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Statistical properties of maximum likelihood estimates for accelerated lifetime data under the Weibull model

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STATISTICAL PROPERTIES OF MAXIMUM LIKELIHOOD ESTIMATES FOR ACCELERATED LIFE-TIME DATA UNDER THE WEIBULL MODEL

by

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We hereby recommend that the dissertation prepared under our supervision by Mahmoud A. Yousef entitled Statistical Properties of Maximum Likelihood Estimates for Accelerated Life-Time Data Under the Weibull Model be accepted in partial fulfillment of the requirements for the Degree of Ph.D. in Computational Analysis and Modeling.

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ABSTRACT

Pipe rehabilitation liners are often installed in host pipes that lie below the water table. As such, they are subjected to external hydrostatic pressure. The external pressure leads to early deformation in the liners, which could ultimately lead to its failing or buckling before its expected service lifetime is achieved. Experiments involving long term buckling behavior of liners are typically accelerated lifetime testing procedures. In an accelerated testing procedure a liner is subjected to a constant external hydrostatic pressure and observed until it fails or for a certain time, t whichever occurs first. Liners that do not fail at time t are deemed censored observations. While a constant pressure is convenient to use in experimental situations, in reality pressure fluctuates under soil conditions over time depending on the water table.

In this study, constant and variable pressures using the Weibull model for time till buckling under different sample sizes and different levels of censoring were investigated. Data were generated through computer simulation and estimates of parameters in the Weibull model were obtained using the Maximum Likelihood and Newton-Raphson methods.

It was concluded that the maximum likelihood estimates under fixed or variable pressure, and for different sample sizes with different levels of censoring, are unbiased. However, the estimates for sample sizes as large as 100 are not normally distributed, especially when the parameter value being estimated is small. It was seen that the

lack of normality was manifested in lack of agreement between the observed variance-covariance matrix and the theoretical variance-covariance matrix. These results cast doubt on the use of normal theory for inference concerning certain parameters.

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DEDICATION

TO THE MEMORY OF MY FATHER ATIEH

TO MY MOTHER GAMILEH

TO MY WIFE GAMILEH

AND

MY CHILDREN AHMED, ABDALA, AND JANNA

WITH LOVE

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CHAPTER 1

INTRODUCTION AND RESEARCH OBJECTIVES

1.1 Introduction

It is known that the underground infrastructure system in the United States is in urgent need for repair or upgrade. It has been the practice, whenever there is a problem with an underground pipeline, to use the open-trench method which includes digging the ground, removal of the deteriorated pipe(s), and replacement with new one(s). It is clear that this method is not desirable because of the amount of work required for the job, the cost associated with it, the time period to finish the work, and the inconvenience to the businesses and the general public.

Recently, with development of trenchless techniques, it has become possible to repair underground pipe(s) without excavating the ground. With the new trenchless methods in effect today, the problems associated with the open-trench method are slowly disappearing.

A relatively recent approach for pipeline rehabilitation which provides significant economical, social and environmental benefits involves pipeline repair by insertion of a lining material through existing manholes. Cured-In-Place Plastic Pipe (CIPP) and Fold-and-Formed Pipe (FFP) are perhaps the most well known of the relining methods (Guice *et al.*, 1994).

The Cured-In-Place Plastic Pipe (CIPP) technique is the most important method in trenchless pipeline rehabilitation. It involves the installation of plastic liners inside the damaged pipeline through existing manholes or other entry points. The process inverts a resin-impregnated fabric tube into the damaged pipe using a hydraulic head or winching it in place. When circulating hot water inside the pipe, the resin will cure and harden into a continuous, snug-fitting tube inside the original host pipe. The CIPP technique not only seals the joints and restores the pipeline integrity, but also increases the strength of the existing pipe and provides improved corrosion resistance for the inner surface of the pipe. This method was introduced by Insituform in Europe in 1971 and then was brought to the United States in 1977 (Li, 1994).

The exact definition of CIPP according to the American Society for Testing and Materials (ASTM) is "a hollow cylinder containing a non-woven material surrounded by the cured thermosetting resin. Plastic coatings may be included. This pipe is formed within an existing pipe. Therefore, it takes the shape of and fits tightly to the existing pipe" (ASTM F1216, 1993).

In the Fold and Formed Pipe systems, the cross-sectional area of the new pipe is temporarily reduced before installation. After installation, it is then expanded to its original size and shape to provide a close fit with the existing pipe. This fitting is accomplished by folding the lining pipe into a U-shape, after which it is inserted inside the old pipe and reverted by heat and pressure (Li, 1994).

Liners are often installed in host pipes that lie below the water table, and as such they are subjected to external hydrostatic pressure. The external pressure leads to early deformation in the liners which could ultimately lead to its failing or buckling before their expected service life is achieved. Insufficient understanding of this buckling phenomenon is a limiting factor in the CIPP liner industry. To design a dependable liner, one needs to have a good knowledge about the long-term buckling behavior of a pipe under external pressure and models to predict such behavior. Many studies exist on predicting short-term buckling behavior of a free or confined pipe (Timoshenko and Gere, 1961; Aggarval and Cooper, 1984; Glock, 1977; Guice and Li, 1994; Omara *et al.*, 1996; Falter, 1996; Welch, 1989, Boot and Welch, 1996; Boot, 1998; Boot and Javadi, 1998; and Hall and Zhu, 2000).

On the other hand, the long-term buckling behavior of a pipe liner under pressure has been under study for a relatively short period of time. Only few models exist for predicting the long-time behavior (Welsh, 1989; Guice *et al.*, 1994; Straughn *et al.*, 1995; Boot and Welch, 1996; Moore, 1998; and Zhao, 1999). There is a need for models to predict the time until failure of a pipe liner system under a hydrostatic pressure load.

Experiments involving long-term buckling behavior of liners are typically accelerated lifetime testing procedures. In an accelerated testing procedure a liner is subjected to a constant external hydrostatic pressure and observed until it fails, or for a certain time, t whichever occurs first. Liners that do not fail at time t are deemed censored observations. While a constant pressure is convenient to use in experimental situations, in reality pressure fluctuates under soil conditions over time depending on the water table. It is desirable then to have accelerated lifetime models to predict time until buckling under constant as well as variable external hydrostatic pressure.

1.2 Objectives

The objectives of this study are

1. to examine accelerated lifetime models for constant and variable pressure for the Weibull lifetime distribution and show how to obtain maximum likelihood estimates (MLE) of model parameters.

2. to study through simulation the statistical properties of the MLE as a function of sample size and percent censoring and compare results for the constant and variable pressure situations.

1.3 Organization of the Dissertation

This dissertation is organized as follows:

In chapter 2, a review of literature is presented. In chapter 3, the maximum likelihood method and the Newton-Raphson technique are discussed. In chapter 4, theory and simulation results for accelerated lifetime testing under constant pressure are presented. In chapter 5, theory and results from simulation for the accelerated lifetime testing under variable pressure are discussed. In chapter 6, a conclusion and future work are provided.

CHAPTER 2

RELATED RESEARCH

2.1 Reliability

The Statistical analysis of lifetime data is of interest to statisticians, engineers, physicians, and researchers in biological sciences.

Applications of lifetime distributions range from investigations into the endurance of items to research involving human diseases. Lifetime distribution methodology has its most frequent application in engineering, and biomedical sciences.

Let T be a nonnegative random variable representing the failure time of an individual from some population. Let $f(t)$ denote the probability density function (p.d.f) of T and let the distribution function be

$$F(t) = Pr(T \leq t) = \int_0^t f(x)\, dx. \tag{2.1}$$

The survivor function is defined as the probability that T is at least as great as t; that is,

$$S(t) = Pr(T \geq t) = \int_t^\infty f(x)\, dx. \tag{2.2}$$

In cases involving lifetimes of manufactured items, $S(t)$ is referred to as the reliability function. $S(t)$ is a monotone decreasing continuous function with $S(0) = 1$ and $S(\infty) = \lim_{t\to\infty} S(t) = 0$.

From Eqs. (2.1) and (2.2), we see that the survivor function or the reliability function is given by

$$S(t) = R(t) = 1 - F(t). \tag{2.3}$$

Another important concept dealing with a lifetime distribution is the hazard function $h(t)$, defined as

$$\begin{aligned} h(t) &= \lim_{\Delta t \to 0} \frac{Pr(t \leq T < t + \Delta t \mid T \geq t)}{\Delta t} \\ &= \frac{f(t)}{S(t)}. \end{aligned} \tag{2.4}$$

The hazard function specifies the instantaneous rate of failure at time $T = t$, given that the individual will survive until time $T = t$.

Now, since $f(t) = -S'(t)$, it is seen from Eq. (2.4) that

$$h(t) = -\frac{d}{dt} \log S(t). \tag{2.5}$$

Hence,

$$\log S(x)|_0^t = -\int_0^t h(x)dx, \tag{2.6}$$

and since $S(0) = 1$, it seen that

$$S(t) = \exp(-\int_0^t h(x)dx). \tag{2.7}$$

The cumulative hazard function is defined as

$$H(t) = \int_0^t h(x)dx. \tag{2.8}$$

Hence, from Eq. (2.6), we have

$$S(t) = \exp[-H(t)]. \tag{2.9}$$

Now, since $S(\infty) = 0$, then $H(\infty) = \lim_{t\to\infty} H(t) = \infty$. Therefore, the hazard function $h(t)$ for a continuous lifetime distribution has the following properties:

1. $h(t) \geq 0$

2. $\int_0^\infty h(t)dt = \infty$

2.2 Some Important Lifetime Models

Throughout the literature on failure time data, numerous parametric models are used to analyze problems related to aging or a failure process. Among these models, few are frequently used because of their demonstrated usefulness in a wide range of situations. The exponential and Weibull models, for example, are often employed. These distributions admit closed-form expressions for tail area probabilities and hence simple formulas for survivor and hazard functions. Also, the lognormal and gamma distributions are frequently used despite the fact that they are generally less convenient computationally. Another important distribution is the extreme value distribution which describes certain extreme phenomena like electrical strength of materials and certain types of lifetime data (Kalbfeisah and Prentice, 1980; Lawless, 1982; Nelson, 1990; and Collett 1999).

2.2.1 The Exponential Distribution

Suppose that the hazard function is constant. Then, the hazard function can be written as

$$h(t) = \lambda \quad \text{for} \ \ 0 \leq t < \infty. \tag{2.10}$$

The parameter λ is a positive constant estimated by fitting the model to observed data. From Eq. (2.7), we have that

$$S(t) = \exp(-\int_0^t \lambda dx) = e^{-\lambda t}. \tag{2.11}$$

Hence, the probability density function (p.d.f) of survival times is given by

$$f(t) = \lambda e^{-\lambda t} \quad \text{for} \ \ 0 \leq t < \infty. \tag{2.12}$$

Equation (2.12) represents the probability density function of a random variable T that has an exponential distribution. It can be easily verified that the mean is $\frac{1}{\lambda}$ and the variance is $\frac{1}{\lambda^2}$ (Ross, 1997).

A very important characteristic of the exponential distribution is its lack of memory. This lack of memory property can be seen as follows:

For $t_2 > t_1 > 0$,

$$P[T > t_2 \mid T > t_1] = \frac{e^{-\lambda t_2}}{e^{-\lambda t_1}} = e^{-\lambda(t_2 - t_1)}. \qquad (2.13)$$

Hence, the survival probability depends on the interval $(t_2 - t_1)$ and is independent of what happened before time t_1.

2.2.2 The Weibull Distribution

The assumption of a constant hazard function is rather restrictive. A more general form of a hazard function is such that

$$h(t) = \lambda\gamma t^{\gamma-1} \quad \text{for} \;\; t \geq 0. \qquad (2.14)$$

Here, the hazard function depends on two parameters, λ and γ, both greater than zero. In the special case when $\gamma = 1$, the hazard function takes the constant value λ, and hence the survival times have the exponential distribution. If $\gamma \neq 1$, the hazard function increases or decreases monotonically. The parameter γ is known as the shape parameter. Hence, the shape of the hazard function depends on γ. The parameter λ is known as the scale parameter. From Eq. (2.7), it is seen that the survivor function is given by

$$S(t) = \exp(-\int_0^t \lambda\gamma x^{\gamma-1}dx)$$
$$= \exp(-\lambda t^{\gamma}). \tag{2.15}$$

Hence, the probability density function is given by

$$f(t) = \lambda\gamma t^{\gamma-1}\exp(-\lambda t^{\gamma}) \quad \text{for} \quad t \geq 0. \tag{2.16}$$

The function in Eq. (2.16) is the density of a random variable that has the Weibull distribution with shape parameter γ and scale parameter λ.

2.2.3 Extreme Value Distribution

This distribution is also known as the Gumbel distribution (Gumbel, 1958). The p.d.f for the extreme value distribution is given by

$$f(x) = \frac{1}{b}\exp[\frac{x-u}{b} - \exp(\frac{x-u}{b})] \quad \text{for} \quad -\infty < x < \infty, \tag{2.17}$$

where $b > 0$ and $-\infty < u < \infty$ are parameters. It can be seen that if T is a random variable with a Weibull distribution, then $X = \log T$ has an extreme value distribution with $b = \frac{1}{\gamma}$ and $u = -\log\lambda^{\gamma}$. Also, the survivor function of the extreme value distribution is given by

$$S(x) = \exp[-\exp(\frac{x-u}{b})]. \tag{2.18}$$

2.2.4 The Gamma Distribution

Another distribution for survival data is the gamma distribution which is defined for a random variable that takes positive values. The hazard function for the gamma distribution is given by

$$h(t) = \frac{\lambda^k t^{k-1} e^{-\lambda t}}{\Gamma(k)(1-\Gamma_{\lambda t}(k))}, \tag{2.19}$$

where $\Gamma(k)$ is a gamma function and $\Gamma_{\lambda t}(k)$ is the incomplete gamma function and is given by

$$\Gamma_{\lambda t}(k) = \frac{1}{\Gamma(k)} \int_0^{\lambda t} x^{k-1} e^{-x} dx. \tag{2.20}$$

The gamma distribution has a probability density function of the form

$$f(t) = \frac{\lambda(\lambda t)^{k-1} e^{-\lambda t}}{\Gamma(k)} \quad \text{for } t > 0, \tag{2.21}$$

where $\lambda > 0$ is called the scale parameter, and $k > 0$ is the index or shape parameter.

