \leq I_i \text{ and } x_i \geq 0.$$

The function $F_i(\xi_i, x_i)$ is the value associated with the best overall policy for the remaining stages of the problem, given that the system is in state ξ_i with i stages to go and the decision variable x_i is selected. The function $F_i(\xi_i, x_i)$ is written in terms of ξ_i, x_i, and $F_{i+1}^*(\cdot)$. For our problem, this recursive relationship can be written as:

$$F_i^*(\xi_i) = \underset{x_i}{Max} \{ R_i(\xi_i, x_i) + F_{i+1}^* (\xi_{i+1}) \} \quad (4.11)$$

subject to

$$x_i T \leq I_i,$$

$$x_i \geq 0.$$

We notice that in maximizing (4.11), ξ_{i+1} is expressed in terms of ξ_i and x_i using the transition functions (4.7) and (4.8). The dynamic programming model formulated above may be solved numerically upon *discretizing* the state variable related to the advertising budget available at the beginning of each stage, I_i.

The Dynamic Programming Model for MP2

The dynamic programming model for MP2 can be similarly formulated by following the same procedure for MP1 presented above. However, several adjustments must be made to account for the time value of money as follows: first, the element I_i of the state vector ξ_i now represents the present value of the advertising budget available at the beginning of period i. Second, the transition function links the sequence$\{I_i\}$ as follows:

$$I_1 = I \text{ (given)}; \; I_i = I_{i-1} - y_{i-1}, \quad i = 2,3,\ldots,n; \quad I_{n+1} = 0. \tag{4.12}$$

Third, the recursive relationship is given by

$$F_i^*(\xi_i) = \underset{x_i}{Max} \{ R_i(\xi_i, x_i) + F_{i+1}^*(\xi_{i+1}) \} \tag{4.11}$$

subject to

$$e^{-(i-1)rT} \int_0^T x_i e^{-rt} dt \le I_i.$$

$x_i \ge 0,$

and $R_i(\xi_i, x_i)$ is given by

$$R_i = e^{-(i-1)rT} \{ \frac{S_i - S(x_i)}{\phi(x_i) + r} [1 - e^{-(\phi(x_i)+r)T}] + \frac{S(x_i)}{r}(1 - e^{-rT}) \}, \tag{4.13}$$

$$i = 1, 2, \ldots, n.$$

$$R_{n+1} = e^{-nrT} S_{n+1} / (a + r). \tag{4.14}$$

Expressions (4.13) and (4.14) are derived in Appendix A.

Given the above discussion, the components of the dynamic programming model of MP2 are represented in exactly the same way as in Figure 4.1, except that x_i is replaced in this case with y_i; i = 1, 2, ..., n+1.

In the next chapter, we explicitly illustrate how to implement the dynamic programming approach discussed in this chapter to solve problems MP1 formulated in (4.3) and MP2 formulated in (4.4).

CHAPTER 5

ILLUSTRATIONS OF APPLICATIONS

The main objectives of this chapter are twofold: (1) illustrating how the dynamic programming approach discussed in Chapter 4 can be applied to solve problems MP1 and MP2, and (2) reporting the results obtained from computing routines especially developed to derive numerically the DP optimal advertising policies related to the two problems mentioned above.

The Considered Planning Horizon and Model Parameters

Four-period budgeting is a common practice in the business world. A firm may wish to plan its advertising spending over a finite time horizon composed of four equal periods (e.g., quarters). The planning horizon considered in our numerical example, therefore, is assumed to consist of four equal time periods to reflect this situation. For illustrative purposes, assume a market potential of $m = 100$ million dollars per year, a decay constant $a = 0.5$ per year and an advertising effectiveness parameter $b = 0.2$. In addition, let us suppose that the firm would allocate exhaustively an advertising budget $I = 4$ million dollars (I stands for the present value of the budget if the time value of money is considered) over a year composed of $n = 4$ equal periods of duration $T = 0.25$ year each.

It should be emphasized here that the initial sales rate, S_1, cannot exceed the market potential m. Therefore, for simplicity and illustration, only 10 alternative values of S_1, smaller than the market potential, and measured in million dollars are considered in the numerical example. They are given by 10k; k = 0, 1, ..., 9. In addition, the alternative values of the convexity (concavity) parameter δ to be investigated are given by 0.05k; k = 1, 2, ..., 60. For the case in which the time value of money is considered, 15 alternative values of the discount rate, r, are considered: 0.01k; k = 1, 2, ..., 12, 100, 200, 300.

The domain of the state variable I_i, the advertising funds available at the beginning of stage i, is discretized as {0.05kI; k = 0, 1, ..., 20} for i = 1, 2, 3, 4.

Formulation of the Dynamic Programming Problems

Before developing computing routines to solve the problems MP1 and MP2 for the planning horizon and model parameters specified above, the corresponding dynamic programming formulations are first developed.

DP Formulation for MP1

According to (4.9) and (4.10), the return of stage i (i = 1, 2, 3, 4), conditioned by the sales rate, S_i, and the advertising funds available at the beginning of the stage, I_i, is a function of the advertising rate over the stage, x_i, and can be explicitly expressed as

$$R_i(S_i, I_i, x_i) = \int_0^T q_i(t)dt = \frac{S_i - S(x_i)}{\phi(x_i)}(1 - e^{-\phi(x_i)T}) + S(x_i)T; \tag{5.1}$$

and return generated over the infinite stage (i.e., stage 5) is given by

$$R_5(S_5, I_5 = 0) = \int_0^{\infty} q_5(t)dt = \int_0^{\infty} S_5 e^{-at} dt = S_5 / a\,; \qquad (5.2)$$

where, S_i and I_i (i = 1, 2, ..., 5) are defined by (4.7) and (4.8) for n = 4, respectively. According to (4.11), the recursive relationship of the DP model is characterized more specifically as follows:

At stage 4,

$$F_4^*(S_4, I_4) = \underset{\forall x_4 = I_4/T}{Max}\{R_4(S_4, I_4, x_4) + S_5 / a\} \qquad (5.3)$$

where, S_5 is stated in terms of S_4 and x_4 using (4.7).

At stage i (i = 1, 2, 3),

$$F_i^*(S_i, I_i) = \underset{\forall x_i \le I_i/T}{Max}\{R_i(S_i, I_i, x_i) + F_{i+1}^*(S_{i+1}, I_{i+1})\} \qquad (5.4)$$

where, S_{i+1} is stated in terms of S_i and x_i using (4.7) and I_{i+1} in terms of I_i and x_i using (4.8).

The solution to the DP model formulated above, x_i^*(i = 1, 2, 3, 4), is functionally dependent upon the two state variables S_i and I_i and thus can be expressed as $x_i^*(S_i, I_i)$. The recursive optimization is carried out backward until the first stage is reached. At stage 1, the maximum total return, $F_1^*(S_1, I_1)$ and the corresponding optimum advertising rate $x_1^* = x_1^*(S_1, I_1)$ are determined. It is noted that $x_1^* = x_1^*(S_1, I_1)$ is a unique value due to the fact that S_1 and $I_1 = I$ are given. It is then possible to backtrack from the first stage through the succeeding stages to obtain the optimum advertising rates for all the other stages in the following manner:

Step 1. Determine the optimum state pair S_2^* and I_2^* using S_1, $I_1 = I$, and x_1^* through (4.7) and (4.8), respectively.

Step 2. Determine the optimum advertising rate for stage 2 through $x_2^* = x_2^*(S_2^*, I_2^*)$.

Step 3. Determine the optimum state pair S_3^* and I_3^* using S_2^*, I_2^*, and x_2^* through (4.7) and (4.8), respectively.

Step 4. Determine the optimum advertising rate for stage 3 through $x_3^* = x_3^*(S_3^*, I_3^*)$.

Step 5. Determine the optimum state pair S_4^* and I_4^* using S_3^*, I_3^*, and x_3^* through (4.7) and (4.8), respectively.

Step 6. Determine the optimum advertising rate for stage 4 through $x_4^* = x_4^*(S_4^*, I_4^*)$.

DP Formulation for MP2

According to (4.13) and (4.14), the returns of the five stages are explicitly specified below:

$$R_i(S_i, I_i, x_i) = e^{-(i-1)rT}\{\frac{S_i - S(x_i)}{\phi(x_i)+r}[1-e^{-(\phi(x_i)+r)T}] + \frac{S(x_i)}{r}(1-e^{-rT})\}, \quad (5.5)$$

$i = 1, 2, 3, 4;$

and $$R_5(S_5, I_5 = 0) = e^{-4rT} S_5 / (a+r); \quad (5.6)$$

where, S_i and I_i ($i = 1, 2, \ldots, 5$) are defined by (4.7) and (4.12) for $n = 4$, respectively. Using (4.5), the advertising rate in current dollars over stage i, x_i, can be expressed as

$$x_i = \frac{r}{e^{-(i-1)rT}(1-e^{-rT})} y_i \quad (5.7)$$

where, y_i is the present value of advertising expenditure over period i. As mentioned in Chapter 4, the decision variable at each stage can be stated in terms of either the advertising rate in current dollars or the present value of advertising expenditure when the time value of money is considered. Therefore, using (5.7) we can restate the stage returns depicted by (5.7) in terms of y_i, i.e., $R_i(S_i, I_i, y_i)$. Consequently, the backward recursive relationship characterized by (4.11) is rephrased as follows:

At stage 4,

$$F_4^*(S_4, I_4) = \underset{\forall y_4 = I_4}{Max}\{R_4(S_4, I_4, y_4) + e^{-4rT} S_5 / (a + r)\} \quad (5.8)$$

where, S_5 is stated in terms of S_4 and y_4 using (4.7) in conjunction with (5.7).

At stage i (i = 1, 2, 3),

$$F_i^*(S_i, I_i) = \underset{\forall y_i \le I_i}{Max}\{R_i(S_i, I_i, y_i) + F_{i+1}^*(S_{i+1}, I_{i+1})\} \quad (5.9)$$

where, S_{i+1} is stated in terms of S_i and y_i using (4.7) in conjunction with (5.7) and I_{i+1} in terms of I_i and y_i using (4.12).

The solution to the DP model for MP2, y_i^*(i = 1, 2, 3, 4), is functionally dependent upon the state variable pair S_i and I_i and can be expressed as $y_i^*(S_i, I_i)$. The recursive optimization is carried out backward until the first stage is reached. At stage 1, the maximum total return, $F_1^*(S_1, I_1)$ and the corresponding optimum present value of advertising expenditure $y_1^* = y_1^*(S_1, I_1)$ are determined. It is noted that $y_1^* = y_1^*(S_1, I_1)$ is a unique value since S_1 and $I_1 = I$ are given. We need to backtrack from the first stage through the succeeding stages to obtain the optimum advertising rates for all the other stages in the following manner:

Step 1a. Determine the optimum advertising rate in current dollars for stage 1, x_1^*, using y_1^* through (5.7) and then the optimum state element S_2^* using S_1 and x_1^* through (4.7).

Step 1b. Determine the optimum state element I_2^* using I_1 and y_1^* through (4.12).

Step 2. Determine the optimum present value of advertising expenditure for stage 2 through $y_2^* = y_2^*(S_2^*, I_2^*)$.

Step 3a. Determine the optimum advertising rate in current dollars for stage 2, x_2^*, using y_2^* through (5.7) and then the optimum state element S_3^* using S_2^* and x_2^* through (4.7).

Step 3b. Determine the optimum state element I_3^* using I_2^* and y_2^* through (4.12).

Step 4. Determine the optimum present value of advertising expenditure for stage 3 through $y_3^* = y_3^*(S_3^*, I_3^*)$.

Step 5a. Determine the optimum advertising rate in current dollars for stage 3, x_3^*, using y_3^* through (5.7) and then the optimum state element S_4^* using S_3^* and x_3^* through (4.7).

Step 5b. Determine the optimum state element I_4^* using I_3^* and y_3^* through (4.12).

Step 6. Determine the optimum present value of advertising expenditure for stage 4 through $y_4^* = y_4^*(S_4^*, I_4^*)$.

The Computing Routines

By defining and calling user-defined functions in C++, two computer routines are developed, using a personal computer (75 MHz processor - 24 MB of RAM), to solve the DP models for MP1 and MP2, respectively. One of the major features of these routines is that they can accommodate various alternative values of the key parameters, i.e., the initial sales rate ,S_1, and the convexity (concavity) parameter, δ. In addition, the DP computing routine for MP2 can determine the optimal advertising policy and the associated return for various values of the discount rate. This feature greatly facilitates the sensitivity analysis of the impact of changes in the model parameters on the behavioral patterns of the optimal advertising policy and the corresponding total returns. Another interesting feature of the computing routines that deserves mentioning is that they are readily extendible to accommodate a more general planning horizon composed of any number of consecutive equal time-periods.

In order to check computational accuracy of these two programs, two computer routines based on exhaustive enumeration are developed for MP1 and MP2, respectively. It is found that no discrepancy whatsoever exists between the computational results generated by the DP programs and those by the enumerating programs.

The developed computing routines based on the dynamic programming approach are exceptionally fast. For example, the time taken to produce the optimum solution for all considered cases for which $r = 0$ (600 cases) was only about 20 minutes. For $r = 0.01$, about 47 minutes were required to arrive at the optimum solution for the same number of cases.

Results

The DP computing routines are developed to find the optimal advertising policy and the associated total return for all the alternative values of S_1, δ, and r, specified in the first section of this chapter. Due to their enormous sizes, the computational results of executing the computing routines are only partially reported in Appendix B. Although the following discussions are based on the partially demonstrated data, they shed interesting light on the behavioral patterns of the DP optimal advertising policy. Tables A1, B1, C1, and D1 in Appendix B illustrate the total returns yielded and the related patterns of advertising spending under the DP optimal policy for selected combinations of the model parameters, namely, the initial sales rate S_1, the convexity (concavity) parameter δ, and the discount rate r.

CHAPTER 6

SENSITIVITY ANALYSIS

In this chapter, the impact of changes in the convexity (concavity) parameter, δ, and the initial sales rate, S_1, on the pattern of the DP optimal advertising policy and its associated return is studied. In addition, the DP optimal advertising policy is compared with and contrasted to its corresponding traditional advertising pulsation counterparts that cost the same in terms of performance. The above analyses are conducted in two cases: (1) the time value of money is not considered ($r = 0$), and (2) the time value of money is considered ($r > 0$). Reference to different tables included in Appendix B is made as deemed appropriate.

DP Optimal Advertising Policy

Case 1: $r = 0$

As Table A1 in Appendix B illustrates, the convexity (concavity) parameter δ and the initial sales rate S_1 significantly influence the pattern of the optimal advertising policy. Let us first consider $\delta \in (0,1)$. It is noted in Table A1 that when δ and S_1 assume smaller values, the pattern of the optimal policy is the same as or close to that of UAP. Given a specific value of $\delta \in (0,1)$, there exists a threshold value for the initial sales rate S_1 such that if S_1 is equal to or larger than the threshold, the pattern of the optimal advertising policy is switched from one of even spending to that of increasing spending over time. The threshold becomes smaller as δ approaches unity from below. For instance, the threshold is between 30 and 60 when $\delta = 0.3$. When δ rises to 0.5, the threshold falls between 10 and 30.

Now consider $\delta \in [1, 3]$. It is interesting to note that the optimal advertising policy is always composed of two pulses with the same magnitude over the first and the last periods of the planning horizon if S_1 equals zero. Given that S_1 is positive, however, various policy patterns may emerge depending on the combination of δ and S_1 values. For certain such combinations, the optimal advertising policy exhibits a BP pattern, with the sole pulse coinciding on the last period of the planning horizon. It is noted that the advertising efforts should be focused on the last, or the first and last quarters, and no advertising resources should be committed over the third quarter under all these optimal policies.

Figure 6.1, derived from Table A.1, graphically demonstrates curves that represent the relationships among the optimum total return, the convexity (concavity) parameter δ and the initial sales rate S_1. For a given specific value of δ, it is observed that higher initial sales rates lead to larger total optimum returns. The vertical differences in total returns across curves get smaller as the value of δ increases. In fact, if the advertising response function is highly convex, the differences become nearly unnoticeable as in the case related to $\delta = 3$. For a specific value of S_1, the optimum total return increases along with δ, implying that a convex advertising response function is much more preferred

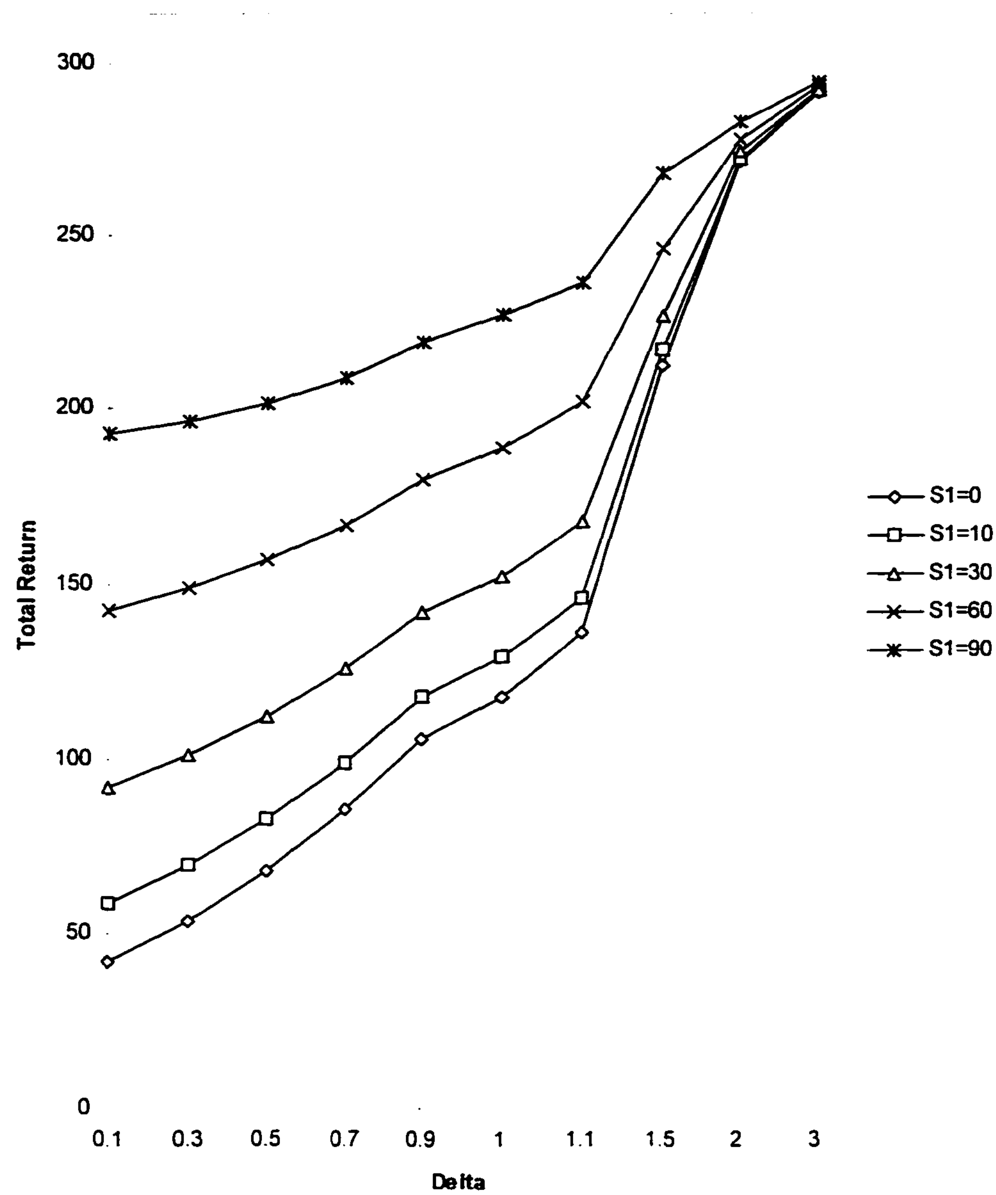

Figure 6.1

The Impacts of δ and S_1 upon the Total Return under the DP Optimal Advertising Policy (r = 0) than a concave one in terms of generating total returns for the considered model parameters.