For $k = 1$, the gamma distribution reduces to the exponential distribution. Integrating Eq. (2.21), one obtains the survival function for the gamma as

$$S(t) = 1 - \Gamma_{\lambda t}(k), \tag{2.22}$$

where $\Gamma_{\lambda t}(k)$ is given in Eq. (2.20).

2.2.5 The Lognormal Distribution

Another important distribution for lifetime data is the lognormal distribution. This distribution has been widely used in engineering and biomedical science.

A random variable T is said to have a lognormal distribution with parameters μ and σ^2 if $y = \log T$ has a normal distribution with mean μ and variance σ^2. The probability density function of Y is given by

$$f(y) = \frac{1}{\sigma\sqrt{2\pi}} \exp[\frac{-1}{2\sigma^2}(y-\mu)^2] \quad \text{for} \quad -\infty < y < \infty. \tag{2.23}$$

From Eq. (2.23), it is seen that the p.d.f for T is given by

$$f(t) = \frac{1}{\sigma t\sqrt{2\pi}} \exp[-\frac{1}{2\sigma^2}(\log t - \mu)^2]; \quad t > 0. \tag{2.24}$$

The survivor and hazard functions for the lognormal distribution involve the standard normal distribution function

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du. \tag{2.25}$$

The lognormal survival function is given by

$$S(t) = 1 - \Phi(\frac{\log t - u}{\sigma}), \tag{2.26}$$

and the hazard function is given by

$$h(t) = \frac{f(t)}{S(t)}. \tag{2.27}$$

2.3 Accelerated Life Testing

The lifetime of a high-reliability device is usually very long. Thus, it is prohibitive time-wise to test such a device under normal conditions. Accelerated lifetime testing (ALT) is a method that exposes devices to higher stress levels than they expect to receive under normal use to induce early failures, and obtain information quickly on their lifetime distribution.

Schabe and Viertl (1995) presented an axiomatic approach to accelerated lifetime testing. Clark, Garganese, and Swarz (1997) presented an approach to designing accelerated-lifetime testing experiments. The basis of their approach is a destructive evaluation performed on a small number of test items to measure the design limits. As such, environmental stress levels can be tailored to achieve the objectives of the accelerated lifetime test.

Accelerated test conditions involve-higher-than usual load or stress (such as temperature, voltage, pressure, etc., or some combination of them) on the device. Accelerated lifetime testing is a common method for assessing the reliability of an item because, for practical reasons, lifetime testing is performed in a relatively short time interval. Two types of accelerated testing exist in the literature, constant-stress testing and step-stress testing.

2.3.1 Constant-Stress Accelerated Test

One way of applying stress to a test device is a constant-stress. Each device is assigned only one stress level in a completely random manner. Regression methods are used to estimate lifetime until failure at a given design stress. A functional relationship between constant-stress and lifetime until failure is assumed. The test data are then used to estimate the parameters of the distribution of time until failure. Estimates of the parameters in the model can be obtained by maximizing the log-likelihood function using the Newton-Raphson Technique.

A very important problem in constant-stress testing is determining the number of devices to be allocated to each stress. Inferential procedures for the constant stress test have been given by Nelson (1980), when lifetime follows a Weibull distribution and by Nelson and Hahn (1972), when the lifetime of an item follows a Lognormal or a Weibull distribution. Kielpinski and Nelson (1975) and Nelson and Kielpinski (1976) presented optimum plans and the theory of optimum plans in the case of ALT for estimating a simple linear relationship between stress and the lifetime of an item, which has a Normal or Lognormal distribution, when the data are to be analyzed before all test devices fail. Their model assumes that the normal distribution location parameter μ (mean) is a linear function of the stress and that the scale parameter σ (standard deviation) does not depend on stress. Nelson (1975) presented simple least-squares methods for analyzing accelerated lifetime test data with the inverse power law model, when all test devices are run to failure. Nelson and Meeker (1978) presented the theory of maximum likelihood for large-sample optimum ALT plans. They showed how the plans can be used to estimate a simple linear relationship between stress and product lifetime in the case of a Weibull or Smallest Extreme Value distribution. They assumed that the smallest extreme-value location parameter μ is a linear function of stress and that the scale parameter is constant. Aitkin and

Calyton (1980) showed how regression models can be fitted to censored survival data by the use of the exponential, Weibull, and extreme value distributions in generalized linear interactive modeling (GLIM). Bugaighis (1990) presented results showing that exchange of censorship types resulted in minor reduction in the efficiency of various estimators. Meeter and Meeker (1994) extended the maximum likelihood theory for test planning to a nonconstant scale parameter σ. They also presented test plans for a large range of practical testing situations. Thiagarajah (1995) considered tests on time-censored data for the equality of several exponential scale parameters in the presence of unspecified location parameters. He derived 3 statistics for testing the homogeneity of $M(M \geq 2)$ exponential scale parameters. He also compared, through a simulation study, the size and power for the 3 developed statistics.

2.3.2 Step-Stress Accelerated Test

Another way of applying stress to a device is a step-stress scheme which allows the stress setting of a device to be changed at prespecified times or upon the occurrence of a fixed number of failures.

Step-stress testing reduces time and assures that failures occur very quickly. A test device starts at a specified low stress. If the device does not fail in a specified time, the stress on it is raised and held at that level for a specified time. If the device does not fail at this stress, its stress is increased and held, and the process continues in the same fashion until all devices fail.

The design problem in step-stress testing is to determine the time to change stresses, provided that a fixed number of stress levels has been selected. The choice of these times will determine how many devices fail at each stress. Constant-stress and step-stress have the same optimality criterion; i.e., they choose the times to change stress that minimize the variance of some estimator of a parameter.

As is the case with constant-stress test, one needs to estimate the parameters of the lifetime model under step-stress. Parameter estimates are then used to determine within reason the lifetime of an item at a constant design stress. As such, one needs a model that relates the lifetime distribution under constant-stress to that under step-stress.

Nelson (1980) presented statistical models and methods for analyzing accelerated lifetime test data from step-stress tests. He used the maximum likelihood estimation technique to estimate the parameters of such models. He applied his method to the Weibull distribution and the Inverse Power Law. Miller and Nelson (1983) obtained optimum plans for two stresses where all devices are run to failure. They obtained an optimal stress test that minimizes the asymptotic variance of the maximum likelihood estimate (MLE) of mean lifetime for an exponential model where the mean lifetime is a log-linear function of stress. Bai, Kim, and Lee (1989) derived an optimum simple (two stresses) stress ALTs for the case where censoring was involved. They obtained an optimum test plan that minimizes the asymptotic variance of the MLE of the mean lifetime at a design stress with censored observations. Bai and Chun (1991) obtained optimum simple step-stress ALT with competing causes of failure. Tyoskin and Krivolapov (1996) developed a nonparametric model for interval estimation, based on results from step-stress ALT. They presented, through simulation, a numerical example to verify their approach.

Optimum simple (two stresses) step-stress ALT plans have some limitations because they depend on the assumption of a linear relationship between stress and time-until failure. Khamis and Higgins (1996) presented 3-step stress plans for ALT. They derived an optimum quadratic plan and evaluated a 3-step stress test plan (the compound linear plan) in lieu of the optimum simple-stress plan. Khamis (1997) obtained optimum M-step, step-stress designs with k-stress variables. Xiong and Milliken (1999) studied statistical models in step stress ALT when the stress-change times are random. They presented the marginal lifetime distribution of a device under a step-stress test plan when the stress-change times are random variables. They also, presented an optimum ALT for simple step-stress (two stresses) when the lifetime under any constant-stress follows the exponential distribution.

The main goal of using step-stress testing is to avoid censoring. One knows that the device may have high reliability and not fail within a reasonable time. By increasing the stress on the device the problem of censoring could be avoided.

2.3.3 Cumulative Exposure Model (CEM)

To analyze data from a step-stress scheme, one needs a model that relates the lifetime distribution of the step-stress to that of the constant-stress. One such model is the Cumulative Exposure Model (CEM) by Nelson (1980).

Suppose there are n increasing levels of stresses $x_1 < x_2 < x_3 < \cdots < x_n$. Let $F_i(t)$ denote the failure distribution under x_i with a constant-stress testing. Let t_i be the time it takes to change a stress from x_i to x_{i+1}; $i = 1, 2, \cdots, n-1$. Then, the CEM is given by

$$F(t) = \begin{cases} F_1(t) & \text{if } 0 \le t < t_1 \\ F_2(t - t_1 + s_1) & \text{if } t_1 \le t < t_2 \\ F_3(t - t_2 + s_2) & \text{if } t_2 \le t < t_3 \\ \vdots & \\ F_n(t - t_{n-1} + s_{n-1}) & \text{if } t_{n-1} \le t < \infty, \end{cases} \tag{2.28}$$

where s_1 is the solution of $F_2(s_1) = F_1(t_1)$.

Here, $F_0(t) = F_1(t), \quad 0 \le t < t_1$.

Hence, $F_0 = F_2[(t - t_1) + s_1] \quad t_1 \le t < t_2$

Also, s_2 is the solution of $F_3(s_2) = F_2(t_2 - t_1 + s_1)$.

Hence, $F_0(t) = F_3[(t - t_2) + s_2], \quad t_2 \le t < t_3$.

If we continue in this manner, we see that s_i is the solution of

$F_i(s_{i-1}) = F_{i-1}[t_{i-1} - t_{i-2} + s_{i-2}]$

and, $F_0 = F_i[(t - t_{i-1}) + s_{i-1}], \quad t_{i-1} \le t < t_i$

This model assumes the following:

1. The remaining lifetime of a device depends on

   (a) the current cumulative fraction failed, and

   (b) the current stress.

2. If held at the current stress, survivors fail according to the cumulative distribution for that stress, but starting at the previously accumulated fraction failed.

3. The change in stress has no effect on lifetime, but the stress level has an effect

on lifetime.

2.3.4 The Khamis-Higgins Model (KHM)

From Eq. (2.16), it is seen that the cumulative distribution function for the Weibull distribution is given by

$$F(w) = 1 - \exp\left(-\lambda w^{\gamma}\right); \qquad w > 0. \tag{2.29}$$

Using the transformation $t = w^{\gamma}$, it is seen that

$$F(t) = 1 - \exp\left(-\lambda t\right); \qquad t > 0. \tag{2.30}$$

The above transformation from W to T transforms the Weibull into an exponential distribution and facilitates many of the inferential results. However, such property does not carry over to step-stress testing for the Weibull CEM; i.e., the above transformation does not result in the exponential exposure model. To over come this difficulty, Khamis and Higgins (1998) obtained a new model for step-stress testing, the Khamis-Higgins Model(KHM). The KHM is based on a time transformation of the exponential. The time transformation enables the user to know results for multiple-step, multiple-stress models developed for the exponential step-stress model. The KHM is given by

$$F(w) = 1 - \begin{cases} \exp(-\lambda_1 w^\gamma) & \text{if } 0 \le w < t_1 \\ \exp(-\lambda_2(w^\gamma - t_1^\gamma) - \lambda_1 t_1^\gamma) & \text{if } t_1 \le w < t_2 \\ \vdots & \\ \exp(-\lambda_n(w^\gamma - t_{n-1}^\gamma) - \cdots - \lambda_2(t_2^\gamma - t_1^\gamma) - \lambda_1 t_1^\gamma) & \text{if } t_{n-1} \le w < \infty, \end{cases} \tag{2.31}$$

and

$$F(t) = 1 - \begin{cases} \exp(-\lambda_1 t) & \text{if } 0 \le t < t_1^* \\ \exp(-\lambda_2(t - t_1^*) - \lambda_1 t_1^*) & \text{if } t_1^* \le t < t_2^* \\ \vdots & \\ \exp(-\lambda_n(t - t_{n-1}^*) - \cdots - \lambda_2(t_2^* - t_1^*) - \lambda_1 t_1^*) & \text{if } t_{n-1}^* \le t < \infty. \end{cases} \tag{2.32}$$

The KHM hazard function is given by

$$h(w) = \begin{cases} \lambda_1 \gamma w^{\gamma-1} & \text{if } 0 \le w < t_1 \\ \lambda_2 \gamma w^{\gamma-1} & \text{if } t_1 \le w < t_2 \\ \vdots & \\ \lambda_n \gamma w^{\gamma-1} & \text{if } t_{n-1} \le w < \infty. \end{cases} \tag{2.33}$$

The KHM assumes the following:

1. $W_{i,j} = T_{i,j}^{\frac{1}{\gamma}}$, where $T_{i,j}$ follows the exponential CEM.

2. The λ_i is related to the stress level x_i by

$$\ln(\lambda_i) = \lambda_0 + \lambda_1 x_i, \tag{2.34}$$

where λ_0, λ_1, and γ are constants, independent of time and stress, and are determined from the test data.

3. All the n devices are initially placed on test at stress level x_1 and run until time τ_1 when the stress is changed to the stress level x_2. At stress level x_2, testing continues until time τ_2 when stress is changed to the stress level x_3, and so on, until stress level x_n. At stress level x_n, testing continues until all remaining devices fail or until time τ_c, whichever occurs first, where τ_c is the censoring time at stress level x_k.

The KHM has a very interesting proportional hazard property. It is also as flexible as the Weibull CEM for data fitting, and its mathematical form makes it easy to obtain parameter estimates and standard deviations of estimates.

CHAPTER 3

METHODOLOGY

In this study, we will

1. Investigate constant and variable pressures using the Weibull model for time until buckling.

2. Use computer simulation to generate data from the Weibull distribution for constant and variable pressures. Apply the maximum likelihood and Newton Raphson methods on the generated data to estimate the parameters of the Weibull distribution under censored observations.

3. Study the effects of

   (a) Sample Size

   and

   (b) percent censorship

on the statistical properties of the estimates. These include

   (a) Bias associated with an estimate,

   (b) The variance of an estimate and the covariance between two estimates as compared to those estimated from the Fisher information matrix, and

(c) The empirical distribution of an estimate as compared to the asymptotic normal distribution of an ML estimate.

4. Compare results for constant-stress vs variable-stress.

3.1 The Maximum Likelihood Method

For a given base line distribution, the log-likelihood function can be established based on the observations of time until failure. Given n observations, r of which are uncensored and $n-r$ censored (for instance, time until failure exceeded 10,000 hours), the log-likelihood function is given by:

$$L(t_0, \Theta) = \sum_{i=1}^{r} \ln f(t_{0i}, \Theta) + \sum_{i=r+1}^{n} \ln(1 - F(t_{0i}, \Theta)), \tag{3.1}$$

where

$f(t_0, \Theta)$= the probability density function of time until failure.

$F(t_0, \Theta)$= cumulative distribution function of time until failure.

Suppose the above model has k parameters, say $\theta_1, \theta_2, \cdots, \theta_k$. The maximum likelihood estimates of the k unknown parameters are the values $\hat{\theta_1}, \hat{\theta_2}, \cdots, \hat{\theta_k}$ which maximize the log-likelihood function L, which are the same values that maximize the likelihood function itself. These estimates are found by solving the k equations.

$$U_i(t_0, \Theta) = \frac{\partial L(t_0, \Theta)}{\partial \theta_i} = 0 \qquad i = 1, 2, 3, \cdots, k. \tag{3.2}$$

The $U_i(\Theta)'s$ are called efficient scores, and the $k \times 1$ vector $\mathbf{U}(\Theta) = [U_1(\Theta), U_2(\Theta), \cdots, U_k(\Theta)]'$ is called the efficient score vector.

The efficient score vector is a sum of *i.i.d* random variables, because

$$L(t_0, \Theta) = \sum_{i=1}^{r} \ln f(t_{0i}, \Theta) + \sum_{i=r+1}^{n} \ln (1 - F(t_{0i}, \Theta)). \tag{3.3}$$

Under mild conditions (Cox and Hinkley, 1974) it is asymptotically normally distributed.

Now, let $\mathbf{H}(\Theta)$ be the $k \times k$ matrix of first partial derivatives of the efficient scores, $U_i(t_0, \Theta)$, or equivalently the second partial derivatives of the log-likelihood function, $L(\Theta, t_0)$. $\mathbf{H}(\Theta)$ is expressed as

$$\mathbf{H}(\Theta) = \begin{bmatrix} \frac{\partial u_1(t_0,\Theta)}{\partial\theta_1} & \frac{\partial u_1(t_0,\Theta)}{\partial\theta_2} & \cdots & \frac{\partial u_1(t_0,\Theta)}{\partial\theta_k} \\ \frac{\partial u_2(t_0,\Theta)}{\partial\theta_1} & \frac{\partial u_2(t_0,\Theta)}{\partial\theta_2} & \cdots & \frac{\partial u_2(t_0,\Theta)}{\partial\theta_k} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial u_k(t_0,\Theta)}{\partial\theta_1} & \frac{\partial u_k(t_0,\Theta)}{\partial\theta_2} & \cdots & \frac{\partial u_k(t_0,\Theta)}{\partial\theta_k} \end{bmatrix}. \tag{3.4}$$

Which is the same as

$$\mathbf{I}(\Theta) = - \begin{bmatrix} \frac{\partial^2 L(t_0,\Theta)}{\partial^2\theta_1} & \frac{\partial^2 L(t_0,\Theta)}{\partial\theta_1\partial\theta_2} & \cdots & \frac{\partial^2 L(t_0,\Theta)}{\partial\theta_1\partial\theta_k} \\ \frac{\partial^2 L(t_0,\Theta)}{\partial\theta_2\partial\theta_1} & \frac{\partial^2 L(t_0,\Theta)}{\partial^2\theta_2} & \cdots & \frac{\partial^2 L(t_0,\Theta)}{\partial\theta_2\partial\theta_k} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial^2 L(t_0,\Theta)}{\partial^2\theta_k\partial\theta_1} & \frac{\partial^2 L(t_0,\Theta)}{\partial^2\theta_k\partial\theta_2} & \cdots & \frac{\partial^2 L(t_0,\Theta)}{\partial^2\theta_k} \end{bmatrix}. \tag{3.5}$$

The (i, j)th element of $\mathbf{H}(\Theta)$ is given by

$$\frac{\partial U(t_0, \Theta)}{\partial \theta_i \partial \theta_j} \qquad i, j = 1, 2, \cdots, k, \tag{3.6}$$

and the (i, j)th element of $\mathbf{I}(\Theta)$ is given by

$$-\frac{\partial^2 L(t_0, \Theta)}{\partial \theta_i \partial \theta_j} \qquad i, j = 1, 2, \cdots, k. \tag{3.7}$$

The matrix $\mathbf{H}(\Theta)$ is called the Hessian matrix, and

$$\mathbf{I}(\Theta) = -\mathbf{H}(\Theta), \tag{3.8}$$

where $\mathbf{I}(\Theta)$ is called the Fisher information matrix.

The (i, j)th element of the corresponding expected information matrix is given by

$$E(\frac{-\partial^2 L(t_0, \Theta)}{\partial \theta_i \partial \theta_j}) \qquad i, j = 1, 2, \cdots, k. \tag{3.9}$$

When the expression in Eq. (3.9) cannot be calculated analytically, a numerical solution is obtained using the Newton Raphson Method.

3.2 Newton Raphson Method

Let $\mathbf{f}(t_0; \Theta)$ be the $k \times 1$ vector of first derivatives of the log-likelihood function in Eq. (3.1) with respect to the Θ-parameters, that is,

$$\mathbf{f}(t_0; \Theta) = \frac{\partial L(t_0, \Theta)}{\partial \Theta}. \tag{3.10}$$

Letting $\mathbf{f}(t_0, \Theta) = 0$ in Eq. (3.10), leads to n non-linear equations. To solve the above equations, define a matrix $\mathbf{I}(\Theta)$ by

$$\mathbf{I}(\Theta) = -\begin{bmatrix} \frac{\partial f_1(t_0,\Theta)}{\partial \theta_1} & \frac{\partial f_1(t_0,\Theta)}{\partial \theta_2} & \cdots & \frac{\partial f_1(t_0,\Theta)}{\partial \theta_k} \\ \frac{\partial f_2(t_0,\Theta)}{\partial \theta_1} & \frac{\partial f_2(t_0,\Theta)}{\partial \theta_2} & \cdots & \frac{\partial f_2(t_0,\Theta)}{\partial \theta_k} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial f_k(t_0,\Theta)}{\partial \theta_1} & \frac{\partial f_k(t_0,\Theta)}{\partial \theta_2} & \cdots & \frac{\partial f_k(t_0,\Theta)}{\partial \theta_k} \end{bmatrix}, \tag{3.11}$$

which is the same as

$$\mathbf{I}(\Theta) = -\begin{bmatrix} \frac{\partial^2 L(t_0,\Theta)}{\partial^2 \theta_1} & \frac{\partial^2 L(t_0,\Theta)}{\partial \theta_1 \partial \theta_2} & \cdots & \frac{\partial^2 L(t_0,\Theta)}{\partial \theta_1 \partial \theta_k} \\ \frac{\partial^2 L(t_0,\Theta)}{\partial \theta_2 \partial \theta_1} & \frac{\partial^2 L(t_0,\Theta)}{\partial^2 \theta_2} & \cdots & \frac{\partial^2 L(t_0,\Theta)}{\partial \theta_2 \partial \theta_k} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial^2 L(t_0,\Theta)}{\partial^2 \theta_k \partial \theta_1} & \frac{\partial^2 L(t_0,\Theta)}{\partial^2 \theta_k \partial \theta_2} & \cdots & \frac{\partial^2 L(t_0,\Theta)}{\partial^2 \theta_k} \end{bmatrix}. \tag{3.12}$$

In other words,

$$\mathbf{I}(\Theta) = -\frac{\partial \mathbf{f}(t_0, \Theta)}{\partial \Theta} = -\frac{\partial^2 L(t_0, \Theta)}{\partial \Theta^2}; \tag{3.13}$$

is the observed information matrix. According to the Newton-Raphson method, an estimate of the Θ-parameters at the $(n+1)th$ cycle of the iterative procedure, $\hat{\Theta}_{n+1}$, is given by

$$\hat{\Theta}_{n+1} = \hat{\Theta}_n + \mathbf{I}^{-1}(\hat{\Theta}_n)\mathbf{f}(\hat{\Theta}_n) \quad \text{for} \quad n = 0, 1, 2, \cdots, \tag{3.14}$$

where $\mathbf{I}^{-1}(\hat{\Theta}_k)$ is the inverse of the information matrix evaluated at $\hat{\Theta}_k$. The iteration in Eq. (3.14) can be started at an initial guess $\hat{\Theta_0}$. The process is terminated when the change in the log-likelihood function is a small number, say ϵ, or when the largest of the relative changes in the values of the parameter estimates is sufficiently small. In other words, the iterative method is continued until convergence is achieved i.e, $| \hat{\Theta}_n - \hat{\Theta}_{n+1} | \leq 0.00001$.

Once convergence is achieved, the variance-covariance matrix of the parameter estimates can be approximated by the inverse of the observed information matrix $\mathbf{I}(\Theta)$, evaluated at $\hat{\Theta}$, i.e., $\mathbf{I}^{-1}(\hat{\Theta})$, so that the variance-covariance matrix of $\hat{\Theta}$ is

$$\sum(\hat{\Theta}) \approx \mathbf{I}^{-1}(\hat{\Theta}). \tag{3.15}$$

The square root of the ith element of this matrix can be taken to be the standard error of $\hat{\theta}_i$, for $i = 1, 2, \cdots, k$. For a complete discussion of the maximum likelihood estimation technique, one may refer to (Mood, Graybill, and Boes, 1963; Lindgren, 1968; Rao, 1973; Serfling, 1980; and Hogg and Craig, 1995) and for the Newton-Raphson technique, to (Ortega, 1972; Johnson and Riess, 1982; Maron and Lopez, 1991; Burden and Faires, 1997; Gautschi, 1997; and Kress, 1998).

3.3 Simulation

In this dissertation, we use the following technique to generate data through simulation. Let U be a uniform $(0, 1)$ random variable, i.e.

$$G(u) = \begin{cases} 0 & \text{if } u \leq 0 \\ u & \text{if } 0 < u < 1 \\ 1 & \text{if } u \geq 1 \end{cases} \tag{3.16}$$

Let $F(t)$ be a strictly increasing continuous distribution function on the interval $(0, 1)$, and let T be a random variable that satisfies the relationship $U = F(T)$. Now, if $0 < F(t) < 1$, then $T \leq t$ and $F(T) \leq F(t)$ are equivalent. Therefore, when $0 < F(t) < 1$, the distribution of T is given by

$$Pr(T \leq t) = Pr[F(T) \leq F(t)] = Pr[U \leq F(t)]. \tag{3.17}$$

However, since

$$Pr(U \leq u) = G(u), \tag{3.18}$$

we have that

$$Pr(T \leq t) = G[F(t)] = F(t), \quad 0 < F(t) < 1 \tag{3.19}$$

Hence, T has a distribution function $F(t)$.

Now, to generate a random value t, we use the computer to generate a random number from a uniform distribution $U(0, 1)$ and let

$$F(t) = u \tag{3.20}$$

After solving the above equation, either explicitly or by numerical techniques, one obtains

$$t = F^{-1}(u) \tag{3.21}$$

By the above argument, it is seen that t is a randomly observed value of T that has a distribution function $F(t)$. The above method of simulation is called the inverse transformation method (Ross, 1997).