Case 2: $r > 0$

In this case, the time value of money is taken into account, and hence the optimal advertising policy is determined by, among other things, the discount factor r. For $\delta \in (0,1)$, as shown in Tables B1, C1, and D1 in Appendix B, respectively, there exists a threshold value for the initial sale rate S_1 such that if S_1 is lower than the threshold, the optimal advertising policy appears to be UAP. However, if S_1 is equal to or greater than the threshold, the advertising spending under the optimal policy, over time, may (1) increase monotonically, or (2) decrease first and then later increase. It is interesting to note, similar to the case $r = 0$, that higher values of δ are associated with lower thresholds.

Given $\delta \in [1, 3]$, various combinations of δ, S_1, and r may lead to different patterns of optimum advertising spending, including that of BP under which the sole pulse occurs during the last period of the planning horizon. Under most of these optimal policy patterns, fewer or no advertising efforts are committed to the second or third quarters of the planning horizon, especially when δ assumes relatively low values.

It is observed in these three tables that, everything else being equal, (1) as the discount rate increases, the optimum total return decreases; (2) a larger initial sales rate leads to a higher optimum total return; (3) a more convex advertising response function brings a greater optimum total return. For a given value of r, as demonstrated in Figure

6.2, the above findings can be very well presented schematically by a graph quite similar to that depicted in Figure 6.1.

DP Optimal Advertising versus Traditional Advertising Pulsation

The tables in Appendix B as a whole reveal that, given any combination of the parameters, δ, S_I, and r, the DP optimal advertising policy produces a total return at least as good as that generated by the best traditional pulsation policy. More specifically, if the DP optimal advertising policy does not belong to the traditional advertising pulsation class, it is superior to any of the corresponding traditional policies that cost the same. For example, as shown in Table A1, for r = 0, δ = 2.0, and S_I = 30, the DP optimum policy does not belong to the traditional advertising pulsation class and yields a total return greater than that generated by any of the corresponding BP, APP-I, APP-II, APMP-I, APMP-II, and UAP that cost the same. If the DP optimal policy does belong to the traditional advertising pulsation class, it is the same as the best traditional policy. For example, for r = 0, δ = 0.3, and S_I = 30, the DP optimal advertising policy appears to be UAP, which yields the highest total return compared to the other traditional advertising pulsation policies. The superiority of the DP optimal advertising policy and the roles of δ and S_I in shaping the performances under the various advertising strategies are demonstrated in Figure 6.3 through 6.14, where APMP-I3, APMP-I7, APMP-II3, and APMP-II7 respectively stand for the corresponding APMP-I and APMP-II policies associated with λ = 0.3 and 0.7, respectively.

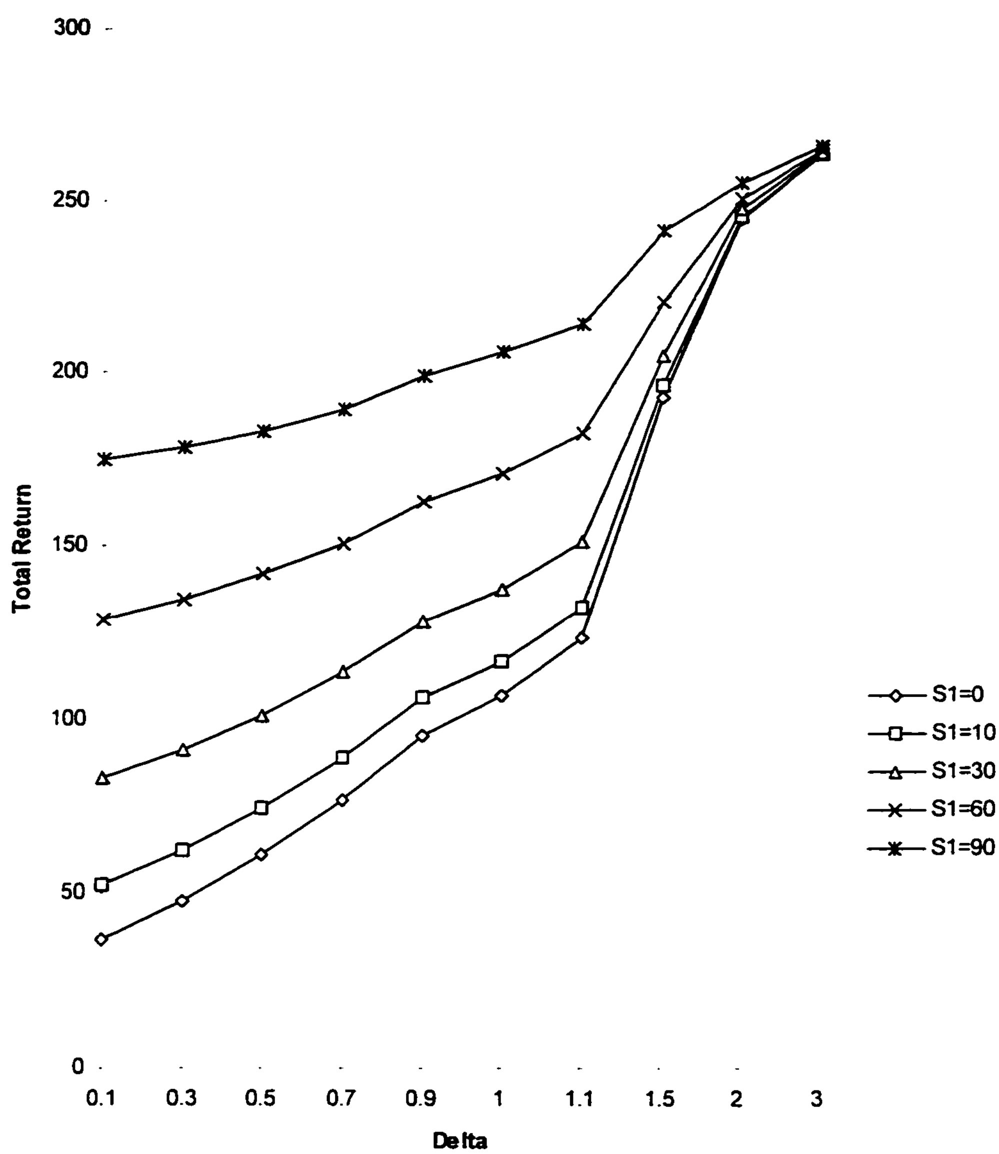

Figure 6.2

The Impacts of δ and S_1 upon the Total Return under the DP Optimal Advertising Policy (r = 0.05)

Table 6.1 provides summary statistics related to the relative effectiveness of the optimal dynamic programming advertising policy, measured in terms of the ratio of DP total return to the best total return among the traditional advertising pulsation policies. Ten groups of the shaping parameter δ and sixteen values of the discount factor r are considered in the analysis. The number of cases examined within each group is 60 (6 values for δ × 10 values for S_1). The descriptive statistics depicted in Table 6.1 reveal that for a total of 9600 considered cases, the mean DP relative effectiveness is about 1.80%, whereas the maximum DP relative effectiveness is as high as 11.16%.

Table 6.1

Descriptive Statistics Related to the Relative Effectiveness of the Optimal DP Policies

	Mean	Std Dev	Maximum	Minimum	n
r = 0.00					
0.05 ≤ δ ≤ 0.30	1.000210238	0.000334395	1.001617475	1.000000000	60
0.35 ≤ δ ≤ 0.60	1.001179207	0.001501643	1.006626046	1.000000000	60
0.65 ≤ δ ≤ 0.90	1.003898937	0.003527939	1.011840265	1.000000000	60
0.95 ≤ δ ≤ 1.20	1.002698618	0.004763358	1.021008968	1.000000000	60
1.25 ≤ δ ≤ 1.50	1.003460613	0.008849142	1.042220703	1.000000000	60
1.55 ≤ δ ≤ 1.80	1.023030399	0.018232275	1.061403523	1.000000000	60
1.85 ≤ δ ≤ 2.10	1.033153216	0.021096539	1.073053108	1.004025604	60
2.15 ≤ δ ≤ 2.40	1.035888192	0.023855356	1.075515754	1.001605633	60
2.45 ≤ δ ≤ 2.70	1.035622383	0.023376465	1.075651048	1.000849404	60
2.75 ≤ δ ≤ 3.00	1.027690594	0.014704295	1.056531929	1.001211667	60
r = 0.01					
0.05 ≤ δ ≤ 0.30	1.000203724	0.000328817	1.001595238	1.000000000	60
0.35 ≤ δ ≤ 0.60	1.001152384	0.001485455	1.006563668	1.000000000	60
0.65 ≤ δ ≤ 0.90	1.003836192	0.003506952	1.011730544	1.000000000	60
0.95 ≤ δ ≤ 1.20	1.002793080	0.004774836	1.020878915	1.000000000	60
1.25 ≤ δ ≤ 1.50	1.003885691	0.009285289	1.042115405	1.000000000	60
1.55 ≤ δ ≤ 1.80	1.023567347	0.018213454	1.061074622	1.000000000	60
1.85 ≤ δ ≤ 2.10	1.033503400	0.021394714	1.072401004	1.003727133	60
2.15 ≤ δ ≤ 2.40	1.036349911	0.024068841	1.074693874	1.001452829	60
2.45 ≤ δ ≤ 2.70	1.035972609	0.023417460	1.074801891	1.000796487	60
2.75 ≤ δ ≤ 3.00	1.027512798	0.014335818	1.055807534	1.001410051	60
r = 0.02					
0.05 ≤ δ ≤ 0.30	1.000197266	0.00032336	1.001572941	1.000000000	60
0.35 ≤ δ ≤ 0.60	1.001126170	0.001468989	1.006500669	1.000000000	60
0.65 ≤ δ ≤ 0.90	1.003775269	0.00348575	1.011620494	1.000000000	60
0.95 ≤ δ ≤ 1.20	1.002886050	0.004805086	1.020744677	1.000000000	60
1.25 ≤ δ ≤ 1.50	1.004377762	0.009757439	1.042000124	1.000000000	60
1.55 ≤ δ ≤ 1.80	1.024098348	0.018213071	1.060731942	1.000000000	60
1.85 ≤ δ ≤ 2.10	1.033875526	0.021699039	1.071741462	1.003438054	60
2.15 ≤ δ ≤ 2.40	1.036819563	0.024293189	1.073874435	1.001328228	60
2.45 ≤ δ ≤ 2.70	1.036299608	0.023435309	1.073956627	1.000820918	60
2.75 ≤ δ ≤ 3.00	1.027327148	0.013959733	1.055105238	1.001608752	60
r = 0.03					
0.05 ≤ δ ≤ 0.30	1.000191120	0.000317862	1.001551249	1.000000000	60
0.35 ≤ δ ≤ 0.60	1.001100075	0.001452820	1.006438512	1.000000000	60
0.65 ≤ δ ≤ 0.90	1.003716573	0.003465198	1.011510903	1.000000000	60
0.95 ≤ δ ≤ 1.20	1.003000003	0.004849367	1.020608889	1.000000000	60
1.25 ≤ δ ≤ 1.50	1.004909481	0.010263373	1.041874904	1.000000000	60
1.55 ≤ δ ≤ 1.80	1.024607793	0.018227120	1.06037565	1.000000000	60
1.85 ≤ δ ≤ 2.10	1.034256303	0.022021753	1.07107534	1.003156487	60

Table 6.1 (Continued)

2.15 ≤ δ ≤ 2.40	1.037296351	0.024528413	1.073058414	1.001248069	60
2.45 ≤ δ ≤ 2.70	1.036610906	0.023445259	1.073116005	1.000767596	60
2.75 ≤ δ ≤ 3.00	1.027117547	0.013597705	1.054424958	1.001807719	60
r = 0.04					
0.05 ≤ δ ≤ 0.30	1.000185069	0.000312401	1.001529211	1.000000000	60
0.35 ≤ δ ≤ 0.60	1.001074483	0.001436583	1.006377134	1.000000000	60
0.65 ≤ δ ≤ 0.90	1.003661135	0.003443809	1.011401509	1.000000000	60
0.95 ≤ δ ≤ 1.20	1.003136384	0.004929368	1.020279624	1.000000000	60
1.25 ≤ δ ≤ 1.50	1.005544958	0.010742322	1.041738554	1.000000000	60
1.55 ≤ δ ≤ 1.80	1.025071421	0.018275106	1.060007115	1.000000000	60
1.85 ≤ δ ≤ 2.10	1.034639660	0.022366518	1.070404138	1.002884121	60
2.15 ≤ δ ≤ 2.40	1.037790228	0.024772340	1.072244702	1.001171073	60
2.45 ≤ δ ≤ 2.70	1.036834968	0.023375550	1.072279548	1.000715514	60
2.75 ≤ δ ≤ 3.00	1.026906268	0.013238566	1.053765755	1.002006556	60
r = 0.05					
0.05 ≤ δ ≤ 0.30	1.000179226	0.000307046	1.001506921	1.000000000	60
0.35 ≤ δ ≤ 0.60	1.001049251	0.001420390	1.006314811	1.000000000	60
0.65 ≤ δ ≤ 0.90	1.003607618	0.003422633	1.011291442	1.000000000	60
0.95 ≤ δ ≤ 1.20	1.003268060	0.005027454	1.019936487	1.000000000	60
1.25 ≤ δ ≤ 1.50	1.006210175	0.011219057	1.041856686	1.000000000	60
1.55 ≤ δ ≤ 1.80	1.025531825	0.018347380	1.059625695	1.000000000	60
1.85 ≤ δ ≤ 2.10	1.034984679	0.022643470	1.069726413	1.002619949	60
2.15 ≤ δ ≤ 2.40	1.038109694	0.024779673	1.071433874	1.001095757	60
2.45 ≤ δ ≤ 2.70	1.036974618	0.023199494	1.071453242	1.000676341	60
2.75 ≤ δ ≤ 3.00	1.026688183	0.012881730	1.052991935	1.002205623	60
r = 0.06					
0.05 ≤ δ ≤ 0.30	1.000173668	0.000301691	1.001485614	1.000000000	60
0.35 ≤ δ ≤ 0.60	1.001024610	0.001404053	1.006252972	1.000000000	60
0.65 ≤ δ ≤ 0.90	1.003555657	0.003401974	1.011181881	1.000000000	60
0.95 ≤ δ ≤ 1.20	1.003420067	0.005130095	1.019588748	1.000000000	60
1.25 ≤ δ ≤ 1.50	1.006844593	0.011703946	1.041969771	1.000000000	60
1.55 ≤ δ ≤ 1.80	1.026016489	0.018419298	1.059616392	1.000000000	60
1.85 ≤ δ ≤ 2.10	1.035210571	0.022727888	1.069043932	1.002363582	60
2.15 ≤ δ ≤ 2.40	1.038389805	0.024743927	1.070626975	1.001023228	60
2.45 ≤ δ ≤ 2.70	1.037067890	0.022991381	1.070635521	1.000682487	60
2.75 ≤ δ ≤ 3.00	1.026454854	0.012526455	1.052110088	1.002404951	60
r = 0.07					
0.05 ≤ δ ≤ 0.30	1.000168331	0.000296326	1.001463697	1.000000000	60
0.35 ≤ δ ≤ 0.60	1.001000642	0.001387777	1.006191443	1.000000000	60
0.65 ≤ δ ≤ 0.90	1.003506775	0.003381498	1.011072944	1.000000000	60
0.95 ≤ δ ≤ 1.20	1.003592993	0.005272916	1.019235008	1.000000000	60
1.25 ≤ δ ≤ 1.50	1.007450708	0.012103609	1.042076619	1.000000000	60
1.55 ≤ δ ≤ 1.80	1.026391848	0.018461102	1.059212278	1.000000000	60
1.85 ≤ δ ≤ 2.10	1.035437369	0.022806104	1.068356991	1.002115736	60
2.15 ≤ δ ≤ 2.40	1.038680440	0.024719346	1.069823533	1.000952661	60

Table 6.1 (Continued)