CHAPTER 4

ACCELERATED LIFETIME UNDER CONSTANT PRESSURE

4.1 Theory

We know from Eq. (3.1) that the log-likelihood function is

$$L = \sum_{i=1}^{r} \ln f_{0i} + \sum_{i=r+1}^{n} \ln (1 - F_{0i}). \tag{4.1}$$

But, since the distribution used in this study is the Weibull, one has

$$f(t_0) = \gamma \lambda t_0^{\gamma - 1} e^{-\lambda t_0^{\gamma}}, \tag{4.2}$$

and

$$1 - F_0 = e^{-\lambda t_0^{\gamma}}. \tag{4.3}$$

Now, let

$$t = t_0 e^{bx}. \tag{4.4}$$

Solving for t_0, one sees that

$$t_0 = te^{-bx}. \tag{4.5}$$

To obtain the new distribution for t, one needs the Jacobian of the transformation $\mid \frac{dt_0}{dt} \mid$ which is given as

$$\frac{dt_0}{dt} = e^{-bx}. \tag{4.6}$$

Hence, the log-likelihood function is

$$\begin{aligned}
L &= \sum_{i=1}^{r} \ln\left[\gamma\lambda t_{0i}^{\gamma-1} e^{-\lambda t_{0i}^{\gamma}}\right] + \sum_{i=r+1}^{n} \ln e^{-\lambda t_{0i}^{\gamma}} \\
&= \sum_{i=1}^{r} \ln\left[\gamma\lambda (t_i e^{-bx})^{\gamma-1} e^{-\lambda(t_i e^{-bx})^{\gamma}} e^{-bx}\right] + \sum_{i=r+1}^{n} \ln e^{-\lambda(t_i e^{-bx})^{\gamma}} \\
&= \sum_{i=1}^{r} [\ln\gamma + \ln\lambda + (\gamma-1)(\ln(t_i) - bx) - \lambda(t_i e^{-bx})^{\gamma} - bx] \\
&- \sum_{i=r+1}^{n} \lambda(t_i e^{-bx})^{\gamma}.
\end{aligned} \tag{4.7}$$

Simplifying Eq. (4.7), it is seen that

$$\begin{aligned}
L &= \sum_{i=1}^{r} [\ln\gamma + \ln\lambda + \gamma\ln t_i - \ln t_i - \gamma bx - \lambda t_i^{\gamma} e^{-\gamma bx}] \\
&- \sum_{i=r+1}^{n} \lambda t_i^{\gamma} e^{-\gamma bx}.
\end{aligned} \tag{4.8}$$

From Eq. (4.8), one obtains

$$\frac{\partial L}{\partial\gamma} = \sum_{i=1}^{r} [\frac{1}{\gamma} + \ln t_i - bx - \lambda t_i^{\gamma} e^{-\gamma bx} \ln t_i + \lambda bx t_i^{\gamma} e^{-\gamma bx}]$$

$$+ \sum_{i=r+1}^{n} [\lambda bxt_i^{\gamma} e^{-\gamma bx} - \lambda t_i^{\gamma} e^{-\gamma bx} \ln t_i], \tag{4.9}$$

$$\frac{\partial L}{\partial \lambda} = \sum_{i=1}^{r} [\frac{1}{\lambda} - t_i^{\gamma} e^{-\gamma bx}] - \sum_{i=r+1}^{n} t_i^{\gamma} e^{-\gamma bx}, \tag{4.10}$$

and

$$\frac{\partial L}{\partial b} = \sum_{i=1}^{r} [-\gamma x + \gamma \lambda x t_i^{\gamma} e^{-\gamma bx}] + \sum_{i=r+1}^{n} \gamma \lambda x t_i^{\gamma} e^{-\gamma bx}. \tag{4.11}$$

Now, let $f_1 = \frac{\partial L}{\partial \gamma}$, $f_2 = \frac{\partial L}{\partial \lambda}$, and $f_3 = \frac{\partial L}{\partial b}$, and set $f_1 = f_2 = f_3 = 0$. Equations (4.9), (4.10), and (4.11) can be solved using the Newton-Raphson method discussed in chapter 3. The Jacobian matrix is

$$\mathbf{I}(t, \gamma, \lambda, b) = - \begin{bmatrix} \frac{\partial f_1}{\partial \gamma} & \frac{\partial f_1}{\partial \lambda} & \frac{\partial f_1}{\partial b} \\ \frac{\partial f_2}{\partial \gamma} & \frac{\partial f_2}{\partial \lambda} & \frac{\partial f_2}{\partial b} \\ \frac{\partial f_3}{\partial \gamma} & \frac{\partial f_3}{\partial \lambda} & \frac{\partial f_3}{\partial b} \end{bmatrix} \tag{4.12}$$

where,

$$\frac{\partial f_1}{\partial \lambda} = \frac{\partial f_2}{\partial \gamma}, \tag{4.13}$$

$$\frac{\partial f_1}{\partial b} = \frac{\partial f_3}{\partial \gamma}, \tag{4.14}$$

and

$$\frac{\partial f_2}{\partial b} = \frac{\partial f_3}{\partial \lambda}. \tag{4.15}$$

From Eq. (4.9), it is seen that

$$\begin{aligned} \frac{\partial f_1}{\partial \gamma} &= \sum_{i=1}^{r}[\frac{-1}{\gamma^2} - \lambda t_i^\gamma e^{-\gamma bx}(\ln t_i)^2 + \lambda bxt_i^\gamma e^{-\gamma bx}\ln t_i \\ &+ \lambda bxt_i^\gamma e^{-\gamma bx}\ln t_i - \lambda b^2x^2t_i^\gamma e^{-\gamma bx}] \\ &+ \sum_{i=r+1}^{n}[-\lambda t_i^\gamma e^{-\gamma bx}(\ln t_i)^2 + \lambda bxt_i^\gamma e^{-\gamma bx}\ln t_i \\ &+ \lambda bxt_i^\gamma e^{-\gamma bx}\ln t_i - \lambda b^2x^2t_i^\gamma e^{-\gamma bx}]. \end{aligned} \tag{4.16}$$

Upon simplification, one has that

$$\frac{\partial f_1}{\partial \gamma} = \sum_{i=1}^{r}\frac{-1}{\gamma^2} + \sum_{i=1}^{n}[\lambda t_i^\gamma e^{-\gamma bx}\ln t_i(bx - \ln t_i) - \lambda bxt_i^\gamma e^{-\gamma bx}(bx - \ln t_i)]. \tag{4.17}$$

Also, from Eq. (4.9)

$$\begin{aligned} \frac{\partial f_1}{\partial \lambda} &= \sum_{i=1}^{r}[bxt_i^\gamma e^{-\gamma bx} - t_i^\gamma e^{-\gamma bx}\ln t_i] \\ &+ \sum_{i=r+1}^{n}[bxt_i^\gamma e^{-\gamma bx} - t_i^\gamma e^{-\gamma bx}\ln t_i], \end{aligned} \tag{4.18}$$

or

$$\frac{\partial f_1}{\partial \lambda} = \sum_{i=1}^{n} t_i^{\gamma} e^{-\gamma b x}(bx - \ln t_i). \tag{4.19}$$

Likewise,

$$\begin{aligned}\frac{\partial f_1}{\partial b} &= \sum_{i=1}^{r}[-x + \gamma\lambda x t_i^{\gamma} e^{-\gamma b x} \ln t_i - \gamma\lambda b x^2 t_i^{\gamma} e^{-\gamma b x} + \lambda x t_i^{\gamma} e^{-\gamma b x}] \\ &+ \sum_{i=r+1}^{n} [\gamma\lambda x t_i^{\gamma} e^{-\gamma b x} \ln t_i - \gamma\lambda b x^2 t_i^{\gamma} e^{-\gamma b x} + \lambda x t_i^{\gamma} e^{-\gamma b x}],\end{aligned} \tag{4.20}$$

or

$$\frac{\partial f_1}{\partial b} = \sum_{i=1}^{r}(-x) + \sum_{i=1}^{n}[-\gamma\lambda x t_i^{\gamma} e^{-\gamma b x}(bx - \ln t_i) + \lambda x t_i^{\gamma} e^{-\gamma b x}]. \tag{4.21}$$

Also,

$$\begin{aligned}\frac{\partial f_2}{\partial \gamma} &= \sum_{i=1}^{r}[b x t_i^{\gamma} e^{-\gamma b x} - t_i^{\gamma} e^{-\gamma b x} \ln t_i] \\ &+ \sum_{i=r+1}^{n} [b x t_i^{\gamma} e^{-\gamma b x} - t_i^{\gamma} e^{-\gamma b x} \ln t_i],\end{aligned} \tag{4.22}$$

which reduces to

$$\frac{\partial f_2}{\partial \gamma} = \sum_{i=1}^{n} t_i^{\gamma} e^{-\gamma b x}(bx - \ln t_i). \tag{4.23}$$

From Eqs. (4.23) and (4.19) one sees that

$$\frac{\partial f_2}{\partial \gamma} = \frac{\partial f_1}{\partial \lambda}. \tag{4.24}$$

For the rest of the derivatives in the Jacobian matrix (4.12), it is seen that

$$\frac{\partial f_2}{\partial \lambda} = \sum_{i=1}^{r} \frac{-1}{\lambda^2}, \tag{4.25}$$

and

$$\begin{aligned} \frac{\partial f_2}{\partial b} &= \sum_{i=1}^{r} \gamma x t_i^{\gamma} e^{-\gamma b x} + \sum_{i=r+1}^{n} \gamma x t_i^{\gamma} e^{-\gamma b x} \\ &= \sum_{i=1}^{n} \gamma x t_i^{\gamma} e^{-\gamma b x} \end{aligned} \tag{4.26}$$

Finally,

$$\begin{aligned} \frac{\partial f_3}{\partial \gamma} &= \sum_{i=1}^{r} [-x + \gamma \lambda x t_i^{\gamma} e^{-\gamma b x} \ln t_i - \gamma \lambda b x^2 t_i^{\gamma} e^{-\gamma b x} + \lambda x t_i^{\gamma} e^{-\gamma b x}] \\ &+ \sum_{i=r+1}^{n} [\gamma \lambda x t_i^{\gamma} e^{-\gamma b x} \ln t_i - \gamma \lambda b x^2 t_i^{\gamma} e^{-\gamma b x} + \lambda x t_i^{\gamma} e^{-\gamma b x}], \end{aligned} \tag{4.27}$$

which reduces to

$$\frac{\partial f_3}{\partial \gamma} = \sum_{i=1}^{r} (-x) + \sum_{i=1}^{n} [-\gamma \lambda x t_i^{\gamma} e^{-\gamma b x} (bx - \ln t_i) + \lambda x t_i^{\gamma} e^{-\gamma b x}]. \tag{4.28}$$

Hence, from Eqs. (4.28) and (4.21) it is seen that

$$\frac{\partial f_3}{\partial \gamma} = \frac{\partial f_1}{\partial b}. \tag{4.29}$$

Also,

$$\begin{aligned} \frac{\partial f_3}{\partial \lambda} &= \sum_{i=1}^{r} \gamma x t_i^{\gamma} e^{-\gamma b x} + \sum_{i=r+1}^{n} \gamma x t_i^{\gamma} e^{-\gamma b x} \\ &= \sum_{i=1}^{n} \gamma x t_i^{\gamma} e^{-\gamma b x}. \end{aligned} \tag{4.30}$$

Hence, from Eqs. (4.30) and (4.26) one sees that

$$\frac{\partial f_3}{\partial \lambda} = \frac{\partial f_2}{\partial b}. \tag{4.31}$$

Finally,

$$\begin{aligned} \frac{\partial f_3}{\partial b} &= \sum_{i=1}^{r} [-\gamma^2 \lambda x^2 t_i^{\gamma} e^{-\gamma b x}] + \sum_{i=r+1}^{n} [-\gamma^2 \lambda x^2 t_i^{\gamma} e^{-\gamma b x}] \\ &= \sum_{i=1}^{n} [-\gamma^2 \lambda x^2 t_i^{\gamma} e^{-\gamma b x}]. \end{aligned} \tag{4.32}$$

4.2 Choices for γ, λ, b, and pressure, x

It is seen from Eq. (4.4) that

$$t = t_0 e^{bx}. \tag{4.33}$$

Taking the expectation of both sides, one obtains

$$E(t) = E(t_0)e^{bx}. \tag{4.34}$$

If one assumes that the liner survives on the average up to 50 years with an external hydrostatic pressure of 10 psi, then from Eq. (4.34)

$$50 = E(t_0)e^{10b}. \tag{4.35}$$

Since t_0 has the Weibull distribution with parameters γ, and λ, it is seen that

$$\begin{aligned} E(t_0) &= \Gamma(1 + \frac{1}{\gamma})\frac{1}{\lambda^{\frac{1}{\gamma}}} \\ &= 50e^{-10b}. \end{aligned} \tag{4.36}$$

Based on estimates from accelerated lifetime data (Guice *et al.* 1994), one may take γ to be 1.5 and $b = -0.25$. Hence, from Eq. (4.36)

$$\begin{aligned} E(t_0) &= \Gamma(1 + \frac{1}{1.5})\frac{1}{\lambda^{\frac{1}{1.5}}} \\ &= \Gamma(1.67)\frac{1}{\lambda^{\frac{1}{1.5}}}. \end{aligned} \tag{4.37}$$

From Eqs. (4.37) and (4.35), it is seen that

$$50 = \frac{0.9}{\lambda^{\frac{1}{1.5}}}e^{-2.5}. \tag{4.38}$$

Solving Eq. (4.38) for λ, one obtains

$$\lambda = (\frac{0.9e^{-.25}}{50})^{1.5}$$
$$= 5.7 \times 10^{-5}. \quad (4.39)$$

In the simulation study the pressure x in psi was calculated for 10%, 20%, 30%, and 40% censoring. Based on the values of γ, λ and b and Eq. (4.3) with the transformation $t_0 = te^{-bx}$, one has for 10% censoring

$$e^{-5.7\times 10^{-5}(1.14e^{.25x})^{1.5}} = 0.1 \quad . \quad (4.40)$$

Solving Eq. (4.40), one obtains

$$x = \frac{\ln(-\ln(0.1)) - \ln(0.000057) - 1.5\ln(1.14)}{1.5 \times 0.25}$$
$$= 27.8 \; psi. \quad (4.41)$$

Likewise, the values for pressure x for 20%, 30%, and 40% censoring are 26.8, 26.0, and 25.3 psi, respectively.