2.45 ≤ δ ≤ 2.70	1.037114427	0.022764447	1.069824646	1.000631229	60
2.75 ≤ δ ≤ 3.00	1.026222387	0.012178734	1.051247742	1.002604203	60
r = 0.08					
0.05 ≤ δ ≤ 0.30	1.000163310	0.000291103	1.001443011	1.000000000	60
0.35 ≤ δ ≤ 0.60	1.000977314	0.001372026	1.006130006	1.000000000	60
0.65 ≤ δ ≤ 0.90	1.003459010	0.003361032	1.010963024	1.000000000	60
0.95 ≤ δ ≤ 1.20	1.003798163	0.005442297	1.018876308	1.000000000	60
1.25 ≤ δ ≤ 1.50	1.008107007	0.012395070	1.042175222	1.000000000	60
1.55 ≤ δ ≤ 1.80	1.026683679	0.018511543	1.058796229	1.000000000	60
1.85 ≤ δ ≤ 2.10	1.035675343	0.022896260	1.067666481	1.001903958	60
2.15 ≤ δ ≤ 2.40	1.038980441	0.024708167	1.069023064	1.000884361	60
2.45 ≤ δ ≤ 2.70	1.037163179	0.022544530	1.069017092	1.000580748	60
2.75 ≤ δ ≤ 3.00	1.025975731	0.011841444	1.050404924	1.002804249	60
r = 0.09					
0.05 ≤ δ ≤ 0.30	1.000158486	0.000285725	1.001421294	1.000000000	60
0.35 ≤ δ ≤ 0.60	1.000955010	0.001355958	1.006069570	1.000000000	60
0.65 ≤ δ ≤ 0.90	1.003415509	0.003339970	1.010854357	1.000000000	60
0.95 ≤ δ ≤ 1.20	1.004003179	0.005648630	1.018838502	1.000000000	60
1.25 ≤ δ ≤ 1.50	1.008655428	0.012616284	1.042267569	1.000000000	60
1.55 ≤ δ ≤ 1.80	1.026970894	0.018536994	1.058741218	1.000000000	60
1.85 ≤ δ ≤ 2.10	1.035943885	0.023016634	1.067013836	1.001775461	60
2.15 ≤ δ ≤ 2.40	1.039277165	0.024707753	1.068266585	1.000817807	60
2.45 ≤ δ ≤ 2.70	1.037130464	0.022258255	1.068258331	1.000593759	60
2.75 ≤ δ ≤ 3.00	1.025718758	0.011516357	1.049597845	1.003003995	60
r = 0.10					
0.05 ≤ δ ≤ 0.30	1.000153869	0.000280594	1.001400421	1.000000000	60
0.35 ≤ δ ≤ 0.60	1.000932670	0.001340054	1.006008711	1.000000000	60
0.65 ≤ δ ≤ 0.90	1.003373290	0.003320513	1.010745232	1.000000000	60
0.95 ≤ δ ≤ 1.20	1.004174038	0.005824112	1.020263822	1.000000000	60
1.25 ≤ δ ≤ 1.50	1.009117176	0.012719736	1.042352646	1.000000000	60
1.55 ≤ δ ≤ 1.80	1.027299416	0.018539958	1.058709879	1.000000000	60
1.85 ≤ δ ≤ 2.10	1.036224899	0.023135718	1.066637778	1.001650876	60
2.15 ≤ δ ≤ 2.40	1.039431376	0.024554548	1.067638953	1.000753262	60
2.45 ≤ δ ≤ 2.70	1.037060259	0.021942257	1.067599024	1.000542660	60
2.75 ≤ δ ≤ 3.00	1.025463831	0.011204060	1.048876260	1.003240376	60
r = 0.11					
0.05 ≤ δ ≤ 0.30	1.000149253	0.000275378	1.001378639	1.000000000	60
0.35 ≤ δ ≤ 0.60	1.000911395	0.001323912	1.005947698	1.000000000	60
0.65 ≤ δ ≤ 0.90	1.003331352	0.003302995	1.010820920	1.000000000	60
0.95 ≤ δ ≤ 1.20	1.004317275	0.005947737	1.021055537	1.000000000	60
1.25 ≤ δ ≤ 1.50	1.009583096	0.012857186	1.042430581	1.000000000	60
1.55 ≤ δ ≤ 1.80	1.027610883	0.018625472	1.058667180	1.000000000	60
1.85 ≤ δ ≤ 2.10	1.036430236	0.023151781	1.066251504	1.001530713	60
2.15 ≤ δ ≤ 2.40	1.039560464	0.024380575	1.067033228	1.000690562	60
2.45 ≤ δ ≤ 2.70	1.036974817	0.021634124	1.066940345	1.000493155	60
2.75 ≤ δ ≤ 3.00	1.025204720	0.010912948	1.048172559	1.003547962	60

Table 6.1 (Continued)

r = 0.12					
0.05 ≤ δ ≤ 0.30	1.000144629	0.000270223	1.001357886	1.000000000	60
0.35 ≤ δ ≤ 0.60	1.000891144	0.001307632	1.005888027	1.000000000	60
0.65 ≤ δ ≤ 0.90	1.003291141	0.003285902	1.010925023	1.000000000	60
0.95 ≤ δ ≤ 1.20	1.004461533	0.006044061	1.021538129	1.000000000	60
1.25 ≤ δ ≤ 1.50	1.010079816	0.012963358	1.042501496	1.000000000	60
1.55 ≤ δ ≤ 1.80	1.027897971	0.018774588	1.058612148	1.000000000	60
1.85 ≤ δ ≤ 2.10	1.036601461	0.023136059	1.065856124	1.001413708	60
2.15 ≤ δ ≤ 2.40	1.039716418	0.024233306	1.066450328	1.000629949	60
2.45 ≤ δ ≤ 2.70	1.036846730	0.021282918	1.066282686	1.000444572	60
2.75 ≤ δ ≤ 3.00	1.024954929	0.010643946	1.047485774	1.003853028	60
r = 1.00					
0.05 ≤ δ ≤ 0.30	1.001019812	0.001632312	1.007553965	1.000000000	60
0.35 ≤ δ ≤ 0.60	1.002725870	0.003594679	1.013712854	1.000000000	60
0.65 ≤ δ ≤ 0.90	1.005729699	0.006796340	1.028208485	1.000000000	60
0.95 ≤ δ ≤ 1.20	1.009761116	0.008318066	1.032978422	1.000000000	60
1.25 ≤ δ ≤ 1.50	1.020611292	0.013979823	1.050064280	1.000000000	60
1.55 ≤ δ ≤ 1.80	1.040799071	0.020226681	1.074819951	1.001746842	60
1.85 ≤ δ ≤ 2.10	1.035135368	0.013711554	1.065294903	1.004847782	60
2.15 ≤ δ ≤ 2.40	1.029077277	0.010510362	1.042531176	1.006770381	60
2.45 ≤ δ ≤ 2.70	1.020608705	0.010324769	1.040447255	1.004053686	60
2.75 ≤ δ ≤ 3.00	1.011902883	0.005775700	1.025701235	1.003169772	60
r = 2.00					
0.05 ≤ δ ≤ 0.30	1.003941644	0.005984454	1.027953398	1.000012751	60
0.35 ≤ δ ≤ 0.60	1.010828282	0.013006044	1.050005857	1.000015341	60
0.65 ≤ δ ≤ 0.90	1.019782666	0.020349680	1.074632845	1.000122435	60
0.95 ≤ δ ≤ 1.20	1.009837693	0.006316470	1.022674595	1.000299500	60
1.25 ≤ δ ≤ 1.50	1.017837183	0.008758395	1.035517633	1.000666062	60
1.55 ≤ δ ≤ 1.80	1.034257536	0.009582528	1.050429606	1.012990296	60
1.85 ≤ δ ≤ 2.10	1.039222305	0.014032922	1.059495711	1.012988169	60
2.15 ≤ δ ≤ 2.40	1.030541680	0.011884753	1.055088066	1.010993435	60
2.45 ≤ δ ≤ 2.70	1.021808189	0.007834027	1.038698557	1.008128530	60
2.75 ≤ δ ≤ 3.00	1.015378339	0.005477497	1.026356009	1.005500724	60
r = 3.00					
0.05 ≤ δ ≤ 0.30	1.007441517	0.011701660	1.056034399	1.000054729	60
0.35 ≤ δ ≤ 0.60	1.021995475	0.025887283	1.102376291	1.000327872	60
0.65 ≤ δ ≤ 0.90	1.027215521	0.025562911	1.111577941	1.000638380	60
0.95 ≤ δ ≤ 1.20	1.007231330	0.004737655	1.017514224	1.000000000	60
1.25 ≤ δ ≤ 1.50	1.011655728	0.006184363	1.026534875	1.000275141	60
1.55 ≤ δ ≤ 1.80	1.022981224	0.006083946	1.034603334	1.011347473	60
1.85 ≤ δ ≤ 2.10	1.031691947	0.005532924	1.039811481	1.018258067	60
2.15 ≤ δ ≤ 2.40	1.032440562	0.009211008	1.044270447	1.013105365	60
2.45 ≤ δ ≤ 2.70	1.024323500	0.008486258	1.041913850	1.008426755	60
2.75 ≤ δ ≤ 3.00	1.015966711	0.006173138	1.028507806	1.005110124	60
All Cases	1.018047016	0.020568000	1.111577941	1.000000000	9600

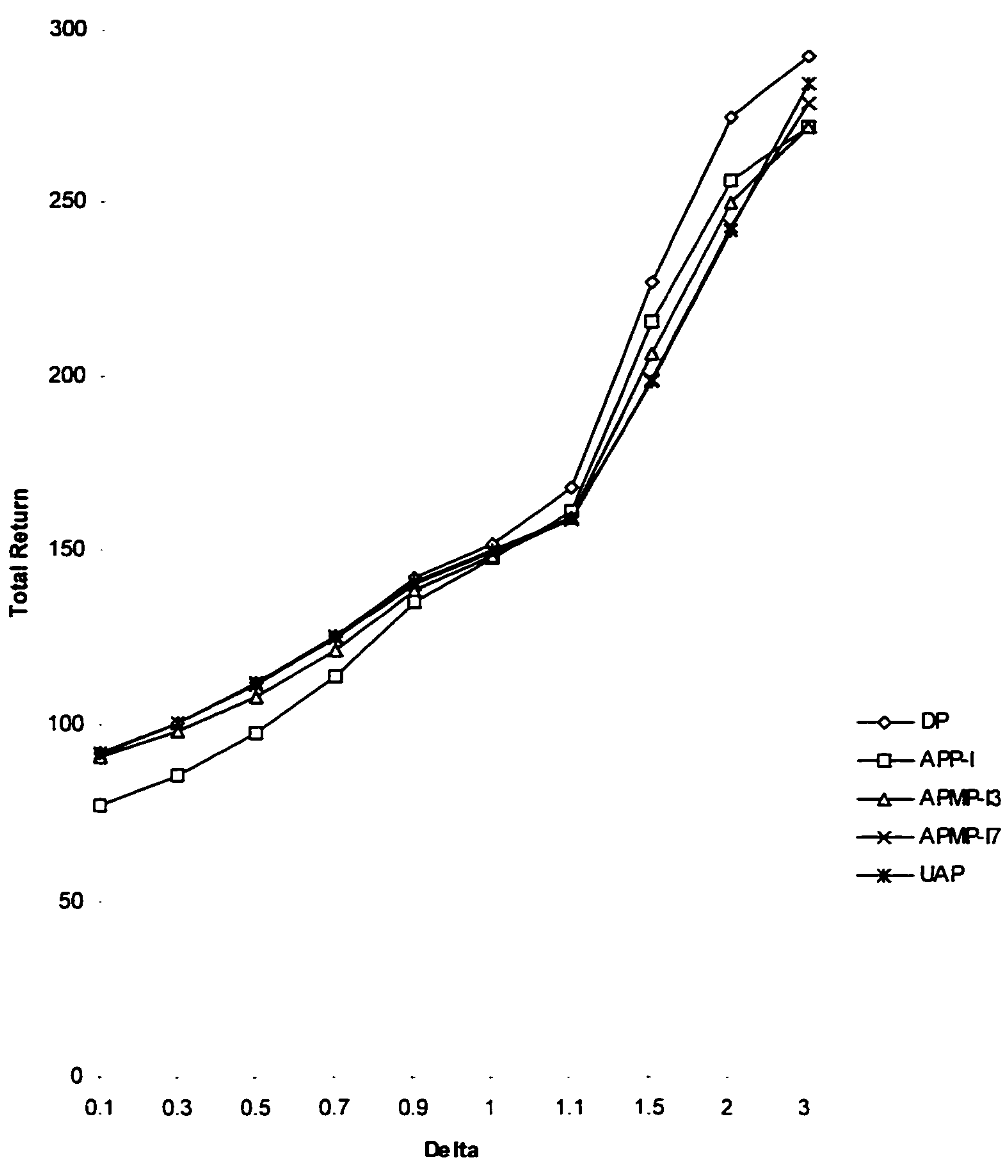

Figure 6.3

DP Policy versus APP-I, APMP-I and UAP (S_1 =30, r = 0)

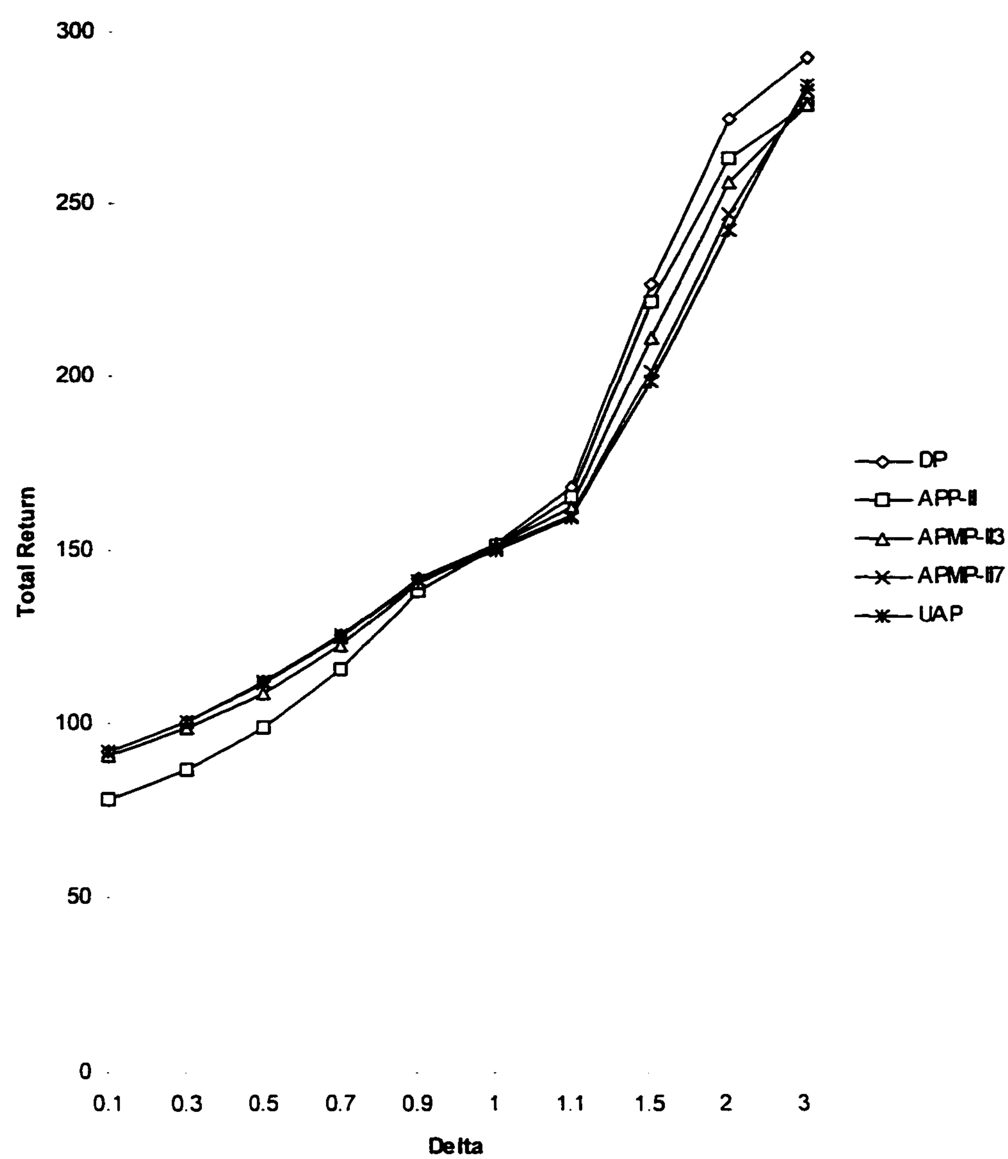

Figure 6.4

DP Policy versus APP-II, APMP-II and UAP (S_I = 30, r = 0)

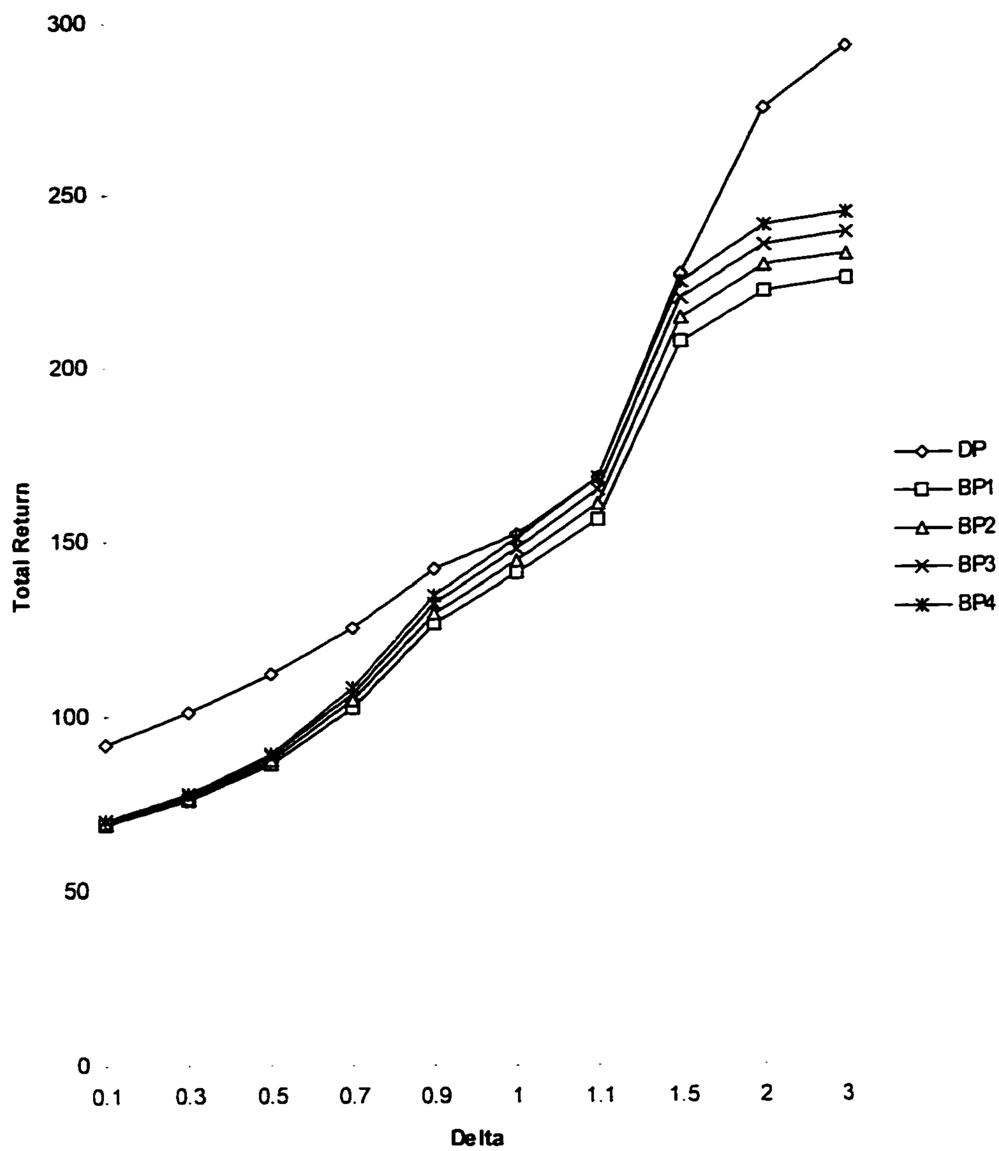

Figure 6.5

DP Policy versus BP Polices ($S_I = 30$, $r = 0$)

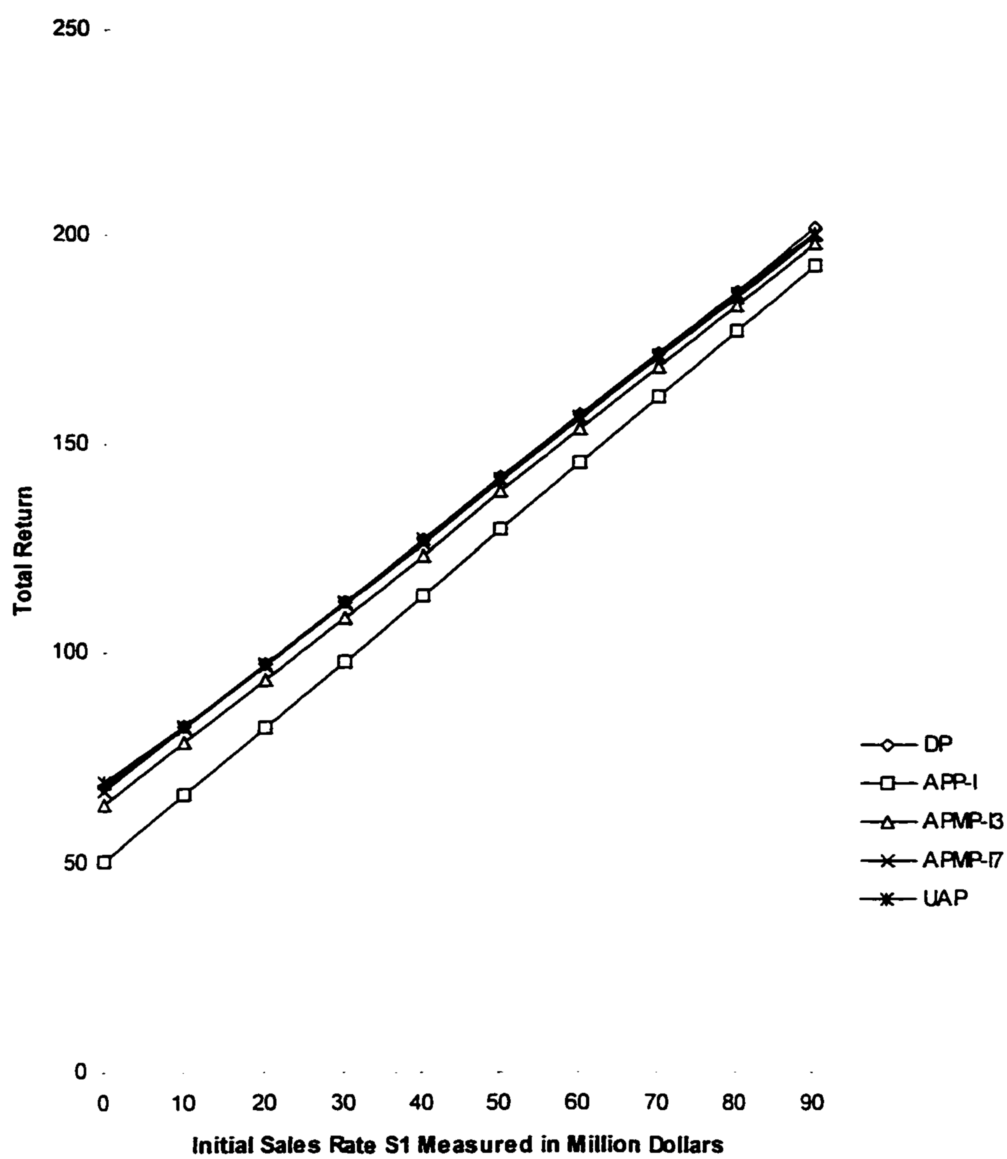

Figure 6.6

DP Policy versus APP-I. APMP-I and UAP (δ = 0.5. r = 0)

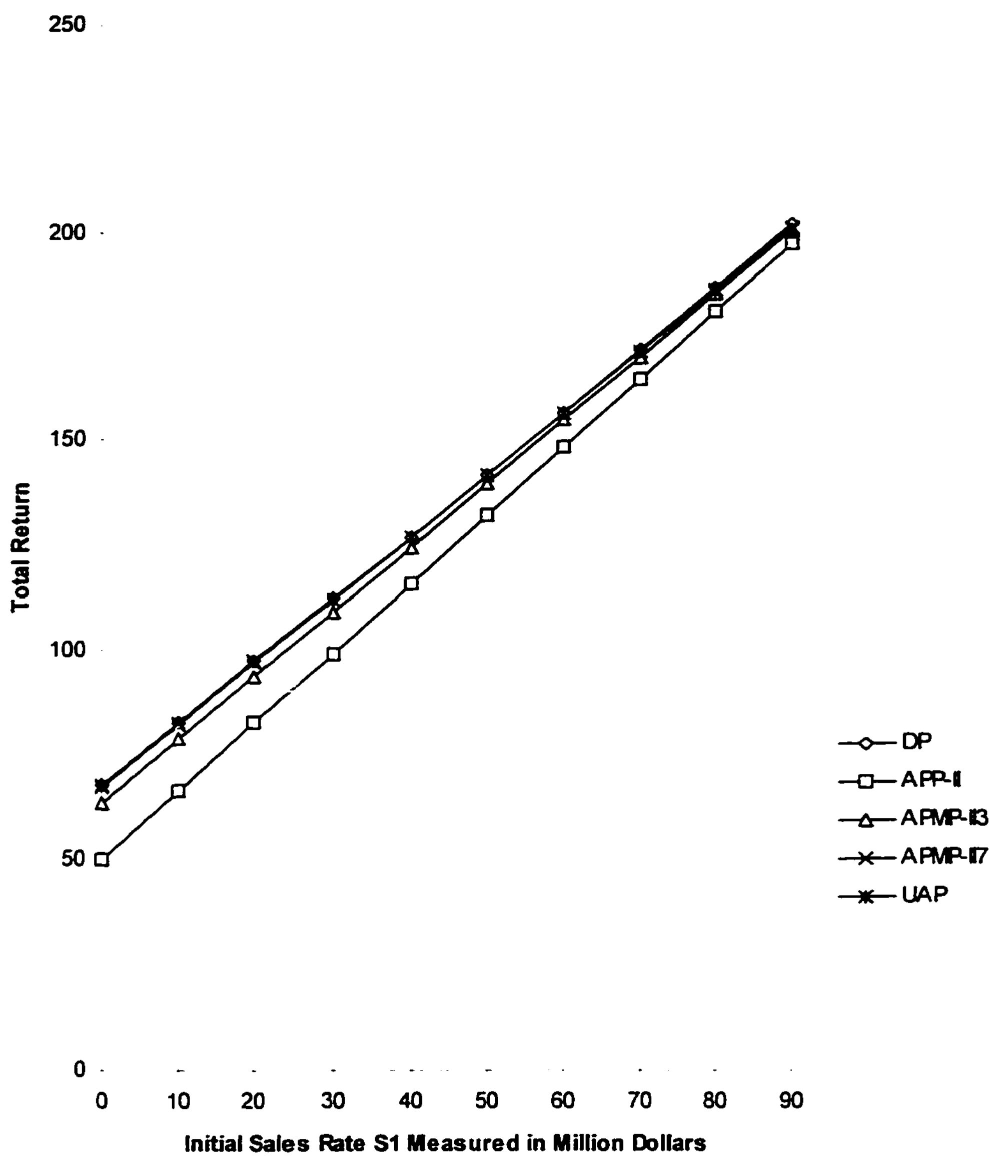

Figure 6.7

DP Policy versus APP-II, APMP-II and UAP ($\delta = 0.5$, $r = 0$)

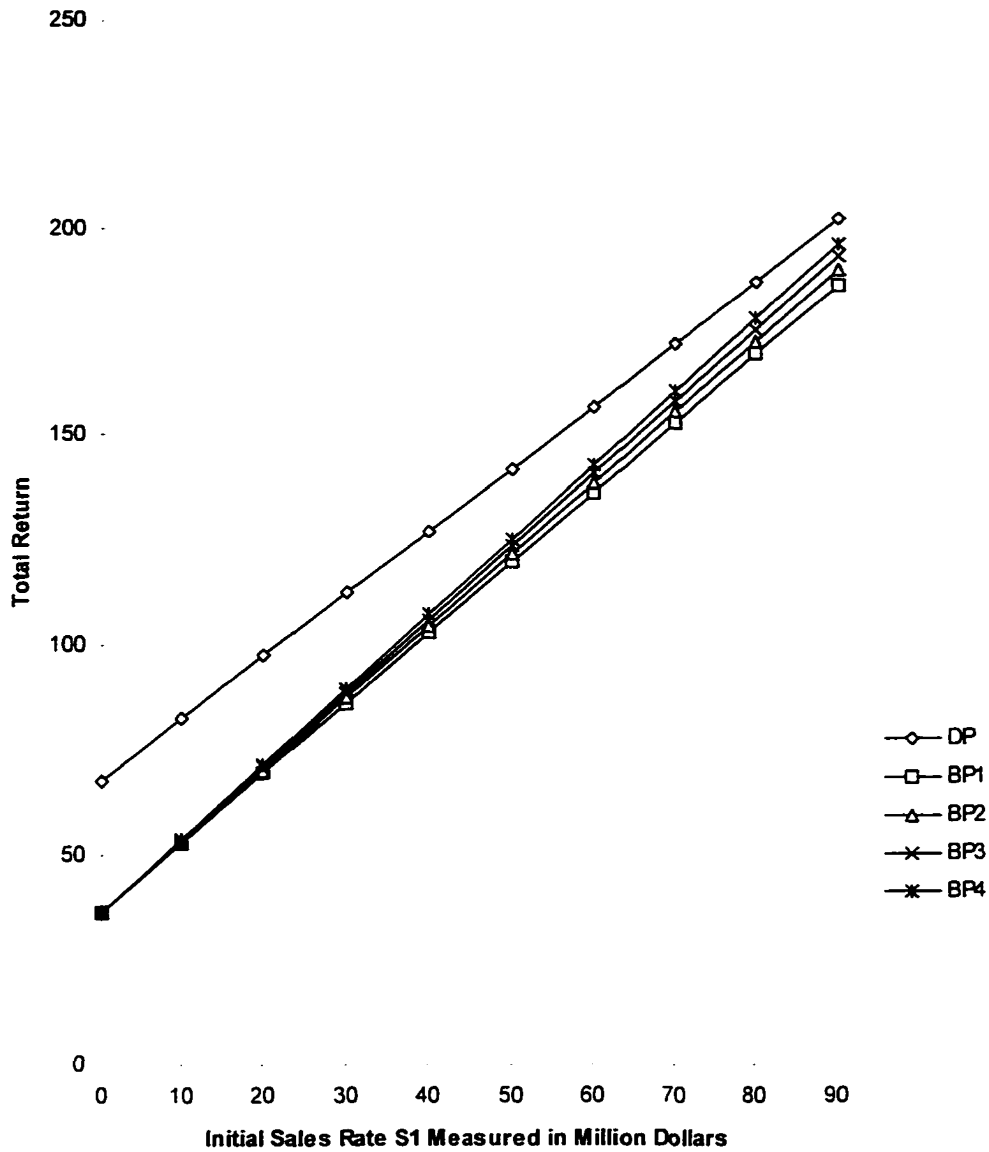

Figure 6.8

DP Policy versus BP Polices ($\delta = 0.5$, $r = 0$)

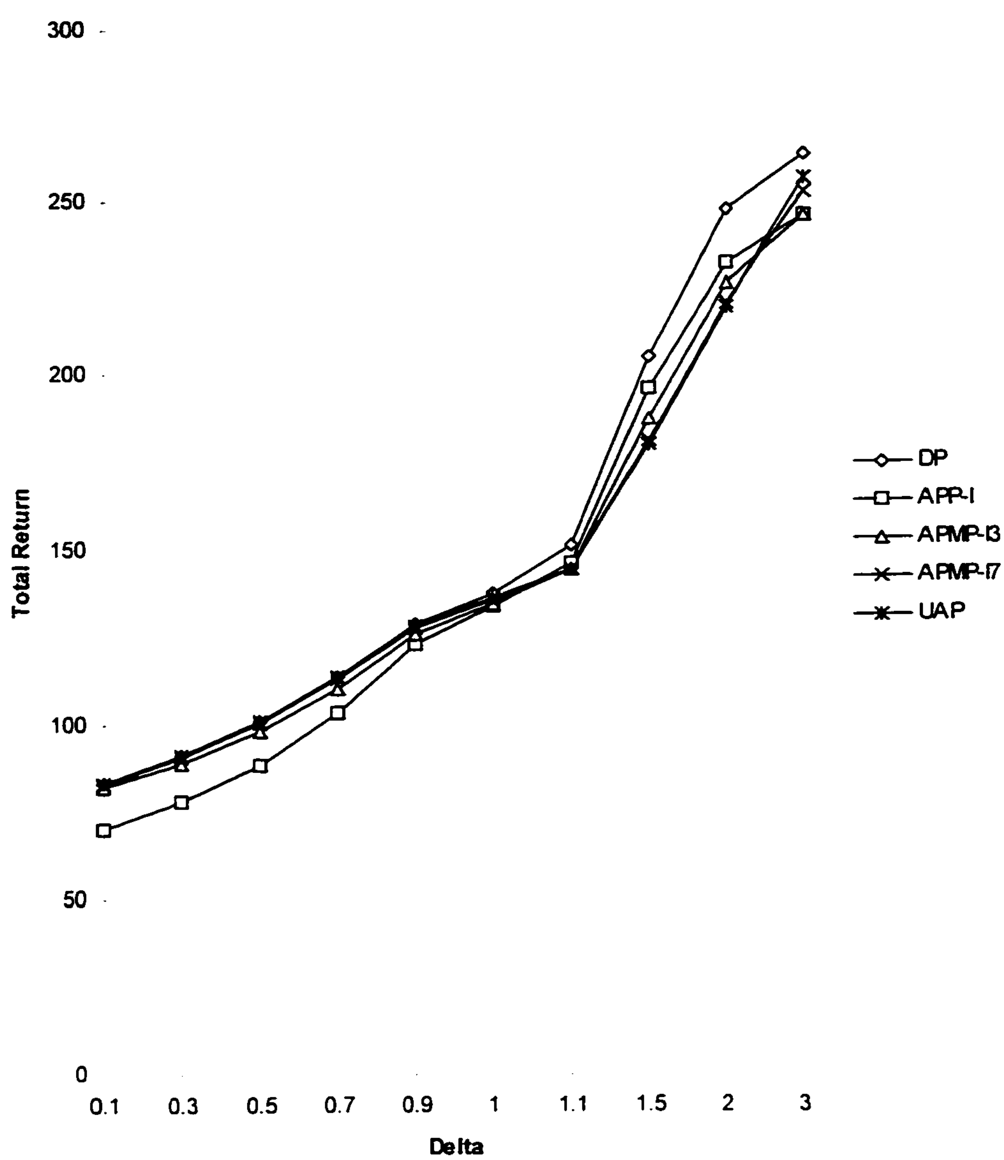

Figure 6.9

DP Policy versus APP-I, APMP-I and UAP (S_1 =30, r = 0.05)

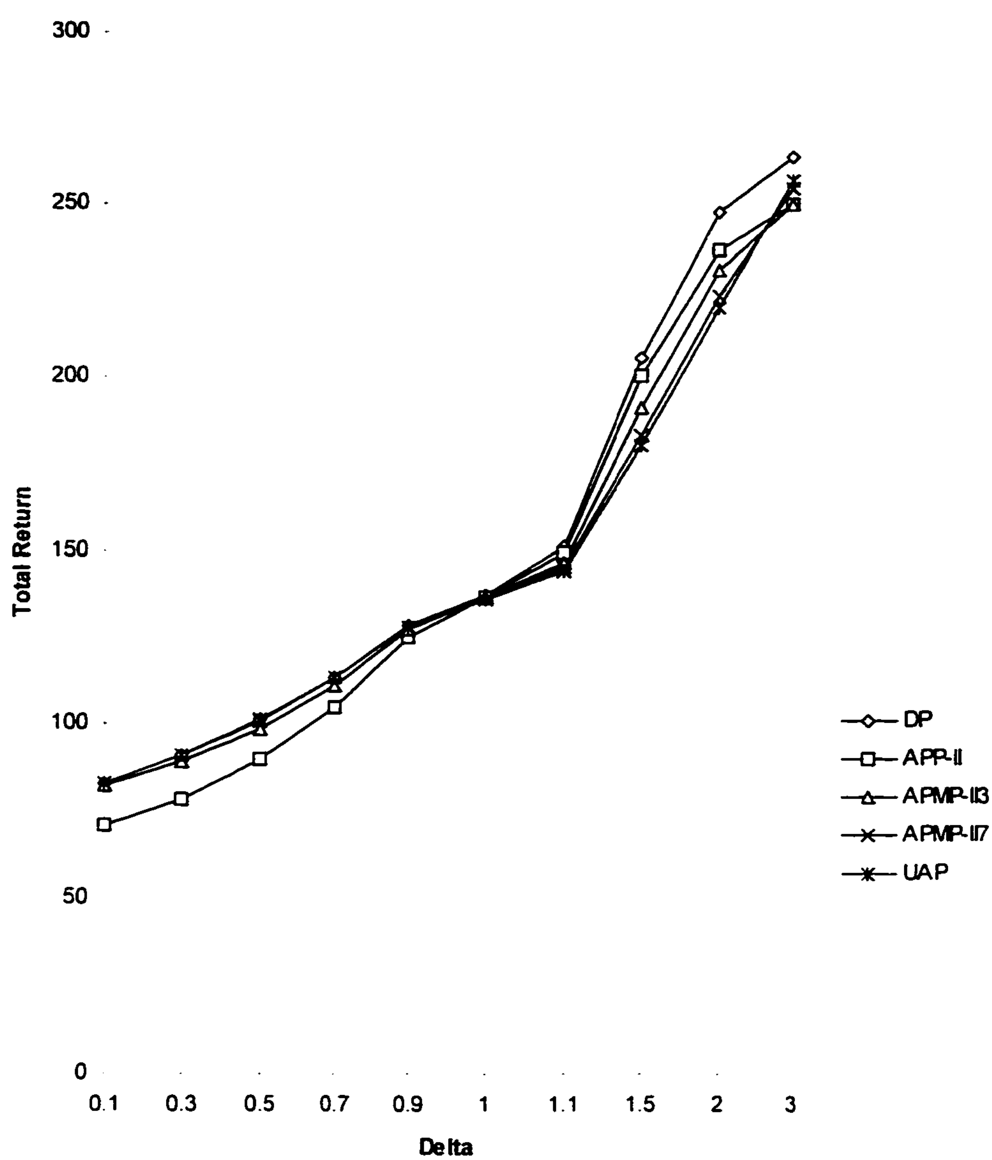

Figure 6.10

DP Policy versus APP-II, APMP-II and UAP ($S_1 = 30$, $r = 0.05$)