4.3 Results From Simulation

It is seen from Tables 4.1 to 4.3 for 10% censoring, fixed pressure of 27.8 psi, and sample sizes 25, 50, and 100, that the maximum likelihood (ML) estimate of the parameter $\gamma = 1.5$ is approximately normally distributed for sample size as small as 25. This result is shown by the D'Agostino Omnibus test and is as expected from asymptotic theory of maximum likelihood estimation. Normality implies that inference about the parameter such as confidence intervals and test of hypothesis can be used based on normal theory. Also, the mean of 1000 estimates is close to the expected value of 1.5 indicating no significant bias in estimation. Figures 4.1 and 4.2 presents histograms of the gamma estimates for sample sizes of 25 and 100. These figures show the approximate normality of the distribution as determined by the D'Agostino Omnibus test for normality. Results in Tables 4.4 to 4.9 for 10% censoring show, on the other hand, that the estimates for lambda and b are not normally distributed. This lack of normality is demonstrated also by the histogram plots in Figs. 4.3 to 4.6. However the means of the 1000 estimates for both lambda and b are close to their respective parameter values of 5.7×10^{-5} and -0.25 indicating no bias in estimation. The lack of normality of the ML estimates even for a sample of size 100 may be due to the small parameter values for lambda and b. These values were chosen because they were close to a real situation as far as a pipe liner lifetime is concerned. The parameter values ($\gamma = 1.5, \lambda = 5.7 \times 10^{-5}$, and $b = -0.25$) correspond to a mean lifetime of 50 years under a fixed pressure of 10 psi. The lack of normality implies that inferences based on normal theory for ML estimates cannot be used in this case. For the normality assumption to hold the sample size needs in all likelihood to be larger than is practically feasible.

Results in Tables 4.10 to 4.12 for 20% censored observations show that estimates for gamma are still approximately normally distributed, but as expected for a larger sample size of 50 and 100. However, the ML estimate as shown by the mean of the 1000 estimates does not show any serious bias. Estimates for lambda and b, as shown in Tables 4.13 to 4.18 for 20% censoring, while not biased do not show normality.

Estimates of gamma for 30% censoring, Tables 4.19 to 4.21, show a more pronounces bias (mean=1.41) and a complete lack of normality for sample size 25. Normality is achieved for larger samples of 50 and 100. Estimates for lambda and b are not normally distributed as seen from Tables 4.22 to 4.27. However, it is seen that these estimates are not biased.

Increasing censoring to 40% (Tables 4.28 to 4.30) seems to increase bias by reducing the mean estimate to a value of about 1.36 but has no effect on the normality of estimates. Figures 4.7 and 4.8 show the empirical distribution of the gamma estimates which appears to be normal in agreement with the D'Agostino tests for normality. Tables 4.31 to 4.36 show that in the case of 40% censoring the ML estimates for lambda and b are not biased. However, these estimates are not normally distributed. Figures 4.9 to 4.12 demonstrate graphically the deviation from normality encountered in the estimates.

Tables 4.37 to 4.48 present the empirical variance-covariance matrices from the 1000 replications from simulation for different pressures, censoring, and sample sizes. These are to be compared with the corresponding theoretical variance-covariance matrices in Tables 4.49 to 4.60 obtained from the Fisher information matrix Eq. (3.8). In each matrix, element $a_{11} = V(\gamma)$, $a_{12} = Cov(\gamma, \lambda)$, $a_{13} = Cov(\gamma, b)$, $a_{22} = V(\lambda)$, $a_{23} = Cov(\lambda, b)$, and $a_{33} = V(b)$. It is clear from these results that the empirical variances of the estimates and covariances between two estimates are larger in value than what one expects from theory using the Fisher matrix. This result is especially true for the lambda and b estimates which deviates significantly from normality. The variance for the gamma estimates which tends to be normal is more in agreement with the theoretical variance than the other estimates.

Table 4.1 Statistical properties of the ML estimate for $\gamma = 1.5$ over 1000 replications. Pressure =27.8 psi, censoring =10%, sample size=25.

Mean	1.556891
Variance	1.976661×10^{-3}
Minimum	1.353129
Maximum	1.677138
Skewness	-2.096338×10^{-2}
Kurtosis	3.196979
D'Agostino Skewness	Accept Normality with prob. level 0.79
D'Agostino Kurtosis	Accept Normality with prob. level 0.20
D'Agostino Omnibus	Accept Normality with prob. level 0.42

Table 4.2 Statistical properties of the ML estimate for $\gamma = 1.5$ over 1000 replications. Pressure =27.8 psi, censoring =10%, sample size=50.

Mean	1.558169
Variance	8.55301×10^{-4}
Minimum	1.400797
Maximum	1.644582
Skewness	-8.590256×10^{-2}
Kurtosis	3.354546
D'Agostino Skewness	Accept Normality with prob. level 0.27
D'Agostino Kurtosis	Reject Normality with prob. level 0.04
D'Agostino Omnibus	Accept Normality with prob. level 0.06

Table 4.3 Statistical properties of the ML estimate for $\gamma = 1.5$ over 1000 replications. Pressure =27.8 psi, censoring =10%, sample size=100.

Mean	1.555082
Variance	4.66491×10^{-4}
Minimum	1.468154
Maximum	1.640457
Skewness	9.118701×10^{-2}
Kurtosis	3.355797
D'Agostino Skewness	Accept Normality with prob. level 0.24
D'Agostino Kurtosis	Reject Normality with prob. level 0.04
D'Agostino Omnibus	Accept Normality with prob. level 0.06

Table 4.4 Statistical properties of the ML estimate for $\lambda = 5.7 \times 10^{-5}$ over 1000 replications. Pressure =27.8 psi, censoring =10%, sample size=25.

Mean	5.70274×10^{-5}
Variance	4.116008×10^{-13}
Minimum	0.000057
Maximum	0.0000761
Skewness	27.49277
Kurtosis	0
D'Agostino Skewness	Reject Normality with prob. level 0.0
D'Agostino Kurtosis	Reject Normality with prob. level 0.0
D'Agostino Omnibus	Reject Normality with prob. level 0.0

Table 4.5 Statistical properties of the ML estimate for $\lambda = 5.7 \times 10^{-5}$ over 1000 replications. Pressure =27.8 psi, censoring =10%, sample size=50.

Mean	5.7017×10^{-5}
Variance	1.651361×10^{-13}
Minimum	0.000057
Maximum	0.000069
Skewness	27.1163
Kurtosis	0
D'Agostino Skewness	Reject Normality with prob. level 0.0
D'Agostino Kurtosis	Reject Normality with prob. level 0.0
D'Agostino Omnibus	Reject Normality with prob. level 0.0

Table 4.6 Statistical properties of the ML estimate for $\lambda = 5.7 \times 10^{-5}$ over 1000 replications. Pressure =27.8 psi, censoring =10%, sample size=100.

Mean	5.70073×10^{-5}
Variance	1.965637×10^{-14}
Minimum	0.000057
Maximum	0.0000604
Skewness	0
Kurtosis	0
D'Agostino Skewness	Accept Normality with prob. level 1.0
D'Agostino Kurtosis	Reject Normality with prob. level 0.0
D'Agostino Omnibus	Reject Normality with prob. level 0.0

Table 4.7 Statistical properties of the ML estimate for $b = -0.25$ over 1000 replications. Pressure =27.8 psi, censoring =10%, sample size=25.

Mean	-0.2500423
Variance	1.457117×10^{-6}
Minimum	-0.2880698
Maximum	-0.2499851
Skewness	-31.32375
Kurtosis	986.9233
D'Agostino Skewness	Reject Normality with prob. level 0.0
D'Agostino Kurtosis	Reject Normality with prob. level 0.0
D'Agostino Omnibus	Reject Normality with prob. level 0.0

Table 4.8 Statistical properties of the ML estimate for $b = -0.25$ over 1000 replications. Pressure =27.8 psi, censoring =10%, sample size=50.

Mean	-0.2500286
Variance	6.444887×10^{-7}
Minimum	-0.2753152
Maximum	-0.2499778
Skewness	-31.31042
Kurtosis	986.3148
D'Agostino Skewness	Reject Normality with prob. level 0.0
D'Agostino Kurtosis	Reject Normality with prob. level 0.0
D'Agostino Omnibus	Reject Normality with prob. level 0.0

Table 4.9 Statistical properties of the ML estimate for $b = -0.25$ over 1000 replications. Pressure =27.8 psi, censoring =10%, sample size=100.

Mean	-0.2500164
Variance	1.789051×10^{-7}
Minimum	-0.2633221
Maximum	-0.2499969
Skewness	-31.19939
Kurtosis	981.3035
D'Agostino Skewness	Reject Normality with prob. level 0.0
D'Agostino Kurtosis	Reject Normality with prob. level 0.0
D'Agostino Omnibus	Reject Normality with prob. level 0.0

Table 4.10 Statistical properties of the ML estimate for $\gamma = 1.5$ over 1000 replications. Pressure =26.8 psi, censoring =20%, sample size=25.

Mean	1.482563
Variance	1.770836×10^{-3}
Minimum	1.355556
Maximum	1.627106
Skewness	0.1470829
Kurtosis	3.271482
D'Agostino Skewness	Accept Normality with prob. level 0.06
D'Agostino Kurtosis	Accept Normality with prob. level 0.09
D'Agostino Omnibus	Reject Normality with prob. level 0.04

Table 4.11 Statistical properties of the ML estimate for $\gamma = 1.5$ over 1000 replications. Pressure =26.8 psi, censoring =20%, sample size=50.

Mean	1.480374
Variance	9.344896×10^{-4}
Minimum	1.378263
Maximum	1.575105
Skewness	-3.742477×10^{-2}
Kurtosis	2.947649
D'Agostino Skewness	Accept Normality with prob. level 0.63
D'Agostino Kurtosis	Accept Normality with prob. level 0.81
D'Agostino Omnibus	Accept Normality with prob. level 0.86

Table 4.12 Statistical properties of the ML estimate for $\gamma = 1.5$ over 1000 replications. Pressure =26.8 psi, censoring =20%, sample size=100.

Mean	1.479459
Variance	4.626646×10^{-4}
Minimum	1.385224
Maximum	1.545953
Skewness	9.868777×10^{-2}
Kurtosis	3.160574
D'Agostino Skewness	Accept Normality with prob. level 0.20
D'Agostino Kurtosis	Accept Normality with prob. level 0.28
D'Agostino Omnibus	Accept Normality with prob. level 0.24

Table 4.13 Statistical properties of the ML estimate for $\lambda = 5.7 \times 10^{-5}$ over 1000 replications. Pressure =26.8 psi, censoring =20%, sample size=25.

Mean	5.70212×10^{-5}
Variance	2.44415×10^{-13}
Minimum	0.000057
Maximum	0.0000716
Skewness	27.05519
Kurtosis	0
D'Agostino Skewness	Reject Normality with prob. level 0.0
D'Agostino Kurtosis	Reject Normality with prob. level 0.0
D'Agostino Omnibus	Reject Normality with prob. level 0.0

Table 4.14 Statistical properties of the ML estimate for $\lambda = 5.7 \times 10^{-5}$ over 1000 replications. Pressure =26.8 psi, censoring =20%, sample size=50.

Mean	5.70067×10^{-5}
Variance	1.516027×10^{-14}
Minimum	0.000057
Maximum	0.0000601
Skewness	0
Kurtosis	0
D'Agostino Skewness	Accept Normality with prob. level 1.0
D'Agostino Kurtosis	Reject Normality with prob. level 0.0
D'Agostino Omnibus	Reject Normality with prob. level 0.0

Table 4.15 Statistical properties of the ML estimate for $\lambda = 5.7 \times 10^{-5}$ over 1000 replications. Pressure =26.8 psi, censoring =20%, sample size=100.

Mean	5.70117×10^{-5}
Variance	6.189501×10^{-14}
Minimum	0.000057
Maximum	0.0000638
Skewness	24.18933
Kurtosis	0
D'Agostino Skewness	Reject Normality with prob. level 0.0
D'Agostino Kurtosis	Reject Normality with prob. level 0.0
D'Agostino Omnibus	Reject Normality with prob. level 0.0

Table 4.16 Statistical properties of the ML estimate for $b = -0.25$ over 1000 replications. Pressure =26.8 psi, censoring =20%, sample size=25.

Mean	-0.2499968
Variance	5.994215×10^{-8}
Minimum	-0.2523384
Maximum	-0.2426388
Skewness	26.30138
Kurtosis	825.8197
D'Agostino Skewness	Reject Normality with prob. level 0.0
D'Agostino Kurtosis	Reject Normality with prob. level 0.0
D'Agostino Omnibus	Reject Normality with prob. level 0.0

Table 4.17 Statistical properties of the ML estimate for $b = -0.25$ over 1000 replications. Pressure =26.8 psi, censoring =20%, sample size=50.

Mean	-0.2499951
Variance	6.621553×10^{-8}
Minimum	-0.2507939
Maximum	-0.2419311
Skewness	30.77648
Kurtosis	966.5384
D'Agostino Skewness	Reject Normality with prob. level 0.0
D'Agostino Kurtosis	Reject Normality with prob. level 0.0
D'Agostino Omnibus	Reject Normality with prob. level 0.0

Table 4.18 Statistical properties of the ML estimate for $b = -0.25$ over 1000 replications. Pressure =26.8 psi, censoring =20%, sample size=100.

Mean	-0.2500237
Variance	3.977499×10^{-7}
Minimum	-0.2698731
Maximum	-0.2499991
Skewness	-31.24174
Kurtosis	983.235
D'Agostino Skewness	Reject Normality with prob. level 0.0
D'Agostino Kurtosis	Reject Normality with prob. level 0.0
D'Agostino Omnibus	Reject Normality with prob. level 0.0

Table 4.19 Statistical properties of the ML estimate for $\gamma = 1.5$ over 1000 replications. Pressure =26.0 psi, censoring =30%, sample size=25.