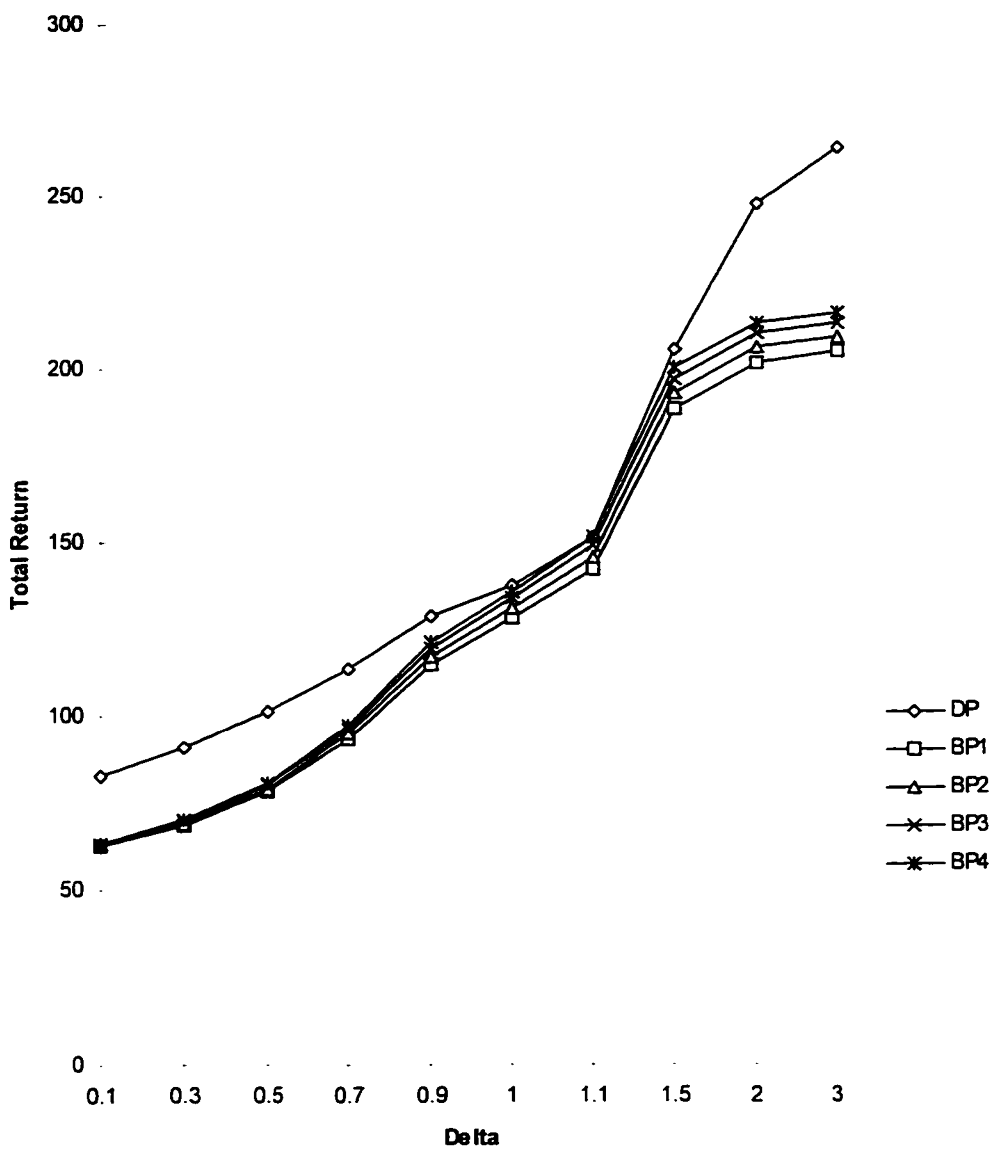

Figure 6.11

DP Policy versus BP Polices (S_1 = 30, r = 0.05)

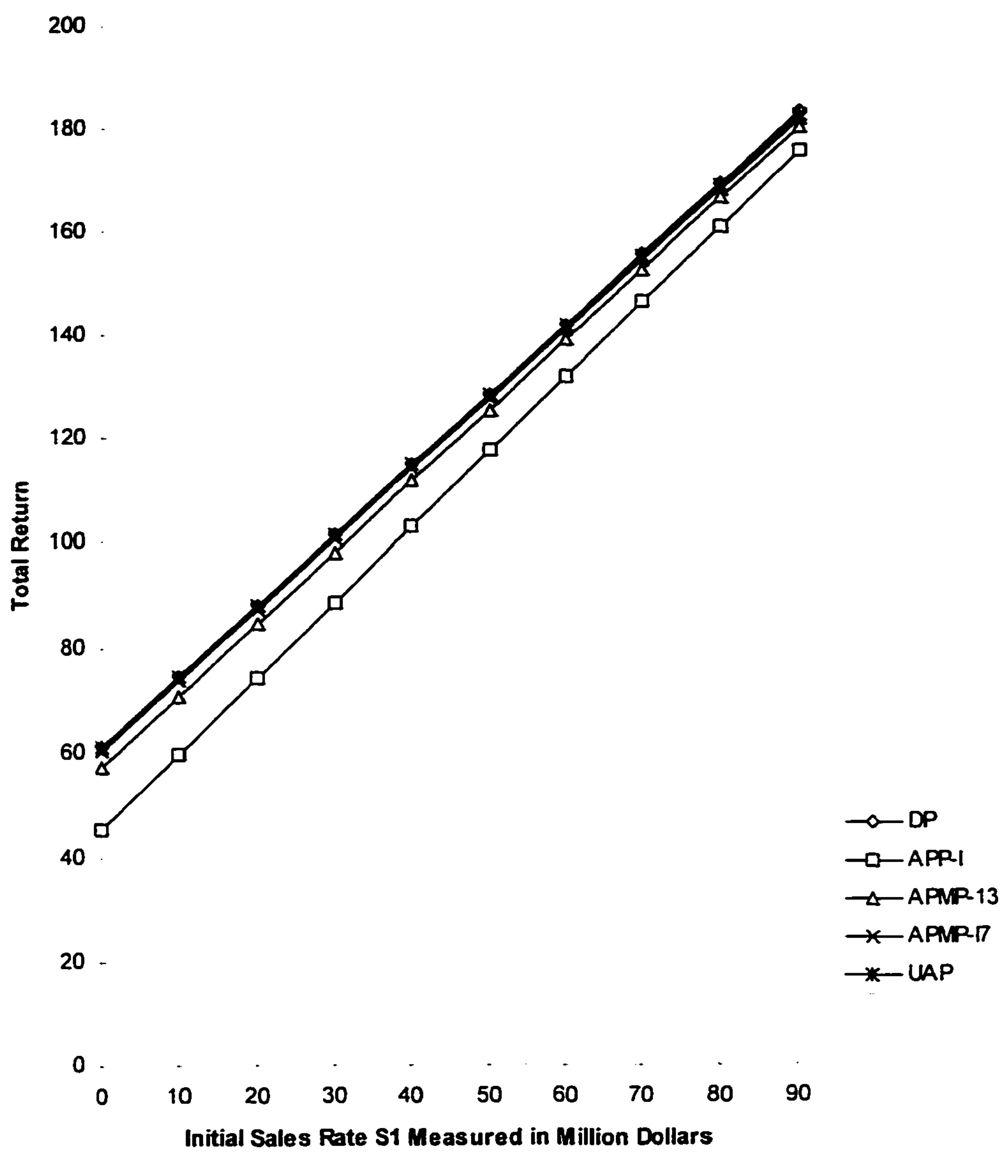

Figure 6.12

DP Policy versus APP-I, APMP-I and UAP ($\delta = 0.5$, $r = 0.05$)

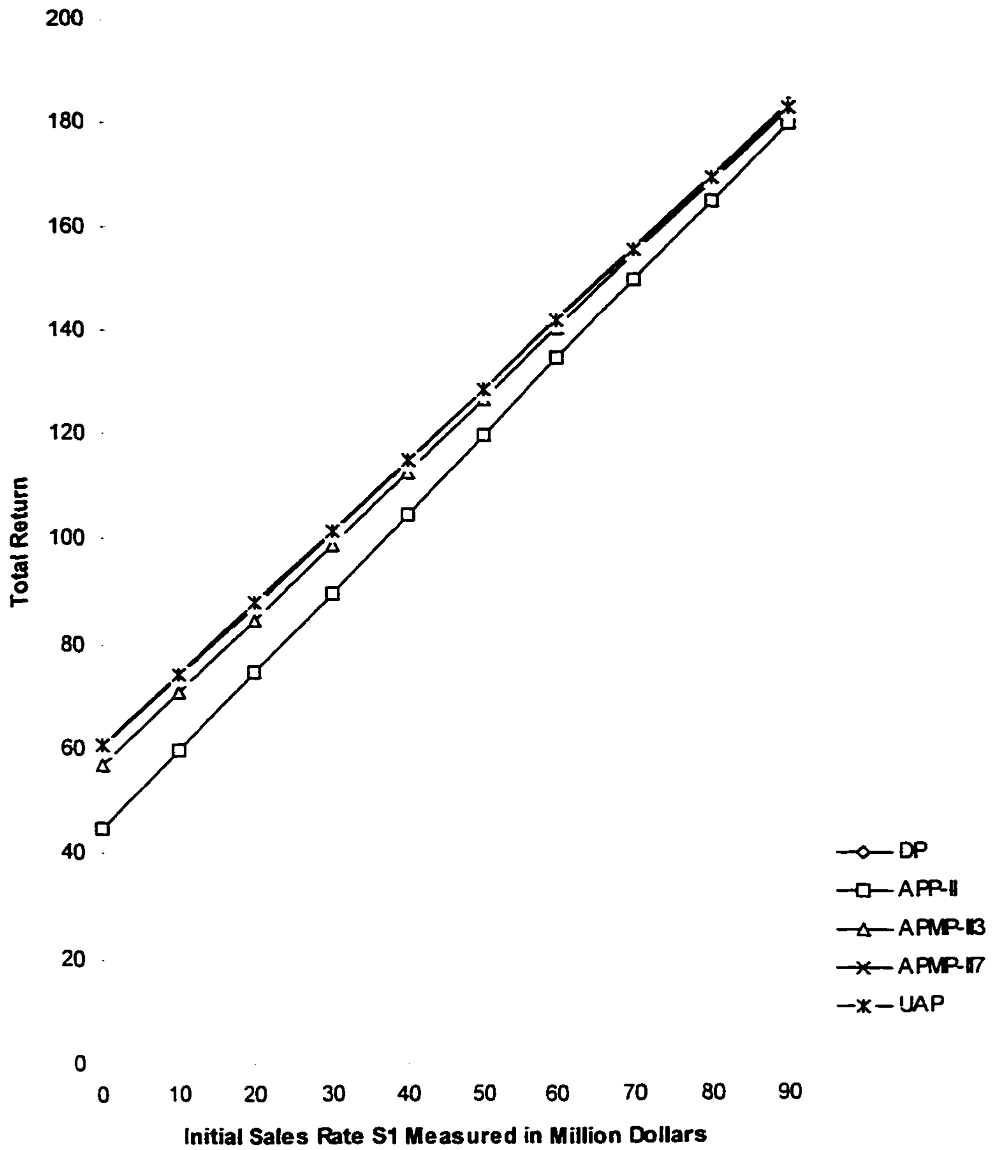

Figure 6.13

DP Policy versus APP-II, APMP-II and UAP ($\delta = 0.5$, $r = 0.05$)

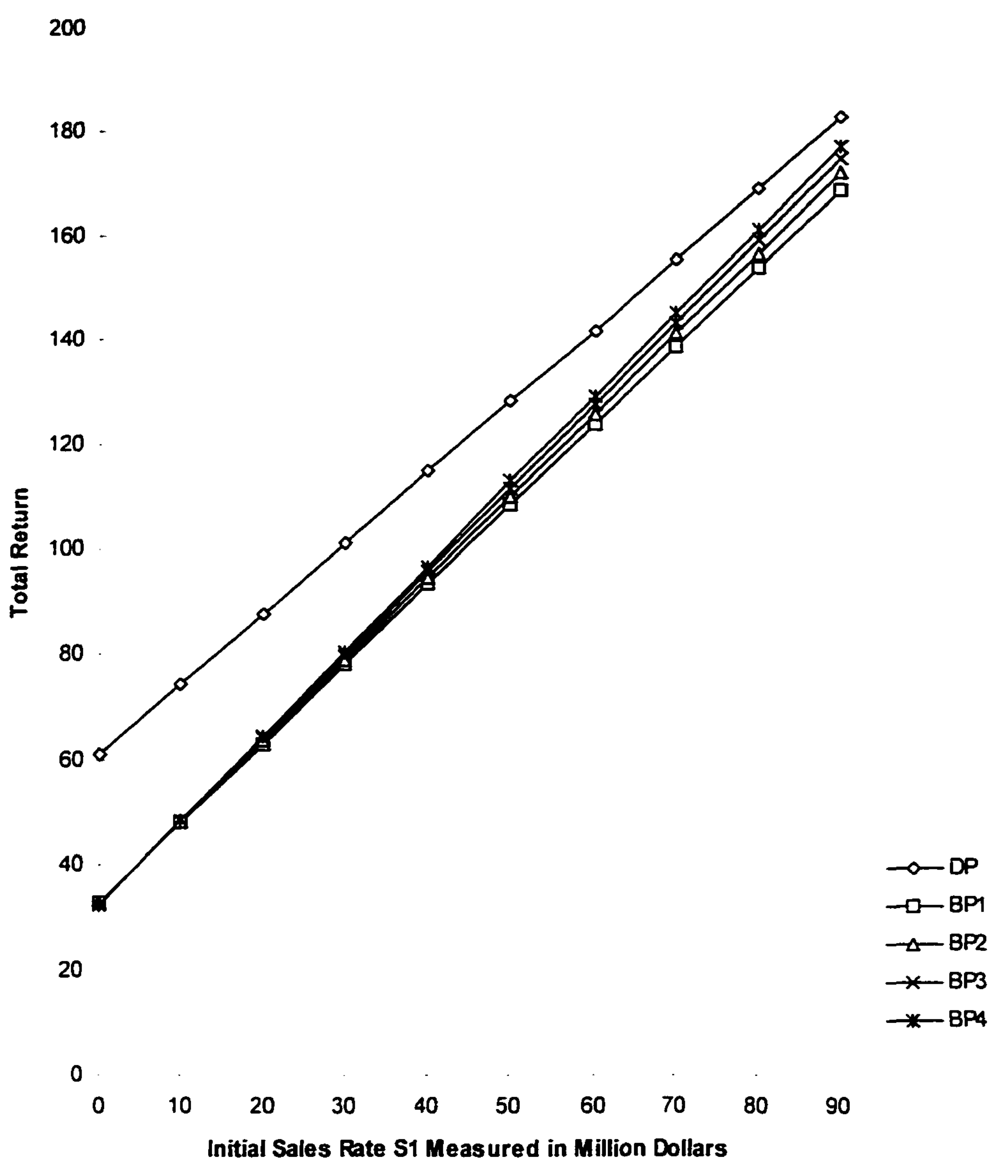

Figure 6.14

DP Policy versus BP Polices ($\delta = 0.5$, $r = 0.05$)

CHAPTER 7

MODEL ESTIMATION

This chapter focuses on assessing empirically the modified Vidale-Wolfe model reviewed in Chapter 2. A discrete analogue of the modified Vidale-Wolfe model is estimated using the Newton-Gauss algorithm of nonlinear regression based on the well-known data of the Lydia E. Pinkham vegetable compound. Two versions of such a model are considered. The first version assumes that the error terms are not autocorrelated, whereas in the second version autocorrelation is assumed to be present. Choice between alternative model specifications is made based on the quality of estimated parameters as well as the predictive power of the proposed models, using the method of one-step-ahead forecasting. Based on the obtained results, the shape of the advertising response function is assessed.

The Data

The firm considered in this empirical study is the frequently studied Lydia E. Pinkham Medicine Company and its product, the Lydia Pinkham vegetable compound, originally examined by Palda (1964). The data used in our empirical investigation are the annual sales and advertising expenditures of the company for the period 1907 through 1960 available in Palda's study. Aaker and Carman (1982) contended that "the Lydia Pinkham data are interesting in many respects: (1) everyone familiar with the situation agrees that advertising caused sales for this product; (2) there are advertising decreases as well as increases; (3) since the product was a monopoly product, competitive effects need not be built into the model; and (4) price was quite stable over long periods of time. Thus it has been possible to focus on the nature of the advertising-to-sales relationship."

Model Discrete Analogue

A discrete analogue of the modified Vidale-Wolfe model was introduced and estimated using OLS by Mesak and Darrat (1992). It can be shown that, using (2.4) and (2.6), the modified Vidale-Wolfe model (2.5) can be restated as in (7.1) upon employing the power function $f(x) = bx^{\delta}$ proposed by Little (1979):

$$dS/dt = mbx^{\delta} - aS - bx^{\delta}S_{t-1}. \tag{7.1}$$

The discrete analogue of (7.1) is given as shown below:

$$S_t - S_{t-1} = mbx_t^{\delta} - aS_{t-1} - bx_t^{\delta}S_{t-1}. \tag{7.2}$$

By incorporating an error term into (7.2) and upon minor rearrangement of terms, (7.2) takes the following form:

$$S_t = mbx_t^{\delta} + (1-a)S_{t-1} - bx_t^{\delta}S_{t-1} + \varepsilon_t \tag{7.3}$$

where, m, a, b, and δ are unknown parameters, and ε_t is a random error term assumed to be normally distributed, serially uncorrelated, and has a zero mean with a constant variance. Following the treatment adopted by Mesak and Darrat (1992), the variables in equation (7.3) are operationalized as follows:

S_t = Annual sales in monetary units in year t divided by the population in year t and divided by a general price deflator in year t (to convert sales to per capita real magnitudes).

x_t = Annual advertising expenditures in monetary units in year t divided by the population in year t and divided by a general price deflator in year t (to convert advertising to per capita real values).

Given the above definition of variables, the related time-series data used in subsequent analyses is found in Appendix C.

Seber and Wild (1989) points out that situations in which data are collected sequentially over time may give rise to substantial serial correlation in the errors. Autocorrelated errors usually exist with economic data in which the response variable and the explanatory variable(s) measure the state of a market at a particular time, and both the response and explanatory variable(s) are time series. If there exists significant evidence of autocorrelation, the order of the autoregressive specification on the random error term ε_t needs to be determined. Bates and Watts (1988) argued that the first order is adequate if time is not the only factor, or the most important factor in the regression situation. The first-order autoregressive specification on ε_t is given by (7.4):

$$\varepsilon_t = \rho\varepsilon_{t-1} + \eta_t \tag{7.4}$$

where, η_t is assumed to be a normally distributed, serially uncorrelated random error with a zero mean and a constant variance and ρ is a parameter such that | ρ | < 1. Substituting for ε_t from (7.4) in (7.3), the annual sales in year t can be expressed in the following form:

$$\begin{aligned} S_t &= \rho S_{t-1} + bx_t^{\delta}(m - S_{t-1}) + (1-a)S_{t-1} \\ &\quad - \rho[bx_{t-1}^{\delta}(m - S_{t-2}) + (1-a)S_{t-2}] + \eta_t . \end{aligned} \tag{7.5}$$

As in linear modeling, autocorrelation in nonlinear modeling is often first detected from plots of regression residuals (Bates and Watts, 1988), or preferably by the formal Durbin-Watson test with the following test statistic (Seber and Wild, 1989):

$$D = \frac{\sum_{i=2}^{n}(e_i - e_{i-1})^2}{\sum_{i=1}^{n} e_i^2} \tag{7.6}$$

where,

e_i = the regression residual associated with the ith observation;

n = the number of observations used in the estimation.

Choosing Among Alternative Model Specifications

Seber and Wild (1989, p.5) argued that even when a linear model approximation is sufficient in modeling nonlinear behavior, a nonlinear model may be used to retain a clear interpretation of the model parameters. Therefore, the nonlinear regression rather than the OLS is used for estimating the nonlinear models (7.3) and (7.5). Since the Newton-Gauss method of nonlinear regression is a much more efficient algorithm (Seber and Wild, 1989, p.621), it is adopted in this empirical study for model estimation. This algorithm represents a non-linear least squares (NLS) method of estimation and is available in SAS. The algorithm was preliminarily performed on (7.3) and (7.5) to estimate all the parameters and the results indicated a singular matrix of partial derivatives, possibly suggesting strong dependency among the parameters. Therefore, instead of estimating all model parameters, (7.3) and (7.5) are estimated assuming reasonable values of m ranging between 0.10 and 1.0 in increments of 0.05.

In order to detect autocorrelation with the Durbin-Watson test procedure, both models (7.3) and (7.5) are estimated over the entire period 1907 through 1960 using the Newton-Gauss method. The Durbin-Watson test is first performed in conjunction with each model specification for selected values of m and the test results are reported in Table 7.1.

As shown in Table 7.1, there is significant evidence of autocorrelation in conjunction with model (7.3) for selected m values. In contrast, no significant evidence of autocorrelation was revealed by the Durbin-Watson test for the 19 m values in relation to (7.5). Therefore, only model (7.5) is further examined to determine the most appropriate value of m.

Mesak and Darrat (1992) suggested one approach to discriminating among alternative model specifications through assessing their predictive power. This is a more rigorous prediction test than forecasting for years on which the estimation is based. In this empirical study, their approach is adopted and in particular, one-period-ahead predictions are made by forecasting sales in year $t+1$ using the data through year t for all the values of m $\in \{0.05k;\ k = 2, 3, \ldots, 20\}$. Only the sales in the years 1956 through 1960 are forecast using this procedure. For example, for m = 0.10, model (7.5) is estimated over the years 1907 through 1955, and the resulting empirical estimates are used to forecast sales for the year 1956 (out-of-sample forecast S_t^f). Then, the actual sales data for the period 1907 through 1956 is used to estimate the model once again and then forecast the sales for the year 1957. The process continues until sales for the year 1960 are forecasted. The root-mean-square percent error (RMSPE) statistic for the case

Table 7.1

Detection of Autocorrelation: The Durbin-Watson Statistic

m	Model (7.3)	Model (7.5)
0.10	1.221379*	2.137237
0.15	1.216154*	2.158653
0.20	1.215014*	2.165598
0.25	1.214634*	2.168914
0.30	1.214481*	2.170835
0.35	1.214414*	2.172083
0.40	1.214384*	2.172957
0.45	1.214373*	2.173602
0.50	1.214370*	2.174098
0.55	1.214372*	2.174490
0.60	1.214377*	2.174808
0.65	1.214383*	2.175072
0.70	1.214389*	2.175293
0.75	1.214395*	2.175482
0.80	1.214402*	2.175645
0.85	1.214408*	2.175787
0.90	1.214414*	2.175912
0.95	1.214420*	2.176022
1.00	1.214425*	2.176121

* significant at the one percent level

of m = 0.10 is calculated from these one-period-ahead forecast series. The same procedure is applied to compute the RMSPE statistic for each of the values of m ranging from 0.10 to 1.0 in increments of 0.05. All the values of the RMSPE statistic are reported in Table 7.2.

Based on the entire data set, it is found that only for the model specifications with the value of m equal to or greater than 0.20, all the estimated parameters, a, b, δ, and ρ, are statistically significant at the 0.05 level. The value of RMSPE is monotonically increasing as the value of m becomes larger. According to the RMSPE criterion, the model specification with m = 0.20 is the best as it minimizes the RMSPE while all the model parameters appear with theoretically expected signs and each is statistically significant at the 0.05 level.

Table 7.3 presents the nonlinear regression results of model (7.5) related to the optimum value of m = 0.20. The goodness of fit, measured by R^2, implies that model (7.5) fits the data quite well. The value of R^2 is approximately found to be equal to 0.93. More importantly, as the estimated parameter δ lies within the interval $0 < \delta < 1$, it is concluded that the shape of the advertising response function is concave.

Table 7.2

Determining the Appropriate Value of m Based on the One-Period-Ahead Forecasting Procedure

Value of m	RMSPE	Value of m	RMSPE
0.10	0.012023155	0.60	0.017021030
0.15	0.014320424	0.65	0.017078418
0.20^{min}	0.015325771	0.70	0.017127180
0.25	0.015877367	0.75	0.017169127
0.30	0.016224054	0.80	0.017205592
0.35	0.016461702	0.85	0.017240572
0.40	0.016634630	0.90	0.017265875
0.45	0.016766048	0.95	0.017291076
0.50	0.016869283	1.00	0.017313665
0.55	0.016952513		

Note: Root-mean-square percent error (RMSPE) is defined as

$$\frac{1}{T}\sum_{t=1}^{T}\left[\frac{S_t^f - S_t^a}{S_t^a}\right]^2,$$

where S_t^f = forecast value of S_t, S_t^a = actual value of S_t, T = the number of forecasting periods = 5. The superscript min indicates the optimum value of m.

Table 7.3

Regression Results of Model (7.5) with the Optimal Value of m = 0.20

Parameter	Estimate	Asymptotic Std. Error	Asymptotic 95 % Confidence Interval Lower	Upper
a	0.6305312924	0.14750070623	0.33396197314	0.9271006116
b	0.8775677200	0.43580077359	0.00133368665	1.7538017534
δ	0.4881177303	0.14670890349	0.19314043331	0.7830950274
ρ	0.8272820385	0.13251434753	0.56084473904	1.0937193380

Source	DF	S.S.	M.S.
Regression	4	0.05665672257	0.01416418064
Residual	48	0.00057312497	0.00001194010
Uncorrected Total	52	0.05722984754	
(Corrected Total)	51	0.00818695539	
R^2	0.93		

Note: R^2 is computed as 1 - (Residual SS/Corrected total SS).

CHAPTER 8

DISCUSSIONS AND IMPLICATIONS

This study has addressed the advertising pulsation problem defined in Chapter 1 and pursued its five objectives: (1) formulation of DP models that represent two versions of the problem, (2) solving the DP models using computing routines, (3) performing sensitivity analyses to assess the role of key model parameters in shaping the optimal policy, (4) comparing the performance of the DP optimal policy with traditional advertising pulsation policies that cost the same, and (5) assessing empirically the plausibility of the modified Vidale-Wolfe model and the shape of the advertising response function.

The purpose of this chapter is to summarize and present the conclusions of the study, highlight its contributions, discuss its managerial implications, state its limitations, and suggest directions for future research.

Summary and Conclusions

Armed with the dynamic programming approach, DP, this study deals with the problem of optimally allocating advertising expenditures over a finite planning horizon comprised of n equal periods to maximize either (1) profits related to the advertising effort or (2) present value of profits related to advertising. The underlying assumption is that the firm is marketing in a monopolistic market a frequently purchased unseasonal product in the mature stage of the life cycle for which advertising is the main element in the marketing mix.

The modified Vidale-Wolfe model is employed to describe the relationship between sales and advertising efforts. Two general dynamic programming models are analytically formulated for finding the optimal allocation of advertising funds over time with respect to whether the time value of money is taken into account. Several numerical examples are provided to illustrate the DP applications. In these examples, two fast computing routines were developed to obtain the results. The performances under traditional advertising pulsation policies, namely, BP(Blitz Policy), APP(Advertising Pulsing Policy), APMP(Advertising Pulsing/Maintenance Policy), and UAP(Uniform Advertising Policy), are also modeled for the purpose of their comparison with that under the DP optimal advertising policy. Computer programs are also developed and run to determine the performances under these traditional policies.

The computational experience associated with these numerical examples reveals that the DP models are properly formulated, and they lead to logically appealing solutions in an adequately short time. The results confirm our expectation that the performance under the DP optimal advertising policy is at least as good as the maximum performance among the four traditional pulsation policies mentioned above (see Appendix B.) The convexity (concavity) parameter, δ, and the initial sales rate, S_1, are found to have significant impacts upon the performance under the DP optimal advertising policy and its behavioral patterns (see Figures 6.1 and 6.2.)

This study has demonstrated that problems of realistic size can be efficiently solved by the DP approach on a microcomputer. The computational efficiency will be enhanced even more by forthcoming hardware and software developments, while permitting the consideration of larger-size problems. It is expected that microcomputer-based approaches, such as those presented in this study, will play an important part in enhancing a firm's profitability through improved allocation of advertising efforts in the future.

The plausibility of the modified Vidale-Wolfe model is empirically investigated using the well-known Lydia Pinkham vegetable compound annual data. Model parameters have been estimated using the Gauss-Newton algorithm of nonlinear regression. The model selected is one corrected for first-order autoregressive residuals. The empirical results show that model parameters are statistically significant and of the expected signs. More importantly, the estimated value of δ is less than unity, implying a concave advertising response function. This latter finding is in line with most previous studies (e.g., Little 1979; Simon and Arndt 1980; Mesak and Darrat 1992).

Contributions

Several distinctive features of this study are highlighted below:

First, to the author's best knowledge, this is the first attempt in the literature in which the DP approach is employed to solve the finite-horizon advertising pulsation problem in which both the initial sales and the discount rates are allowed to be different from zero. In addition, the modeling framework is significantly more flexible than the rigid ones already found in the literature (see Table 2.1.)

Second, computer programs are developed to solve the DP models. These programs are capable of accommodating various combinations of key model parameters. Therefore, this feature offers remarkable flexibility for conducting sensitivity analyses related to the role of the key parameters in shaping the optimal advertising policy. Computer programs with similar flexibility are also designed and implemented for assessing the performance under the traditional advertising pulsation policies.

Third, the modified Vidale-Wolfe model is empirically estimated for the first time using a nonlinear regression procedure to examine directly the statistical properties of model parameters. From this point of view, it is believed that the estimation procedure employed in this dissertation represents an improvement over that used by Mesak and Darrat (1992) who use OLS to estimate the same model. To reduce the problem of dependency among model parameters, a method is proposed such that the value of the market potential, m, is held constant at a particular value each time the model is estimated. The estimation procedure is performed repeatedly for a collection of reasonable m values. Based on the results obtained for assessing the predictive power of alternative model specifications and the quality of their estimated parameters, the optimum value of m is determined.

Managerial Implications

This study provides the marketing manager of a monopolistic firm with an implementable framework for structuring and solving the problem of optimally allocating advertising funds over time, given that the sales-advertising relationship can be captured by the modified Vidale-Wolfe model. The first step is to empirically estimate the sales-advertising relationship using historical data as discussed in Chapter 7. After obtaining statistically sound estimates of the model parameters, a DP model, aimed at achieving maximum advertising productivity over the planning horizon, is developed to identify the optimal advertising policy.

The numerical example presented in this study suggests opportunities for increasing advertising effectiveness through properly allocating advertising funds over a finite multiperiod planning horizon. For alternative advertising policies that cost the same, the optimum total return generated by DP could significantly exceed that under different advertising policies. For firms operating on a large scale, small improvements in allocation of advertising resources can produce substantial monetary returns. From a managerial point of view, however, the implementation of the optimal advertising policy requires a multiperiod orientation and the willingness to accept temporarily low profits or even losses, depending on the situation (Simon, 1980).

Limitations and Directions for Future Research

The modeling framework developed in this study is exploratory, revealing several limitations and many possibilities for future research.

First, advertising expenditures are treated here as the sole decision variable in the DP models. Incorporating other marketing mix variables such as price and distribution would offer avenues for future research. The modeling efforts by Robinson and Lakhani (1975), Lodish (1980), Boronico and Bland (1996), and Ladany (1996) should shed interesting light on this proposed extension.

Second, this study deals with stationary markets for which the parameters of the advertising response function are assumed to be constant over time. In addition, the sales response structure is assumed to be deterministic. Relaxing these assumptions would offer further topics for future research. The studies of Schmalensee (1978), Jagpal and Brick (1982), Mesak (1985), and Aykac, Corstjens, Gautshi, and Horowitz (1989) provide valuable insights in this respect.

Third, Simon (1982), Mesak (1985), and Mahajan and Muller (1986) raised several interesting issues regarding the complex dynamics of advertising effects about which little is known. This study has addressed some of the issues, but many more remain unresolved. Examples of such unresolved inquiries are, Should fresh copy be introduced with new pulses? Which is superior, a time pulsation or a media pulsation policy? The experimental work of Eastlack and Rao (1986), for example, could be used as a reference in this avenue of future research.

Fourth, the sales-advertising relationship is modeled in this study under the assumption that the firm is operating in a monopolistic market. A plausible extension would be to incorporate competition in the modeling framework. In so doing, we will be in a position to address, among other things, the question raised by Simon (1982) and Mesak (1985): What is the effect of different competitive interference patterns? Several pioneering studies have attempted to incorporate competition in conjunction with the original or the modified Vidale-Wolfe model (Deal 1979; Little 1979; Jones 1983; Erickson 1985; Monahan 1987; Park and Hahn 1991). More recently, Villas-Boas (1993), Mesak and Calloway (1995a, b) and Mesak and Means (1998) used game theory to analyze pulsing models of advertising competition.

Finally, the computer programs presented in the numerical example are flexible in that various values of certain key parameters can be dealt with, but they are designed to solve the problem only for a four-period planning horizon. Although four-period budgeting is a commonplace phenomenon, future efforts should be geared towards developing more flexible programs that allow the user to choose the number of periods in a planning horizon within a reasonable range of options. We feel that such an extension can be achieved with a relative ease through following the general approach of dynamic programming discussed herein.

In summary, the main thrust of this study lies in providing a guide to marketing managers in determining an optimal advertising pulsing policy, given a fixed amount for the advertising budget. Moreover, it demonstrates a practical means to assess the plausibility of a model representing the dynamic relationship between advertising and sales as well as the shape of the advertising response function. The application of a DP modeling framework may provide an effective means through which the allocation of advertising funds over time can be determined and implemented.

APPENDIX A

DERIVATION OF KEY EXPRESSIONS

Derivation of Expression (3.10)

The present value of advertising spending over period i is given by

$$I = e^{-(i-1)rT}\int_0^T xe^{-rt}dt = e^{-(i-1)rT}(1-e^{-rT})x/r. \qquad (A.1)$$

(3.10) is obtained by expressing x in terms of I.

Derivation of Expression (4.13)

$$R_i(\xi_i, x_i) = e^{-(i-1)rT}\int_0^T q_i(t)e^{-rt}dt. \qquad (A.2)$$

Substituting for $q_i(t)$ from (4.1) gives

$$R_i(\xi_i, x_i) = e^{-(i-1)rT}\int_0^T [S_i e^{-\phi(xi)t} + S(x_i)(1-e^{-\phi(xi)t})]e^{-rt}dt$$

$$= e^{-(i-1)rT}\{\int_0^T S_i e^{-(\phi(x_i)+r)t}dt + \int_0^T S(x_i)e^{-rt}dt - \int_0^T S(x_i)e^{-(\phi(x_i)+r)t}dt\}$$

$$= (-1)e^{-(i-1)rT}\{\frac{S_i - S(x_i)}{(\phi(x_i)+r)}e^{-(\phi(x_i)+r)t}\Big|_0^T + \frac{S(x_i)}{r}e^{-rt}\Big|_0^T\}$$

$$= e^{-(i-1)rT}\{\frac{S_i - S(x_i)}{\phi(x_i)+r}[1-e^{-(\phi(x_i)+r)T}] + \frac{S(x_i)}{r}(1-e^{-rT})\}.$$

Derivation of Expression (4.14)

$$R_{n+1} = e^{-nrT} \int_0^{\infty} q_{n+1}(t) e^{-rt} dt. \tag{A.3}$$

Substituting for $q_{n+1}(t)$ from (4.2) yields

$$R_{n+1} = e^{-nrT} \int_0^{\infty} S_{n+1} e^{-(a+r)t} dt$$

$$= (-1) e^{-nrT} [(S_{n+1} / (a+r)] e^{-(a+r)t} \Big|_0^{\infty}$$

$$= e^{-nrT} S_{n+1} / (a+r).$$

APPENDIX B

PERFORMANCE OF ADVERTISING POLICIES

Table A1
Returns Of DP
(r = 0.00)

	S_1	x_1^*	x_2^*	x_3^*	x_4^*	Total Return
δ = 0.1	0	4	4	4	4	41.74997
	10	4	4	4	4	58.48861
	30	4	4	4	4	91.96588
	60	3.2	4	4	4.8	142.2
	90	1.6	3.2	4.8	6.4	192.4782
δ = 0.3	0	4	4	4	4	53.47512
	10	4	4	4	4	69.3075
	30	4	4	4	4	100.9723
	60	3.2	4	4	4.8	148.5551
	90	1.6	3.2	4.8	6.4	196.3634
δ = 0.5	0	4	4	4	4	67.8886
	10	4	4	4	4	82.61367
	30	3.2	4	4	4.8	112.0946
	60	2.4	3.2	4.8	5.6	156.5099
	90	0.8	2.4	4.8	8	201.5184
δ = 0.7	0	4	4	4	4	85.22369
	10	4	4	4	4	98.62789
	30	3.2	4	4	4.8	125.5673
	60	2.4	3.2	4	6.4	166.4323
	90	0.8	1.6	4	9.6	208.5651
δ = 0.9	0	4.8	3.2	3.2	4.8	105.6334
	10	4	3.2	3.2	5.6	117.5369
	30	3.2	2.4	4	6.4	141.6997
	60	0.8	1.6	4	9.6	179.3472
	90	0	0	2.4	13.6	219.2776

Table A1 (Continued)