Mean	1.41782
Variance	2.626594×10^{-3}
Minimum	1.229722
Maximum	1.80538
Skewness	0.4435235
Kurtosis	5.831409
D'Agostino Skewness	Reject Normality with prob. level 0.0
D'Agostino Kurtosis	Reject Normality with prob. level 0.0
D'Agostino Omnibus	Reject Normality with prob. level 0.0

Table 4.20 Statistical properties of the ML estimate for $\gamma = 1.5$ over 1000 replications. Pressure =26.0 psi, censoring =30%, sample size=50.

Mean	1.414278
Variance	1.177458×10^{-3}
Minimum	1.296862
Maximum	1.516428
Skewness	2.046373×10^{-2}
Kurtosis	3.007486
D'Agostino Skewness	Accept Normality with prob. level 0.79
D'Agostino Kurtosis	Accept Normality with prob. level 0.87
D'Agostino Omnibus	Accept Normality with prob. level 0.95

Table 4.21 Statistical properties of the ML estimate for $\gamma = 1.5$ over 1000 replications. Pressure =26.0 psi, censoring =30%, sample size=100.

Mean	1.413975
Variance	5.686668×10^{-4}
Minimum	1.345811
Maximum	1.490464
Skewness	-4.843413×10^{-2}
Kurtosis	3.01112
D'Agostino Skewness	Accept Normality with prob. level 0.53
D'Agostino Kurtosis	Accept Normality with prob. level 0.85
D'Agostino Omnibus	Accept Normality with prob. level 0.81

Table 4.22 Statistical properties of the ML estimate for $\lambda = 5.7 \times 10^{-5}$ over 1000 replications. Pressure =26.0 psi, censoring =30%, sample size=25.

Mean	5.70465×10^{-5}
Variance	2.116004×10^{-12}
Minimum	0.000057
Maximum	0.000103
Skewness	31.57421
Kurtosis	0
D'Agostino Skewness	Reject Normality with prob. level 0.0
D'Agostino Kurtosis	Reject Normality with prob. level 0.0
D'Agostino Omnibus	Reject Normality with prob. level 0.0

Table 4.23 Statistical properties of the ML estimate for $\lambda = 5.7 \times 10^{-5}$ over 1000 replications. Pressure =26.0 psi, censoring =30%, sample size=50.

Mean	5.70131×10^{-5}
Variance	8.418257×10^{-14}
Minimum	0.000057
Maximum	0.0000657
Skewness	27.86605
Kurtosis	0
D'Agostino Skewness	Reject Normality with prob. level 0.0
D'Agostino Kurtosis	Reject Normality with prob. level 0.0
D'Agostino Omnibus	Reject Normality with prob. level 0.0

Table 4.24 Statistical properties of the ML estimate for $\lambda = 5.7 \times 10^{-5}$ over 1000 replications. Pressure =26.0 psi, censoring =30%, sample size=100.

Mean	5.7018×10^{-5}
Variance	2.658018×10^{-13}
Minimum	0.000057
Maximum	0.0000733
Skewness	31.54558
Kurtosis	0
D'Agostino Skewness	Reject Normality with prob. level 0.0
D'Agostino Kurtosis	Reject Normality with prob. level 0.0
D'Agostino Omnibus	Reject Normality with prob. level 0.0

Table 4.25 Statistical properties of the ML estimate for $b = -0.25$ over 1000 replications. Pressure =26.0 psi, censoring =30%, sample size=25.

Mean	-0.2499399
Variance	3.841903×10^{-6}
Minimum	-0.2500303
Maximum	-0.1880189
Skewness	31.57522
Kurtosis	997.9966
D'Agostino Skewness	Reject Normality with prob. level 0.0
D'Agostino Kurtosis	Reject Normality with prob. level 0.0
D'Agostino Omnibus	Reject Normality with prob. level 0.0

Table 4.26 Statistical properties of the ML estimate for $b = -0.25$ over 1000 replications. Pressure =26.0 psi, censoring =30%, sample size=50.

Mean	-0.2499906
Variance	1.782981×10^{-7}
Minimum	-0.2513529
Maximum	-0.2367204
Skewness	31.05284
Kurtosis	977.5618
D'Agostino Skewness	Reject Normality with prob. level 0.0
D'Agostino Kurtosis	Reject Normality with prob. level 0.0
D'Agostino Omnibus	Reject Normality with prob. level 0.0

Table 4.27 Statistical properties of the ML estimate for $b = -0.25$ over 1000 replications. Pressure =26.0 psi, censoring =30%, sample size=100.

Mean	-0.2499906
Variance	1.46998×10^{-7}
Minimum	-0.2500373
Maximum	-0.2378791
Skewness	31.57
Kurtosis	997.7775
D'Agostino Skewness	Reject Normality with prob. level 0.0
D'Agostino Kurtosis	Reject Normality with prob. level 0.0
D'Agostino Omnibus	Reject Normality with prob. level 0.0

Table 4.28 Statistical properties of the ML estimate for $\gamma = 1.5$ over 1000 replications. Pressure =25.3 psi, censoring =40%, sample size=25.

Mean	1.360228
Variance	2.520586×10^{-3}
Minimum	1.179837
Maximum	1.498348
Skewness	-4.401014×10^{-2}
Kurtosis	3.054044
D'Agostino Skewness	Accept Normality with prob. level 0.57
D'Agostino Kurtosis	Accept Normality with prob. level 0.65
D'Agostino Omnibus	Accept Normality with prob. level 0.77

Table 4.29 Statistical properties of the ML estimate for $\gamma = 1.5$ over 1000 replications. Pressure =25.3 psi, censoring =40%, sample size=50.

Mean	1.355946
Variance	1.177019×10^{-3}
Minimum	1.247187
Maximum	1.462618
Skewness	-7.620224×10^{-4}
Kurtosis	2.969444
D'Agostino Skewness	Accept Normality with prob. level 0.99
D'Agostino Kurtosis	Accept Normality with prob. level 0.93
D'Agostino Omnibus	Accept Normality with prob. level 1.0

Table 4.30 Statistical properties of the ML estimate for $\gamma = 1.5$ over 1000 replications. Pressure =25.3 psi, censoring =40%, sample size=100.

Mean	1.356143
Variance	5.586523×10^{-4}
Minimum	1.275957
Maximum	1.42983
Skewness	-5.645929×10^{-2}
Kurtosis	3.153582
D'Agostino Skewness	Accept Normality with prob. level 0.46
D'Agostino Kurtosis	Accept Normality with prob. level 0.30
D'Agostino Omnibus	Accept Normality with prob. level 0.44

Table 4.31 Statistical properties of the ML estimate for $\lambda = 5.7 \times 10^{-5}$ over 1000 replications. Pressure =25.3 psi, censoring =40%, sample size=25.

Mean	5.70147×10^{-5}
Variance	4.005397×10^{-14}
Minimum	0.000057
Maximum	0.0000618
Skewness	18.73638
Kurtosis	0
D'Agostino Skewness	Reject Normality with prob. level 0.0
D'Agostino Kurtosis	Reject Normality with prob. level 0.0
D'Agostino Omnibus	Reject Normality with prob. level 0.0

Table 4.32 Statistical properties of the ML estimate for $\lambda = 5.7 \times 10^{-5}$ over 1000 replications. Pressure =25.3 psi, censoring =40%, sample size=50.

Mean	5.70296×10^{-5}
Variance	6.144583×10^{-13}
Minimum	0.000057
Maximum	0.0000817
Skewness	31.23557
Kurtosis	0
D'Agostino Skewness	Reject Normality with prob. level 0.0
D'Agostino Kurtosis	Reject Normality with prob. level 0.0
D'Agostino Omnibus	Reject Normality with prob. level 0.0

Table 4.33 Statistical properties of the ML estimate for $\lambda = 5.7 \times 10^{-5}$ over 1000 replications. Pressure =25.3 psi, censoring =40%, sample size=100.

Mean	5.70373×10^{-5}
Variance	1.110569×10^{-12}
Minimum	0.000057
Maximum	0.0000903
Skewness	31.49409
Kurtosis	0
D'Agostino Skewness	Reject Normality with prob. level 0.0
D'Agostino Kurtosis	Reject Normality with prob. level 0.0
D'Agostino Omnibus	Reject Normality with prob. level 0.0

Table 4.34 Statistical properties of the ML estimate for $b = -0.25$ over 1000 replications. Pressure =25.3 psi, censoring =40%, sample size=25.

Mean	-0.2499959
Variance	1.059915×10^{-7}
Minimum	-0.251498
Maximum	-0.2398885
Skewness	29.84311
Kurtosis	931.4003
D'Agostino Skewness	Reject Normality with prob. level 0.0
D'Agostino Kurtosis	Reject Normality with prob. level 0.0
D'Agostino Omnibus	Reject Normality with prob. level 0.0

Table 4.35 Statistical properties of the ML estimate for $b = -0.25$ over 1000 replications. Pressure =25.3 psi, censoring =40%, sample size=50.

Mean	-0.2499985
Variance	3.063113×10^{-8}
Minimum	-0.2508691
Maximum	-0.2445563
Skewness	29.97229
Kurtosis	937.4557
D'Agostino Skewness	Reject Normality with prob. level 0.0
D'Agostino Kurtosis	Reject Normality with prob. level 0.0
D'Agostino Omnibus	Reject Normality with prob. level 0.0

Table 4.36 Statistical properties of the ML estimate for $b = -0.25$ over 1000 replications. Pressure =25.3 psi, censoring =40%, sample size=100.

Mean	-0.2499993
Variance	1.788854×10^{-8}
Minimum	-0.2505954
Maximum	-0.2458177
Skewness	30.5179
Kurtosis	957.7864
D'Agostino Skewness	Reject Normality with prob. level 0.0
D'Agostino Kurtosis	Reject Normality with prob. level 0.0
D'Agostino Omnibus	Reject Normality with prob. level 0.0

Table 4.37 The Empirical variance-covariance matrix of the ML estimates over 1000 replications. Pressure =27.8 psi, censoring =10%, sample size=25.

$$\begin{bmatrix} 1.98 \times 10^{-3} & -4.30 \times 10^{-9} & 7.96 \times 10^{-6} \\ -4.30 \times 10^{-9} & 4.12 \times 10^{-13} & -7.00 \times 10^{-10} \\ 7.96 \times 10^{-6} & -7.00 \times 10^{-10} & 1.46 \times 10^{-6} \end{bmatrix}$$

Table 4.38 The Empirical variance-covariance matrix of the ML estimates over 1000 replications. Pressure =26.8 psi, censoring =20%, sample size=25.

$$\begin{bmatrix} 1.77 \times 10^{-3} & -1.40 \times 10^{-9} & -6.81 \times 10^{-7} \\ -1.40 \times 10^{-9} & 2.44 \times 10^{-13} & 1.00 \times 10^{-10} \\ -6.81 \times 10^{-7} & 1.00 \times 10^{-10} & 5.99 \times 10^{-8} \end{bmatrix}$$

Table 4.49 The Empirical variance-covariance matrix of the ML estimates over 1000 replications. Pressure =26.0 psi, censoring =30%, sample size=25.

$$\begin{bmatrix} 2.63 \times 10^{-3} & 1.78 \times 10^{-8} & 2.41 \times 10^{-5} \\ 1.78 \times 10^{-8} & 2.12 \times 10^{-12} & 2.90 \times 10^{-9} \\ 2.41 \times 10^{-5} & 2.90 \times 10^{-9} & 3.84 \times 10^{-6} \end{bmatrix}$$

Table 4.40 The Empirical variance-covariance matrix of the ML estimates over 1000 replications. Pressure =25.3 psi, censoring =40%, sample size=25.

$$\begin{bmatrix} 2.52 \times 10^{-3} & -6.00 \times 10^{-10} & -4.29 \times 10^{-7} \\ -6.00 \times 10^{-10} & 4.01 \times 10^{-14} & 2.32 \times 10^{-11} \\ -4.29 \times 10^{-7} & 2.32 \times 10^{-11} & 1.06 \times 10^{-7} \end{bmatrix}$$

Table 4.41 The Empirical variance-covariance matrix of the ML estimates over 1000 replications. Pressure =27.8 psi, censoring =10%, sample size=50.

$$\begin{bmatrix} 8.55 \times 10^{-4} & -2.20 \times 10^{-9} & 4.13 \times 10^{-6} \\ -2.20 \times 10^{-9} & 1.65 \times 10^{-13} & -3.00 \times 10^{-10} \\ 4.13 \times 10^{-6} & -3.00 \times 10^{-10} & 6.45 \times 10^{-7} \end{bmatrix}$$

Table 4.42 The Empirical variance-covariance matrix of the ML estimates over 1000 replications. Pressure =26.8 psi, censoring =20%, sample size=50.

$$\begin{bmatrix} 9.35 \times 10^{-4} & 1.00 \times 10^{-10} & 1.06 \times 10^{-7} \\ 1.00 \times 10^{-10} & 1.52 \times 10^{-14} & 2.03 \times 10^{-11} \\ 1.06 \times 10^{-7} & 2.03 \times 10^{-11} & 6.62 \times 10^{-8} \end{bmatrix}$$

Table 4.43 The Empirical variance-covariance matrix of the ML estimates over 1000 replications. Pressure =26.0 psi, censoring =30%, sample size=50.