δ = 1.0	0	8	0	0	8	117.8516
	10	7.2	0	0	8.8	128.864
	30	4	0	0	12	151.7229
	60	0	0	0	16	188.8544
	90	0	0	0	16	227.3318
δ = 1.1	0	8	0	0	8	136.0667
	10	6.4	0	0	9.6	145.8376
	30	0	0	0	16	167.5943
	60	0	0	0	16	202.046
	90	0	0	0	16	236.4976
δ = 1.5	0	8	0	0	8	212.4199
	10	7.2	0	0	8.8	216.8173
	30	5.6	0	0	10.4	226.5549
	60	0	0	0	16	246.1226
	90	0	0	0	16	267.8632
δ = 2.0	0	8	0	0	8	271.2067
	10	8	0	0	8	272.1312
	30	8	0	0	8	273.98
	60	7.2	0	0	8.8	277.5468
	90	0	6.4	0	9.6	282.9146
δ = 3.0	0	8	0	0	8	291.6291
	10	8	0	0	8	291.7263
	30	5.6	4	0	6.4	292.2
	60	5.6	4	0	6.4	293.0424
	90	4	4.8	0	7.2	294.2862

Table A2
Returns of BP
(r = 0.00)

	S_1	BP_1	BP_2	BP_3	BP_4
δ = 0.1	0	12.78624	12.78624	12.78624	12.78624
	10	31.58508	31.72622	31.85077	31.96069
	30	69.18275	69.60617	69.97984	70.3096
	60	125.5793	126.4261	127.1734	127.833
	90	181.9758	183.246	184.3671	185.3563
δ = 0.3	0	21.75393	21.75393	21.75393	21.75393
	10	39.7113	39.95132	40.16313	40.35005
	30	75.62604	76.34608	76.98152	77.5423
	60	129.4982	130.9382	132.2091	133.3307
	90	183.3703	185.5304	187.4367	189.119
δ = 0.5	0	36.4003	36.4003	36.4003	36.4003
	10	52.98525	53.38652	53.74065	54.05317
	30	86.15515	87.35899	88.42137	89.35892
	60	135.91	138.3177	140.4424	142.3175
	90	185.6649	189.2764	192.4635	195.2762
δ = 0.7	0	59.22458	59.22458	59.22458	59.22458
	10	73.67615	74.3281	74.90345	75.4112
	30	102.5793	104.5351	106.2612	107.7844
	60	145.934	149.8457	153.2978	156.3443
	90	189.2887	195.1563	200.3345	204.9042
δ = 0.9	0	92.03432	92.03432	92.03432	92.03432
	10	103.4334	104.444	105.3359	106.123
	30	126.2315	129.2634	131.9391	134.3003
	60	160.4287	166.4925	171.8438	176.5663
	90	194.6259	203.7217	211.7486	218.8324
δ = 1.0	0	111.8995	111.8995	111.8995	111.8995
	10	121.4611	122.6876	123.7701	124.7253
	30	140.5843	144.264	147.5112	150.3769
	60	169.2692	176.6285	183.123	188.8544
	90	197.9541	208.993	218.7347	227.3318
δ = 1.1	0	133.1427	133.1427	133.1427	133.1427
	10	140.7518	142.2077	143.4926	144.6266
	30	155.97	160.3379	164.1926	167.5943
	60	178.7974	187.5332	195.2425	202.046
	90	201.6248	214.7285	226.2924	236.4976

Table A2 (Continued)

δ = 1.5	0	202.6415	202.6415	202.6415	202.6415
	10	204.0858	206.2661	208.1903	209.8883
	30	206.9743	213.5154	219.2879	224.382
	60	211.3072	224.3893	235.9342	246.1226
	90	215.64	235.2632	252.5806	267.8632
δ = 2.0	0	220.908	220.908	220.908	220.908
	10	221.1015	223.4288	225.4826	227.2952
	30	221.4884	228.4704	234.632	240.0695
	60	222.0688	236.0328	248.3559	259.2311
	90	222.6492	243.5952	262.0799	278.3926
δ = 3.0	0	224.7408	224.7408	224.7408	224.7408
	10	224.753	227.1017	229.1743	231.0034
	30	224.7774	231.8233	238.0413	243.5286
	60	224.814	238.9058	251.3418	262.3164
	90	224.8506	245.9883	264.6422	281.1042

Table A3
Returns of APMP-I
(r = 0.00)

	S_1	λ=0.0	λ=0.3	λ=0.7	λ=1.0
δ = 0.1	0	23.35345	40.63125	41.58986	41.74997
	10	41.40785	57.43946	58.33464	58.48861
	30	77.51665	91.05589	91.8242	91.96589
	60	131.6799	141.4805	142.0585	142.1818
	90	185.8431	191.9051	192.2929	192.3977
δ = 0.3	0	34.4602	50.37573	53.01354	53.47512
	10	51.59447	66.38467	68.85764	69.3075
	30	85.86303	98.40256	100.5458	100.9723
	60	137.2659	146.4294	148.0781	148.4694
	90	188.6687	194.4562	195.6105	195.9666
δ = 0.5	0	50.1809	63.55943	67.21983	67.8886
	10	66.01917	78.48949	81.94602	82.61367
	30	97.69569	108.3496	111.3984	112.0638
	60	145.2105	153.1398	155.5769	156.239
	90	192.7253	197.93	199.7555	200.4142
δ = 0.7	0	71.6776	81.00655	84.54839	85.22369
	10	85.75716	94.51606	97.91648	98.62788
	30	113.9163	121.5351	124.6527	125.4363
	60	156.155	162.0636	164.7569	165.6489
	90	198.3937	202.5921	204.8612	205.8615
δ = 0.9	0	99.60522	103.2907	105.124	105.4923
	10	111.4274	115.0019	116.8909	117.3694
	30	135.0719	138.4241	140.4247	141.1236
	60	170.5386	173.5575	175.7253	176.7549
	90	206.0052	208.6909	211.0259	212.3862
δ = 1.0	0	115.8432	116.2482	116.5654	116.6367
	10	126.3701	126.9239	127.4488	127.6832
	30	147.4239	148.2753	149.2155	149.7763
	60	179.0047	180.3024	181.8658	182.9159
	90	210.5854	212.3294	214.516	216.0556
δ = 1.1	0	133.2669	130.2772	128.6743	128.3503
	10	142.421	139.842	138.6292	138.5318
	30	160.7291	158.9717	158.5391	158.8948
	60	188.1914	187.6662	188.4039	189.4393
	90	215.6536	216.3606	218.2687	219.9839

Table A3 (Continued)

$\delta = 1.5$	0	203.8147	191.1266	180.5877	178.0731
	10	207.6952	196.0541	186.6719	184.701
	30	215.4561	205.909	198.8404	197.9568
	60	227.0974	220.6914	217.093	217.8405
	90	238.7388	235.4738	235.3457	237.7242
$\delta = 2.0$	0	253.3547	245.8676	235.7489	232.3861
	10	254.2099	247.1867	238.1101	235.5165
	30	255.9203	249.8249	242.8323	241.7772
	60	258.486	253.7821	249.9156	251.1683
	90	261.0516	257.7393	256.999	260.5593
$\delta = 3.0$	0	271.0909	270.7823	277.3505	281.4854
	10	271.1881	270.9401	277.7008	282.2373
	30	271.3825	271.2556	278.4014	283.7411
	60	271.674	271.7289	279.4524	285.9968
	90	271.9656	272.2021	280.5033	288.2526

Table A4
Returns of APMP-II
(r = 0.00)

	S_1	λ=0.0	λ=0.3	λ=0.7	λ=1.0
δ = 0.1	0	23.35345	40.63125	41.58986	41.74997
	10	41.63646	57.4737	58.34717	58.48861
	30	78.20249	91.15859	91.86179	91.96589
	60	133.0515	141.6859	142.1337	142.1818
	90	187.9006	192.2133	192.4057	192.3977
δ = 0.3	0	34.4602	50.37573	53.01354	53.47512
	10	51.93121	66.50926	68.90531	69.3075
	30	86.87321	98.77632	100.6888	100.9723
	60	139.2862	147.1769	148.3641	148.4694
	90	191.6993	195.5775	196.0394	195.9666
δ = 0.5	0	50.1809	63.55942	67.21983	67.8886
	10	66.50819	78.74099	82.04586	82.61367
	30	99.16274	109.1041	111.6979	112.0638
	60	148.1446	154.6488	156.176	156.239
	90	197.1264	200.1935	200.6541	200.4142
δ = 0.7	0	71.6776	81.00656	84.54839	85.22369
	10	86.45284	94.9398	98.0901	98.62788
	30	116.0033	122.8063	125.1736	125.4363
	60	160.329	164.606	165.7987	165.6489
	90	204.6547	206.4057	206.4239	205.8615
δ = 0.9	0	99.60522	103.2907	105.124	105.4923
	10	112.3884	115.649	117.1638	117.3694
	30	137.9546	140.3656	141.2435	141.1236
	60	176.304	177.4405	177.3631	176.7549
	90	214.6535	214.5154	213.4826	212.3862
δ = 1.0	0	115.8432	116.2482	116.5654	116.6367
	10	127.4832	127.7018	127.7817	127.6832
	30	150.7633	150.6089	150.2144	149.7763
	60	185.6833	184.9696	183.8633	182.9159
	90	220.6034	219.3302	217.5123	216.0556
δ = 1.1	0	133.2669	130.2772	128.6743	128.3503
	10	143.6954	140.7616	139.0289	138.5318
	30	164.5524	161.7304	159.7382	158.8948
	60	195.838	193.1836	190.8022	189.4393
	90	227.1235	224.6368	221.8661	219.9839

Table A4 (Continued)

$\delta = 1.5$	0	203.8147	191.1266	180.5877	178.0731
	10	209.5893	197.5826	187.3923	184.701
	30	221.1384	210.4945	201.0016	197.9568
	60	238.462	229.8624	221.4156	217.8405
	90	255.7857	249.2303	241.8295	237.7242
$\delta = 2.0$	0	253.3547	245.8676	235.7489	232.3861
	10	256.4595	249.2267	239.2225	235.5165
	30	262.669	255.9449	246.1695	241.7772
	60	271.9834	266.0222	256.5901	251.1683
	90	281.2978	276.0995	267.0106	260.5593
$\delta = 3.0$	0	271.0909	270.7823	277.3504	281.4854
	10	273.5268	273.1634	278.8963	282.2373
	30	278.3984	277.9257	281.988	283.7411
	60	285.7059	285.0691	286.6256	285.9968
	90	293.0133	292.2124	291.2632	288.2526

Table B1
Returns Of DP
(r = 0.01)

	S_1	y_1^*	y_2^*	y_3^*	y_4^*	Total Return
δ = 0.1	0	1	1	1	1	40.751839
	10	1	1	1	1	57.174976
	30	1	1	1	1	90.021255
	60	0.8	1	1	1.2	139.30771
	90	0.4	0.8	1.2	1.6	188.63734
δ = 0.5	0	1	1	1	1	66.382095
	10	1	1	1	1	80.830719
	30	0.8	1	1	1.2	109.75255
	60	0.6	0.8	1.2	1.4	153.32967
	90	0.2	0.6	1.2	2	197.48863
δ = 0.9	0	1.2	0.8	0.8	1.2	103.43779
	10	1	0.8	0.8	1.4	115.10532
	30	0.8	0.8	0.8	1.6	138.79926
	60	0.2	0.4	1	2.4	175.70822
	90	0	0	0.6	3.4	214.87711
δ = 1.0	0	2	0	0	2	115.44118
	10	1.8	0	0	2.2	126.23062
	30	1	0.2	0	2.8	148.60855
	60	0	0	0	4	184.96655
	90	0	0	0	4	222.74831
δ = 1.1	0	2	0	0	2	133.30789
	10	1.6	0	0	2.4	142.85614
	30	0	0	0	4	163.99878
	60	0	0	0	4	197.84996
	90	0	0	0	4	231.70116
δ = 1.5	0	2	0	0	2	208.10933
	10	1.8	0	0	2.2	212.3688
	30	1.4	0	0	2.6	221.79996
	60	0	0	0	4	240.57756
	90	0	0	0	4	262.1207
δ = 3.0	0	2	0	0	2	285.3616
	10	1.6	0	0.8	1.6	285.483
	30	1.4	1	0	1.6	285.9906
	60	1.4	1	0	1.6	286.82959
	90	1	1.2	0	1.8	288.03204

Table B2
Returns of BP
(r = 0.01)

	S_1	BP_1	BP_2	BP_3	BP_4
δ = 0.1	0	12.52155	12.49331	12.465134	12.437022
	10	30.953064	31.065388	31.16098	31.241846
	30	67.816093	68.209541	68.552673	68.851494
	60	123.11063	123.92577	124.64021	125.26596
	90	178.40518	179.64201	180.72774	181.68044
δ = 0.5	0	35.663597	35.614922	35.566299	35.517742
	10	51.925419	52.273945	52.575382	52.835323
	30	84.449059	85.591995	86.593544	87.470482
	60	133.23453	135.56908	137.62077	139.42322
	90	182.01997	185.54616	188.64803	191.37596
δ = 0.9	0	90.202538	90.127014	90.051453	89.97583
	10	101.38041	102.30175	103.10519	103.80463
	30	123.73614	126.65123	129.21266	131.46222
	60	157.26973	163.17545	168.37387	172.94861
	90	190.80333	199.69966	207.53508	214.435
δ = 1.0	0	109.67669	109.58562	109.49442	109.40311
	10	119.05322	120.17203	121.14766	121.99702
	30	137.8063	141.34485	144.45412	147.18484
	60	165.9359	173.10408	179.41382	184.96657
	90	194.06549	204.86331	214.3735	222.74829
δ = 1.1	0	130.49966	130.38254	130.26518	130.14761
	10	137.96228	139.28214	140.43179	141.43134
	30	152.88747	157.08136	160.76503	163.99878
	60	175.27527	183.78018	191.26486	197.84998
	90	197.66309	210.479	221.76469	231.70114
δ = 1.5	0	198.58894	198.22327	197.85741	197.49136
	10	200.01152	201.81337	203.35674	204.67239
	30	202.85669	208.99355	214.35538	219.03445
	60	207.12444	219.76385	230.85335	240.57758
	90	211.39218	230.53413	247.35132	262.1207
δ = 3.0	0	220.30212	219.75395	219.20714	218.66168
	10	220.31429	222.11177	223.62982	224.90207
	30	220.33859	226.82738	232.47517	237.38283
	60	220.37506	233.9008	245.74323	256.10397
	90	220.41151	240.97423	259.01126	274.82513

Table B3
Returns of APMP-I
(r = 0.01)

	S_1	λ=0.0	λ=0.3	λ=0.7	λ=1.0
δ = 0.1	0	22.819628	39.663631	40.59697	40.751839
	10	40.525734	56.154419	57.025997	57.174976
	30	75.93795	89.135986	89.884048	90.021255
	60	129.05626	138.60834	139.17113	139.29068
	90	182.17461	188.0807	188.45821	188.56009
δ = 0.5	0	49.102043	62.167351	65.735481	66.382095
	10	64.63678	76.815063	80.184631	80.830719
	30	95.706245	106.11048	109.08292	109.72794
	60	142.31046	150.0536	152.43034	153.07379
	90	188.91464	193.99673	195.77779	196.41965
δ = 0.9	0	97.568619	101.16647	102.94724	103.29643
	10	109.1651	112.6551	114.49162	114.94964
	30	132.35806	135.63231	137.58035	138.25607
	60	167.14748	170.09814	172.21347	173.21573
	90	201.9369	204.56398	206.84659	208.17538
δ = 1.0	0	113.49461	113.88375	114.18141	114.23943
	10	123.82073	124.3563	124.85853	125.07732
	30	144.47296	145.30141	146.21277	146.7531
	60	175.45131	176.71906	178.24413	179.26677
	90	206.42966	208.13672	210.2755	211.78043
δ = 1.1	0	130.58154	127.65027	126.07008	125.74082
	10	139.56146	137.03296	135.83592	135.72961
	30	157.5213	155.79836	155.36758	155.70724
	60	184.46104	183.94644	184.6651	185.67366
	90	211.40079	212.09454	213.9626	215.64009
δ = 1.5	0	199.69888	187.30551	177.00272	174.52792
	10	203.51224	192.14278	182.97266	181.03186
	30	211.13893	201.81735	194.91258	194.0397
	60	222.57896	216.32918	212.82243	213.5515
	90	234.01904	230.84103	230.73232	233.06329
δ = 3.0	0	265.48431	265.20349	271.71353	275.61432
	10	265.58112	265.36063	272.06244	276.36276
	30	265.77475	265.67493	272.76019	277.85965
	60	266.06516	266.14639	273.80685	280.10498
	90	266.35559	266.61786	274.85349	282.35028

Table B4
Returns of APMP-II
(r = 0.01)

	S_1	λ=0.0	λ=0.3	λ=0.7	λ=1.0
δ = 0.1	0	22.768044	39.655846	40.59412	40.751839
	10	40.701405	56.180668	57.035603	57.174976
	30	76.56813	89.230324	89.918564	90.021255
	60	130.36823	138.80481	139.24303	139.29068
	90	184.1683	188.37929	188.56747	188.56009
δ = 0.5	0	49.03355	62.130985	65.720917	66.382095
	10	65.051941	77.027512	80.268845	80.830719
	30	97.088715	106.82056	109.36473	109.72794
	60	145.14386	151.51015	153.00854	153.07379
	90	193.19901	196.19974	196.65234	196.41965
δ = 0.9	0	97.489265	101.11154	102.92376	103.29643
	10	110.03313	113.23829	114.7373	114.94964
	30	135.12085	137.49179	138.3644	138.25607
	60	172.75246	173.87204	173.80502	173.21573
	90	210.38403	210.25229	209.24567	208.17538
δ = 1.0	0	113.40963	113.82433	114.15598	114.23943
	10	124.83295	125.06361	125.16125	125.07732
	30	147.67957	147.54214	147.1718	146.7531
	60	181.94949	181.25996	180.18761	179.26677
	90	216.21941	214.97778	213.20343	211.78043
δ = 1.1	0	130.48647	127.5846	126.04228	125.74082
	10	140.72266	137.87355	136.20198	135.72961
	30	161.19507	158.45145	156.52138	155.70724
	60	191.9037	189.31828	187.00046	185.67366
	90	222.61232	220.18512	217.47958	215.64009
δ = 1.5	0	199.4725	187.16014	176.94817	174.52792
	10	205.15981	193.50708	183.62865	181.03186
	30	216.53442	206.20099	196.98961	194.0397
	60	233.59633	225.24182	217.03104	213.5515
	90	250.65825	244.28267	237.07249	233.06329
δ = 3.0	0	264.83887	264.57944	271.28113	275.61432
	10	267.27063	266.95569	272.82132	276.36276
	30	272.13419	271.70825	275.90167	277.85965
	60	279.4295	278.83704	280.52216	280.10498
	90	286.72482	285.96585	285.14267	282.35028

Table C1
Returns of DP
(r = 0.05)