$$\begin{bmatrix} 1.18 \times 10^{-3} & 5.00 \times 10^{-10} & 1.36 \times 10^{-6} \\ 5.00 \times 10^{-10} & 8.42 \times 10^{-14} & 1.00 \times 10^{-10} \\ 1.36 \times 10^{-6} & 1.00 \times 10^{-10} & 1.78 \times 10^{-7} \end{bmatrix}$$

Table 4.44 The Empirical variance-covariance matrix of the ML estimates over 1000 replications. Pressure =25.3 psi, censoring =40%, sample size=50.

$$\begin{bmatrix} 2.52 \times 10^{-3} & -6.00 \times 10^{-10} & -4.29 \times 10^{-7} \\ -6.00 \times 10^{-10} & 6.15 \times 10^{-13} & 1.08 \times 10^{-11} \\ -4.29 \times 10^{-7} & 1.08 \times 10^{-11} & 3.06 \times 10^{-7} \end{bmatrix}$$

Table 4.45 The Empirical variance-covariance matrix of the ML estimates over 1000 replications. Pressure =27.8 psi, censoring =10%, sample size=100.

$$\begin{bmatrix} 4.67 \times 10^{-4} & -4.00 \times 10^{-10} & 1.23 \times 10^{-6} \\ -4.00 \times 10^{-10} & 1.97 \times 10^{-14} & -6.15 \times 10^{-11} \\ 1.23 \times 10^{-6} & -6.15 \times 10^{-11} & 1.79 \times 10^{-7} \end{bmatrix}$$

Table 4.46 The Empirical variance-covariance matrix of the ML estimates over 1000 replications. Pressure =26.8 psi, censoring =20%, sample size=100.

$$\begin{bmatrix} 4.63 \times 10^{-4} & -7.00 \times 10^{-10} & 1.93 \times 10^{-6} \\ -7.00 \times 10^{-10} & 6.19 \times 10^{-14} & -1.00 \times 10^{-10} \\ 1.93 \times 10^{-6} & -1.00 \times 10^{-10} & 3.98 \times 10^{-7} \end{bmatrix}$$

Table 4.47 The Empirical variance-covariance matrix of the ML estimates over 1000 replications. Pressure =26.0 psi, censoring =30%, sample size=100.

$$\begin{bmatrix} 5.69 \times 10^{-4} & 7.00 \times 10^{-10} & 5.71 \times 10^{-7} \\ 7.00 \times 10^{-10} & 2.66 \times 10^{-13} & 2.00 \times 10^{-10} \\ 5.71 \times 10^{-7} & 2.00 \times 10^{-10} & 1.47 \times 10^{-7} \end{bmatrix}$$

Table 4.48 The Empirical variance-covariance matrix of the ML estimates over 1000 replications. Pressure =25.3 psi, censoring =40%, sample size=100.

$$\begin{bmatrix} 5.59 \times 10^{-4} & -2.00 \times 10^{-9} & -2.31 \times 10^{-7} \\ -2.00 \times 10^{-9} & 1.11 \times 10^{-12} & 1.00 \times 10^{-10} \\ -2.31 \times 10^{-7} & 1.00 \times 10^{-10} & 1.79 \times 10^{-8} \end{bmatrix}$$

Table 4.49 The Fisher variance-covariance matrix of the ML estimates over 1000 replications. Pressure =27.8 psi, censoring =10%, sample size=25.

$$\begin{bmatrix} 1.08 \times 10^{-3} & -2.99 \times 10^{-10} & 1.27 \times 10^{-7} \\ -2.99 \times 10^{-10} & 4.61 \times 10^{-18} & -4.79 \times 10^{-11} \\ 1.27 \times 10^{-7} & -4.79 \times 10^{-11} & 1.51 \times 10^{-12} \end{bmatrix}$$

Table 4.50 The Fisher variance-covariance matrix of the ML estimates over 1000 replications. Pressure =26.8 psi, censoring =20%, sample size=25.

$$\begin{bmatrix} 1.15 \times 10^{-3} & -3.36 \times 10^{-10} & -1.50 \times 10^{-7} \\ -3.36 \times 10^{-10} & 4.97 \times 10^{-18} & 5.83 \times 10^{-11} \\ -1.50 \times 10^{-7} & 5.83 \times 10^{-11} & 1.96 \times 10^{-12} \end{bmatrix}$$

Table 4.51 The Fisher variance-covariance matrix of the ML estimates over 1000 replications. Pressure =26.0 psi, censoring =30%, sample size=25.

$$\begin{bmatrix} 1.27 \times 10^{-3} & 2.76 \times 10^{-10} & 1.29 \times 10^{-7} \\ 2.76 \times 10^{-10} & 2.99 \times 10^{-18} & 5.12 \times 10^{-11} \\ 1.29 \times 10^{-7} & 5.12 \times 10^{-11} & 1.45 \times 10^{-12} \end{bmatrix}$$

Table 4.52 The Fisher variance-covariance matrix of the ML estimates over 1000 replications. Pressure =25.3 psi, censoring =40%, sample size=25.

$$\begin{bmatrix} 1.43 \times 10^{-3} & -2.93 \times 10^{-10} & -1.43 \times 10^{-7} \\ -2.93 \times 10^{-10} & 9.32 \times 10^{-18} & 5.85 \times 10^{-11} \\ -1.43 \times 10^{-7} & 5.85 \times 10^{-11} & 4.21 \times 10^{-12} \end{bmatrix}$$

Table 4.53 The Fisher variance-covariance matrix of the ML estimates over 1000 replications. Pressure =27.8 psi, censoring =10%, sample size=50.

$$\begin{bmatrix} 5.33 \times 10^{-4} & -1.71 \times 10^{-10} & 7.28 \times 10^{-8} \\ -1.71 \times 10^{-10} & 3.09 \times 10^{-18} & -2.74 \times 10^{-11} \\ 7.28 \times 10^{-8} & -2.74 \times 10^{-11} & 9.11 \times 10^{-13} \end{bmatrix}$$

Table 4.54 The Fisher variance-covariance matrix of the ML estimates over 1000 replications. Pressure =26.8 psi, censoring =20%, sample size=50.

$$\begin{bmatrix} 5.69 \times 10^{-4} & 2.03 \times 10^{-10} & 9.10 \times 10^{-8} \\ 2.03 \times 10^{-10} & 4.82 \times 10^{-18} & 3.54 \times 10^{-11} \\ 9.10 \times 10^{-8} & 3.54 \times 10^{-11} & 1.78 \times 10^{-12} \end{bmatrix}$$

Table 4.55 The Fisher variance-covariance matrix of the ML estimates over 1000 replications. Pressure =26.0 psi, censoring =30%, sample size=50.

$$\begin{bmatrix} 6.32 \times 10^{-4} & 1.59 \times 10^{-10} & 7.44 \times 10^{-8} \\ 1.59 \times 10^{-10} & 2.71 \times 10^{-18} & 2.98 \times 10^{-11} \\ 7.44 \times 10^{-8} & 2.98 \times 10^{-11} & 1.44 \times 10^{-12} \end{bmatrix}$$

Table 4.56 The Fisher variance-covariance matrix of the ML estimates over 1000 replications. Pressure =25.3 psi, censoring =40%, sample size=50.

$$\begin{bmatrix} 7.12 \times 10^{-4} & -1.34 \times 10^{-10} & -6.53 \times 10^{-8} \\ -1.34 \times 10^{-10} & 2.07 \times 10^{-18} & 2.68 \times 10^{-11} \\ -6.53 \times 10^{-8} & 2.68 \times 10^{-11} & 1.18 \times 10^{-12} \end{bmatrix}$$

Table 4.57 The Fisher variance-covariance matrix of the ML estimates over 1000 replications. Pressure =27.8 psi, censoring =10%, sample size=100.

$$\begin{bmatrix} 2.67 \times 10^{-4} & -9.60 \times 10^{-11} & 4.09 \times 10^{-8} \\ -9.60 \times 10^{-11} & 2.53 \times 10^{-18} & -1.54 \times 10^{-11} \\ 4.09 \times 10^{-8} & -1.54 \times 10^{-11} & 8.65 \times 10^{-13} \end{bmatrix}$$

Table 4.58 The Fisher variance-covariance matrix of the ML estimates over 1000 replications. Pressure=26.8 psi, censoring =20%, sample size=100.

$$\begin{bmatrix} 2.83 \times 10^{-4} & -1.07 \times 10^{-10} & 4.80 \times 10^{-8} \\ -1.07 \times 10^{-10} & 3.10 \times 10^{-18} & -1.88 \times 10^{-11} \\ 4.80 \times 10^{-8} & -1.88 \times 10^{-11} & 1.16 \times 10^{-12} \end{bmatrix}$$

Table 4.59 The Fisher variance-covariance matrix of the ML estimates over 1000 replications. Pressure =26.0 psi, censoring =30%, sample size=100.

$$\begin{bmatrix} 3.14 \times 10^{-4} & 8.48 \times 10^{-11} & 3.98 \times 10^{-8} \\ 8.48 \times 10^{-11} & 1.66 \times 10^{-18} & 1.59 \times 10^{-11} \\ 3.98 \times 10^{-8} & 1.59 \times 10^{-11} & 8.64 \times 10^{-13} \end{bmatrix}$$

Table 4.60 The Fisher variance-covariance matrix of the ML estimates over 1000 replications. Pressure =25.3 psi, censoring =40%, sample size=100.

$$\begin{bmatrix} 3.52 \times 10^{-4} & -6.69 \times 10^{-11} & -3.27 \times 10^{-8} \\ -6.69 \times 10^{-11} & 1.07 \times 10^{-18} & 1.35 \times 10^{-11} \\ -3.27 \times 10^{-8} & 1.35 \times 10^{-11} & 6.37 \times 10^{-13} \end{bmatrix}$$

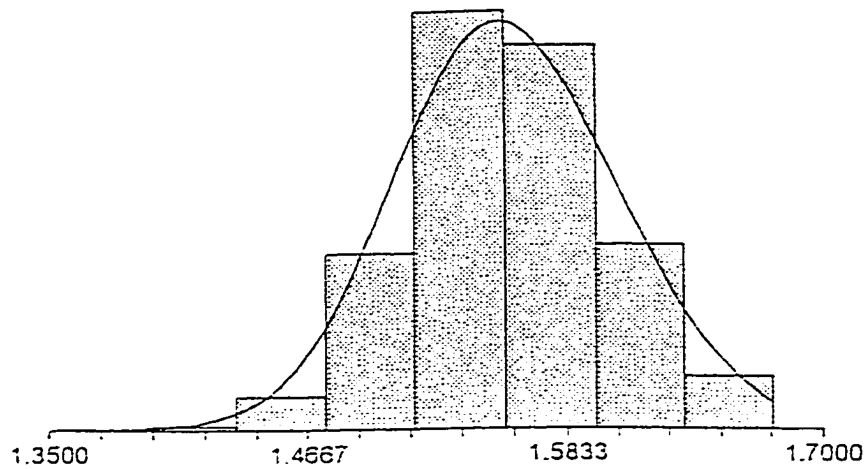

Fig. 4.1 Relative frequency histogram of the ML estimate for gamma=1.5, pressure=27.8, censoring=10%, sample size=25.

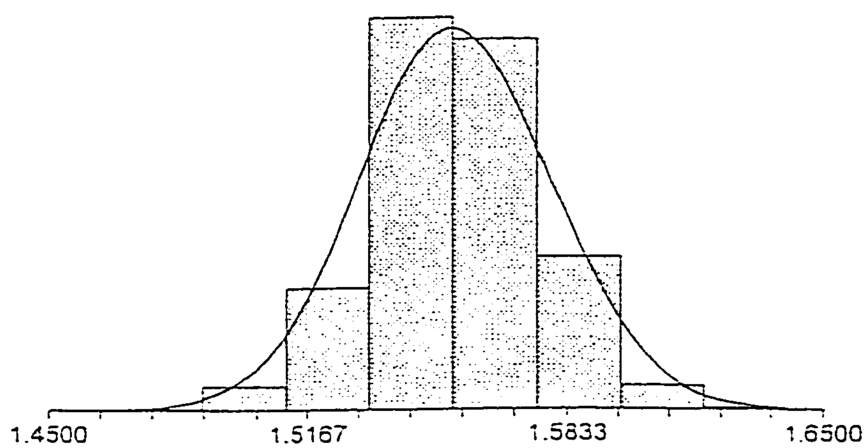

Fig. 4.2 Relative frequency histogram of the ML estimate for gamma=1.5, pressure=27.8, censoring=10%, sample size=100.

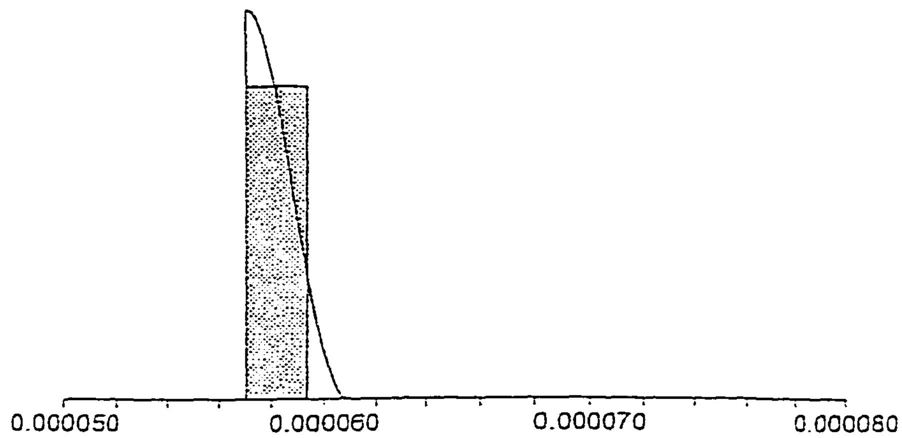

Fig. 4.3 Relative frequency histogram of the ML estimate for lambda=0.000057, pressure=27.8, censoring=10%, sample size=25.