	S_1	y_1^*	y_2^*	y_3^*	y_4^*	Total Return
δ = 0.1	0	1	1	1	1	37.132111
	10	1	1	1	1	52.407322
	30	1	1	1	1	82.957741
	60	0.8	1	1	1.2	128.79625
	90	0.4	0.8	1.2	1.6	174.67461
δ = 0.5	0	1	1	1	1	60.909611
	10	1	1	1	1	74.352577
	30	0.8	1	1	1.2	101.24081
	60	0.8	0.8	1	1.4	141.77051
	90	0.2	0.6	1.2	2	182.83798
δ = 0.9	0	1.2	0.8	0.8	1.2	95.453339
	10	1.2	0.8	0.8	1.2	106.2773
	30	0.8	0.8	0.8	1.6	128.26515
	60	0.2	0.4	1	2.4	162.47449
	90	0	0	0.6	3.4	198.87384
δ = 1.0	0	2.2	0	0	1.8	106.67902
	10	1.8	0	0	2.2	116.65031
	30	1	0.2	0	2.8	137.28438
	60	0	0	0	4	170.82224
	90	0	0	0	4	206.07626
δ = 1.1	0	2.2	0	0	1.8	123.29905
	10	1.8	0	0	2.2	132.04623
	30	0	0	0	4	150.90657
	60	0	0	0	4	182.57849
	90	0	0	0	4	214.2504
δ = 1.5	0	2.2	0	0	1.8	192.45015
	10	2	0	0	2	196.31834
	30	1.6	0	0	2.4	204.74609
	60	0	0	0	4	220.40544
	90	0	0	0	4	241.23158
δ = 3.0	0	1.6	0	1	1.4	262.83133
	10	1.6	0	1	1.4	263.01663
	30	1.4	1	0	1.6	263.43219
	60	1.4	1	0	1.6	264.25797
	90	1.2	1.2	0	1.6	265.33319

Table C2
Returns of BP
(r = 0.05)

	S_1	BP_1	BP_2	BP_3	BP_4
δ = 0.1	0	11.55924	11.429481	11.301179	11.174315
	10	28.655018	28.66363	28.656073	28.634567
	30	62.846569	63.131927	63.365856	63.555069
	60	114.1339	114.83437	115.43053	115.93583
	90	165.42123	166.53682	167.49521	168.31657
δ = 0.5	0	32.98494	32.760372	32.537228	32.315483
	10	48.071712	48.229454	48.341396	48.413361
	30	78.245262	79.16761	79.949738	80.609108
	60	123.50558	125.57485	127.36225	128.90274
	90	168.7659	171.98209	174.77477	177.19638
δ = 0.9	0	83.541718	83.19104	82.839378	82.48674
	10	93.915192	94.511383	94.991959	95.370773
	30	114.66214	117.15208	119.29713	121.13885
	60	145.78256	151.11313	155.75488	159.79097
	90	176.90298	185.07417	192.21265	198.44308
δ = 1.0	0	101.59404	101.17012	100.74348	100.31417
	10	110.29751	111.023	111.60674	112.06551
	30	127.70444	130.72878	133.33324	135.56819
	60	153.81485	160.28745	165.92299	170.82222
	90	179.92525	189.8461	198.51274	206.07625
δ = 1.1	0	120.88892	120.3426	119.79116	119.23467
	10	127.81873	128.64072	129.29178	129.79198
	30	141.6783	145.23697	148.293	150.90657
	60	162.46768	170.13133	176.79483	182.57849
	90	183.25706	195.0257	205.29668	214.2504
δ = 1.5	0	183.85376	182.15671	180.45627	178.75317
	10	185.19731	185.62115	185.77585	185.69522
	30	187.88438	192.55005	196.41501	199.5793
	60	191.91501	202.94337	212.37373	220.40544
	90	195.94562	213.3367	228.33247	241.23158
δ = 3.0	0	204.16197	201.63463	199.13824	196.67247
	10	204.17395	203.98042	203.5181	202.82512
	30	204.19789	208.672	212.2778	215.13045
	60	204.23381	215.70937	225.41736	233.58842
	90	204.26973	222.74673	238.55692	252.04642

Table C3
Returns of APMP-I
(r = 0.05)

	S_1	λ=0.0	λ=0.3	λ=0.7	λ=1.0
δ = 0.1	0	20.881966	36.154289	36.996124	37.132111
	10	37.321289	51.490185	52.27631	52.407318
	30	70.199936	82.161964	82.83667	82.957741
	60	119.51791	128.16965	128.67722	128.78337
	90	168.83586	174.17731	174.51776	174.60902
δ = 0.5	0	45.181507	57.10976	60.343159	60.909615
	10	59.612312	70.7304	73.784531	74.352585
	30	88.473923	97.971672	100.66727	101.23852
	60	131.76633	138.83357	140.99138	141.56743
	90	175.05873	179.6955	181.31549	181.89635
δ = 0.9	0	90.162491	93.441666	95.031288	95.310951
	10	100.93809	104.12109	105.76656	106.15009
	30	122.48927	125.47999	127.2371	127.82837
	60	154.81606	157.5183	159.4429	160.34578
	90	187.14284	189.55661	191.64871	192.8632
δ = 1.0	0	104.95271	105.28371	105.51024	105.51959
	10	114.54876	115.01772	115.4374	115.59891
	30	133.74086	134.48572	135.29173	135.75751
	60	162.52901	163.68774	165.07323	165.99542
	90	191.31718	192.88977	194.85472	196.23334
δ = 1.1	0	120.81322	118.09403	116.59577	116.24647
	10	129.16005	126.81468	125.67427	125.5349
	30	145.85368	144.256	143.83127	144.11174
	60	170.89413	170.41797	171.06676	171.97701
	90	195.9346	196.57994	198.30225	199.84229
δ = 1.5	0	184.71838	173.39583	163.94905	161.61482
	10	188.28836	177.90599	169.50452	167.66924
	30	195.42833	186.92627	180.61546	179.77806
	60	206.13829	200.4567	197.28189	197.94131
	90	216.84822	213.98714	213.94832	216.10454
δ = 3.0	0	245.10788	244.92851	251.20897	254.23994
	10	245.20322	245.08325	251.55217	254.97479
	30	245.39391	245.39273	252.23857	256.44446
	60	245.67993	245.85695	253.26817	258.64902
	90	245.96596	246.32115	254.29778	260.85352

Table C4
Returns of APMP-II
(r = 0.05)

	S_1	λ=0.0	λ=0.3	λ=0.7	λ=1.0
δ = 0.1	0	20.647017	36.118839	36.983143	37.132111
	10	37.308395	51.48801	52.275505	52.407318
	30	70.631157	82.226341	82.860229	82.957741
	60	120.6153	128.33385	128.73732	128.78337
	90	170.59944	174.44138	174.61443	174.60902
δ = 0.5	0	44.867107	56.942822	60.276291	60.909615
	10	59.761864	70.802437	73.812576	74.352585
	30	89.551376	98.521667	100.88515	101.23852
	60	134.23564	140.10049	141.494	141.56743
	90	178.91991	181.67934	182.10287	181.89635
δ = 0.9	0	89.793854	93.186417	94.922195	95.310951
	10	101.46768	104.47123	105.9129	106.15009
	30	124.81536	127.04085	127.89426	127.82837
	60	159.83687	160.89528	160.86632	160.34578
	90	194.85837	194.74973	193.83838	192.8632
δ = 1.0	0	104.55592	105.00627	105.39143	105.51959
	10	115.19143	115.46658	115.62943	115.59891
	30	136.46245	136.38722	136.10544	135.75751
	60	168.369	167.76817	166.81944	165.99542
	90	200.27554	199.14912	197.53345	196.23334
δ = 1.1	0	120.36673	117.78555	116.4651	116.24647
	10	129.90422	127.36425	125.91628	125.5349
	30	148.97919	146.52167	144.81865	144.11174
	60	177.59164	175.25781	173.1722	171.97701
	90	206.2041	203.99396	201.52576	199.84229
δ = 1.5	0	183.6489	172.7027	163.68636	161.61482
	10	189.02057	178.65508	169.9169	167.66924
	30	199.76392	190.5598	182.37804	179.77806
	60	215.87891	208.4169	201.0697	197.94131
	90	231.99393	226.274	219.76138	216.10454
δ = 3.0	0	242.14111	242.05727	249.22659	254.23994
	10	244.55692	244.41438	250.74445	254.97479
	30	249.38849	249.12862	253.78012	256.44446
	60	256.63586	256.19998	258.33365	258.64902
	90	263.88324	263.27133	262.88718	260.85352

Table D1
Returns of DP
(r = 0.12)

	S_1	y_1^*	y_2^*	y_3^*	y_4^*	Total Return
δ = 0.1	0	1	1	1	1	31.954189
	10	1	1	1	1	45.574726
	30	1	1	1	1	72.815804
	60	0.8	1	1	1.2	113.68452
	90	0.6	0.8	1.2	1.4	154.59016
δ = 0.5	0	1	1	1	1	53.051079
	10	1	1	1	1	65.045113
	30	1	1	1	1	89.03318
	60	0.8	0.8	1	1.4	125.15721
	90	0.2	0.6	1.2	2	161.75784
δ = 0.9	0	1.4	0.8	0.8	1	83.993568
	10	1.2	0.8	0.8	1.2	93.609375
	30	0.8	0.8	0.8	1.6	113.09856
	60	0.2	0.6	0.8	2.4	143.43826
	90	0	0	0.6	3.4	175.83282
δ = 1.0	0	2.2	0	0	1.8	94.119278
	10	2	0	0	2	102.89659
	30	1.2	0.2	0	2.6	121.03543
	60	0	0.4	0	3.6	150.51781
	90	0	0	0	4	182.06085
δ = 1.1	0	2.4	0	0	1.6	108.93945
	10	2	0	0	2	116.56384
	30	1.2	0	0	2.8	132.71341
	60	0	0	0	4	160.55052
	90	0	0	0	4	189.09933
δ = 1.5	0	2.2	0	0	1.8	170.03952
	10	2.2	0	0	1.8	173.27808
	30	1.8	0	0	2.2	180.38171
	60	1.2	0	0	2.8	192.98451
	90	0	0	0	4	211.14946
δ = 3.0	0	1.6	0	1	1.4	230.67084
	10	1.6	0	1	1.4	230.85115
	30	1.6	0	1	1.4	231.21182
	60	1.4	1	0	1.6	231.83711
	90	1.2	1.2	0	1.6	232.84981

Table D2
Returns of BP
(r = 0.12)

	S_1	BP_1	BP_2	BP_3	BP_4
δ = 0.1	0	10.174658	9.9026899	9.637989	9.3803606
	10	25.347557	25.210493	25.061665	24.903559
	30	55.693352	55.826099	55.909008	55.949951
	60	101.21205	101.74951	102.18003	102.51954
	90	146.73074	147.67291	148.45105	149.08913
δ = 0.5	0	29.129759	28.655846	28.189066	27.729317
	10	42.524963	42.411606	42.25787	42.069908
	30	69.315369	69.923119	70.39547	70.751091
	60	109.50098	111.19038	112.60187	113.77287
	90	149.6866	152.45766	154.80827	156.79463
δ = 0.9	0	73.95401	73.205482	72.452026	71.69368
	10	83.169487	83.296028	83.307808	83.218544
	30	101.60045	103.47711	105.01936	106.26829
	60	129.2469	133.74873	137.5867	140.8429
	90	156.89334	164.02036	170.15402	175.4175
δ = 1.0	0	89.959518	89.050789	88.12841	87.192566
	10	97.694038	97.848183	97.855797	97.733482
	30	113.16309	115.44296	117.31057	118.81533
	60	136.36667	141.83514	146.49272	150.43811
	90	159.57025	168.22733	175.67488	182.06087
δ = 1.1	0	107.05463	105.87946	104.67869	103.4529
	10	113.2174	113.31261	113.22314	112.96916
	30	125.54292	128.17888	130.31206	132.00171
	60	144.03119	150.47832	155.9454	160.55052
	90	162.51945	172.77774	181.57877	189.09933
δ = 1.5	0	162.64684	159.0298	155.40273	151.77466
	10	163.87614	162.31303	160.46326	158.37184
	30	166.33473	168.87952	170.58432	171.56625
	60	170.02261	178.72925	185.76593	191.35786
	90	173.71053	188.57896	200.94753	211.14948
δ = 3.0	0	180.92999	175.60124	170.42789	165.40558
	10	180.94165	177.92625	174.73427	171.409
	30	180.96498	182.57628	183.34705	183.41582
	60	180.99997	189.55132	196.26622	201.42607
	90	181.03496	196.52635	209.18538	219.43631

Table D3
Returns of APMP-I
(r = 0.12)

	S_1	λ=0.0	λ=0.3	λ=0.7	λ=1.0
δ = 0.1	0	18.104332	31.133383	31.844883	31.954191
	10	32.719254	44.804699	45.469112	45.57473
	30	61.949093	72.147324	72.717575	72.815804
	60	105.79386	113.16127	113.59027	113.67742
	90	149.63863	154.17522	154.46297	154.53905
δ = 0.5	0	39.546467	49.844341	52.598682	53.051079
	10	52.387527	61.985458	64.588135	65.045113
	30	78.069649	86.2677	88.567055	89.03318
	60	116.59283	122.69106	124.53542	125.0153
	90	155.11601	159.11443	160.50378	160.99739
δ = 0.9	0	79.50016	82.320488	83.63472	83.814003
	10	89.094139	91.835175	93.205345	93.481346
	30	108.2821	110.86454	112.3466	112.81604
	60	137.06403	139.40858	141.05846	141.81808
	90	165.84598	167.95264	169.77032	170.82013
δ = 1.0	0	92.650871	92.897476	93.02018	92.958138
	10	101.19634	101.56874	101.86825	101.94602
	30	118.28728	118.91125	119.56442	119.92178
	60	143.92371	144.92502	146.10866	146.88542
	90	169.56012	170.93878	172.65291	173.84906
δ = 1.1	0	106.73998	104.32508	102.94215	102.56123
	10	114.17618	112.09354	111.03219	110.84259
	30	129.04857	127.63044	127.21227	127.40529
	60	151.35713	150.93579	151.48239	152.24936
	90	173.66571	174.24115	175.7525	177.09343
δ = 1.5	0	163.10716	153.32245	145.09822	142.95197
	10	166.32898	157.36446	150.06064	148.36375
	30	172.77258	165.44843	159.98549	159.18733
	60	182.43802	177.57439	174.87277	175.42271
	90	192.10344	189.70035	189.76004	191.65807
δ = 3.0	0	215.81424	215.78328	221.66666	223.4023
	10	215.90707	215.9339	222.00021	224.11438
	30	216.09274	216.23514	222.66734	225.53854
	60	216.37123	216.68698	223.66801	227.6748
	90	216.64973	217.13884	224.6687	229.81105

Table D4
Returns of APMP-II
(r = 0.12)

	S_1	λ=0.0	λ=0.3	λ=0.7	λ=1.0
δ = 0.1	0	17.619318	31.060184	31.818079	31.954191
	10	32.44796	44.76355	45.454041	45.57473
	30	62.105247	72.170288	72.725975	72.815804
	60	106.59118	113.2804	113.63387	113.67742
	90	151.07712	154.39049	154.54175	154.53905
δ = 0.5	0	38.88876	49.495132	52.458824	53.051079
	10	52.164753	61.860703	64.537476	65.045113
	30	78.716736	86.591858	88.694778	89.03318
	60	118.54472	123.68858	124.93073	125.0153
	90	158.3727	160.78531	161.16669	160.99739
δ = 0.9	0	78.712662	81.775253	83.401703	83.814003
	10	89.13427	91.848236	93.207977	93.481346
	30	109.9775	111.99419	112.82053	112.81604
	60	141.24234	142.21312	142.23936	141.81808
	90	172.50719	172.43205	171.65819	170.82013
δ = 1.0	0	91.79586	92.299515	92.764099	92.958138
	10	101.29825	101.63932	101.89825	101.94602
	30	120.30302	120.3189	120.16656	119.92178
	60	148.81018	148.33829	147.56905	146.88542
	90	177.31735	176.35767	174.97153	173.84906
δ = 1.1	0	105.76888	103.65324	102.65728	102.56123
	10	114.3024	112.21119	111.08988	110.84259
	30	131.36945	129.3271	127.95509	127.40529
	60	156.97002	155.00096	153.25291	152.24936
	90	182.57059	180.67484	178.55074	177.09343
δ = 1.5	0	160.76045	151.7778	144.50266	142.95197
	10	165.68288	157.16826	150.09097	148.36375
	30	175.52771	167.94914	161.26762	159.18733
	60	190.29495	184.12047	178.03261	175.42271
	90	205.06221	200.29181	194.79758	191.65807
δ = 3.0	0	209.59293	209.75212	217.54279	223.4023
	10	211.98151	212.07683	219.02303	224.11438
	30	216.75867	216.72621	221.98351	225.53854
	60	223.92441	223.70032	226.42421	227.6748
	90	231.09015	230.67442	230.86493	229.81105

APPENDIX C

THE SERIES OF ANNUAL SALES AND ADVERTISING EXPENDITURES

$\{S_t\}$ and $\{x_t\}$

Year t	S_t	x_t
1907	0.041703883	0.024956654
1908	0.038452385	0.018829561
1909	0.038228083	0.021651666
1910	0.037721323	0.020986351
1911	0.03538592	0.019975922
1912	0.038050938	0.019857381
1913	0.041003157	0.018181299
1914	0.036503922	0.019374901
1915	0.035562408	0.019924109
1916	0.034611783	0.015116411
1917	0.033539351	0.018963603
1918	0.042537826	0.013169539
1919	0.04106154	0.015922199
1920	0.034488373	0.01355739
1921	0.043213423	0.017464136
1922	0.049344191	0.024617792
1923	0.055677032	0.025906864
1924	0.057356754	0.027523026
1925	0.056536545	0.029600285
1926	0.046881723	0.031195551
1927	0.038110963	0.019855182
1928	0.036233574	0.02220924
1929	0.035154859	0.025789835
1930	0.034303729	0.025479984
1931	0.031929429	0.017379085
1932	0.032197692	0.02048588
1933	0.037229614	0.029820634

1934	0.034927794	0.029678759
1935	0.029024995	0.015430284
1936	0.020755713	0.006379136
1937	0.022854151	0.010145366
1938	0.026886357	0.013598327
1939	0.02613595	0.01375673
1940	0.031883405	0.015553761
1941	0.036810321	0.017613083
1942	0.035744364	0.016127808
1943	0.0374179	0.01673883
1944	0.035955816	0.015736024
1945	0.036929019	0.016034784
1946	0.026570948	0.012351768
1947	0.020007216	0.008711475
1948	0.01813296	0.008933568
1949	0.018691094	0.009241917
1950	0.016388416	0.008932466
1951	0.014160532	0.006422124
1952	0.015076209	0.007433072
1953	0.014958362	0.007605412
1954	0.012980104	0.006251107
1955	0.012392334	0.005987478
1956	0.012167086	0.005888958
1957	0.010924453	0.005361268
1958	0.009260794	0.0042573
1959	0.008969285	0.004164542
1960	0.008074348	0.003532919

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