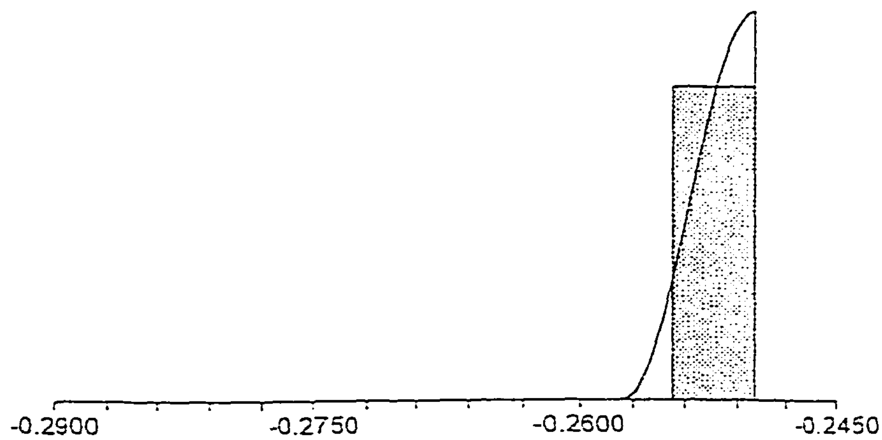

Fig. 4.4 Relative frequency histogram of the ML estimate for b=-0.25, pressure=27.8, censoring=10%, sample size=25.

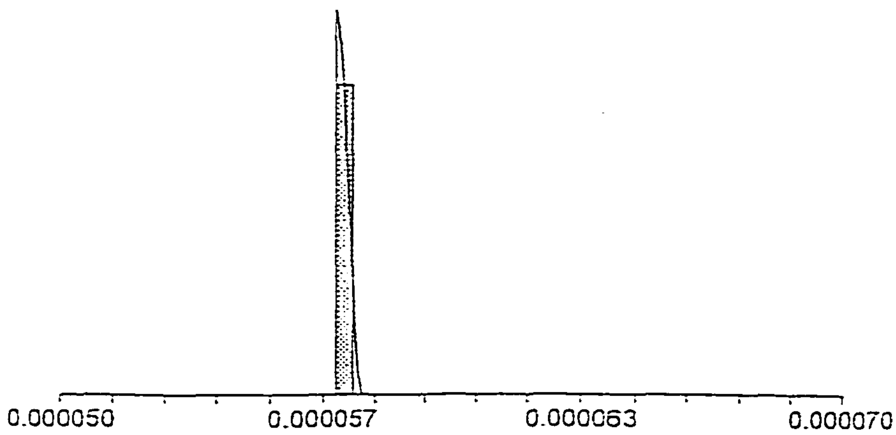

Fig. 4.5 Relative frequency histogram of the ML estimate for lambda=0.000057, pressure=27.8, censoring=10%, sample size=100.

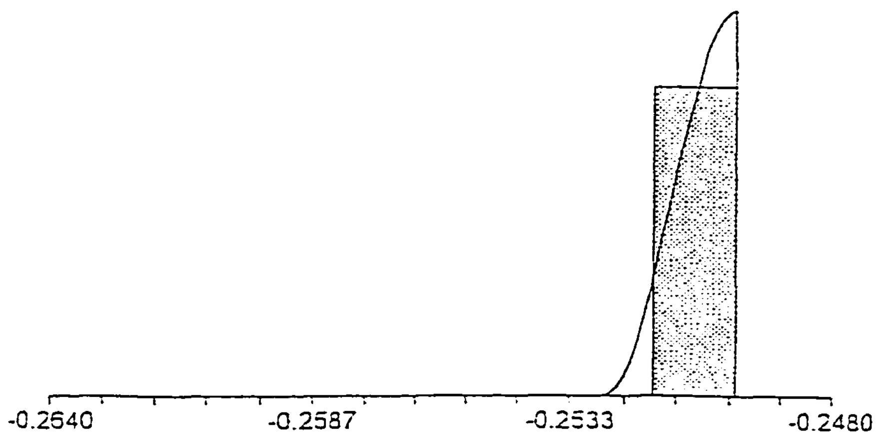

Fig. 4.6 Relative frequency histogram of the ML estimate for b=-0.25, pressure=27.8, censoring=10%, sample size=100.

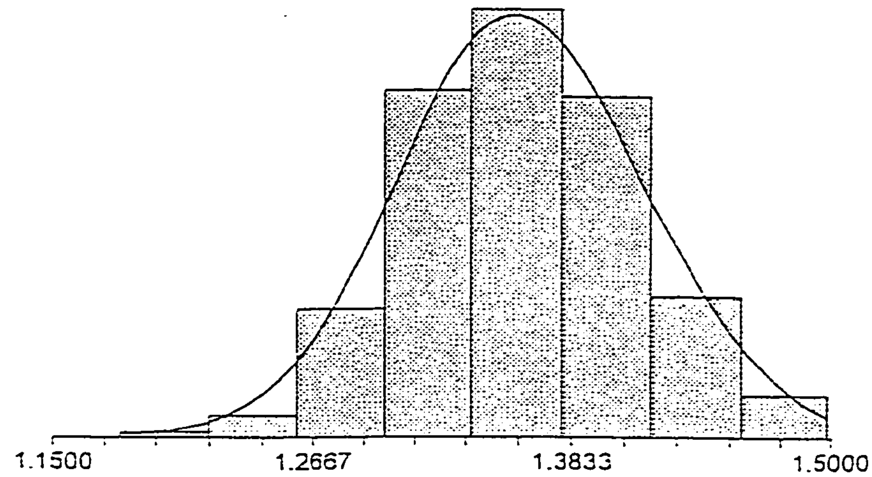

Fig. 4.7 Relative frequency histogram of the ML estimate for gamma=1.5, pressure=25.3, censoring=40%, sample size=25.

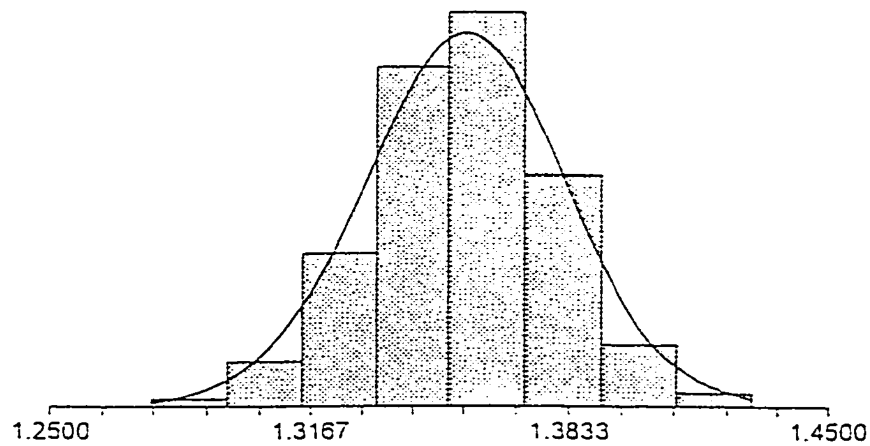

Fig. 4.8 Relative frequency histogram of the ML estimate for gamma=1.5, pressure=25.3, censoring=40%, sample size=100.

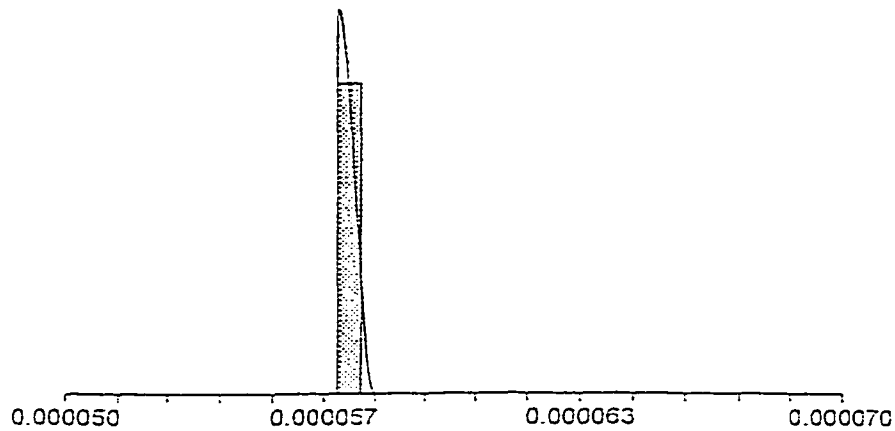

Fig. 4.9 Relative frequency histogram of the ML estimate for lambda=0.000057, pressure=25.3, censoring=40%, sample size=25.

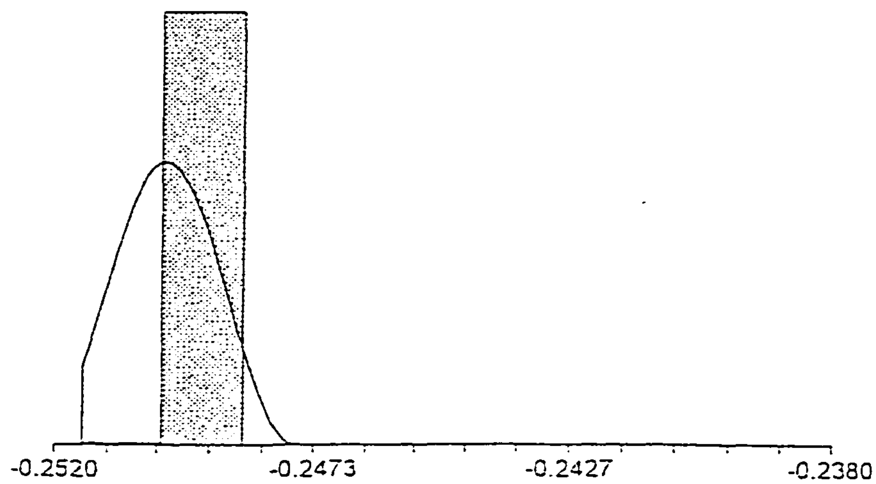

Fig. 4.10 Relative frequency histogram of the ML estimate for b=-0.25, pressure=25.3, censoring=40%, sample size=25.

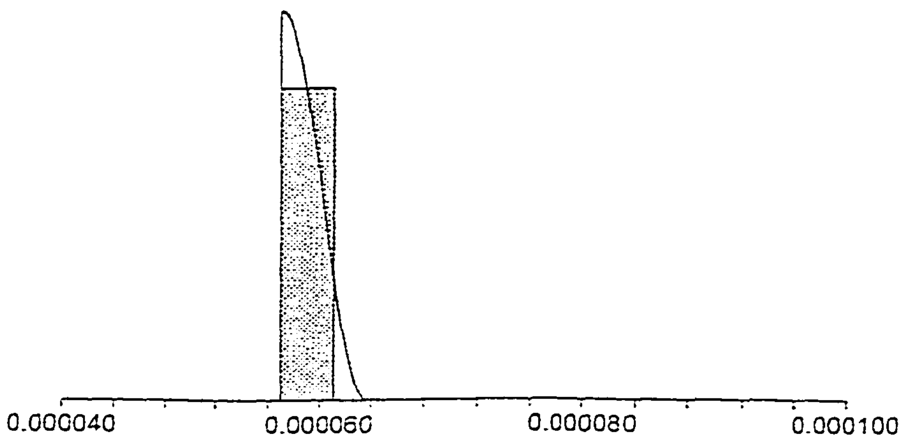

Fig. 4.11 Relative frequency histogram of the ML estimate for lambda=0.000057, pressure=25.3, censoring=40%, sample size=100.

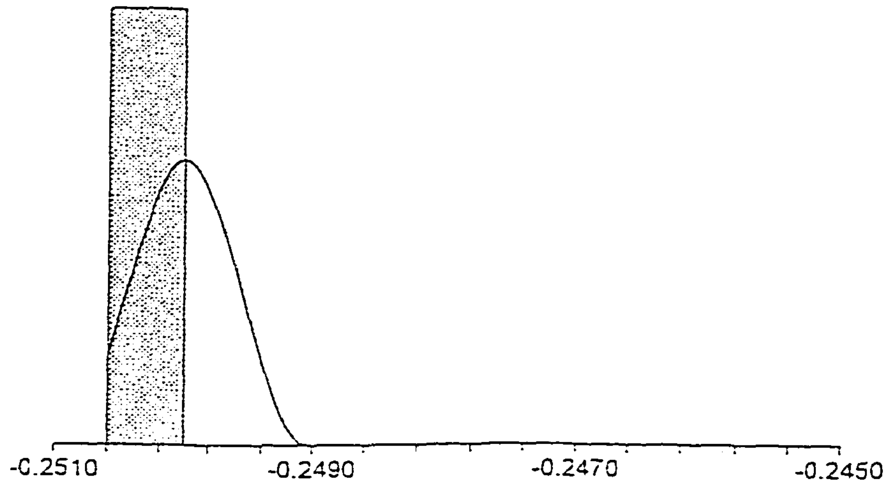

Fig. 4.12 Relative frequency histogram of the ML estimate for b=-0.25, pressure=25.3, censoring=40%, sample size=100.

CHAPTER 5

ACCELERATED LIFETIME UNDER VARIABLE PRESSURE

5.1 Theory

It is well known that the external hydrostatic pressure acting on a liner is not constant. In this chapter, variable pressure is considered.

From Eq. (3.1), the log-likelihood function is

$$L = \sum_{i=1}^{r} \ln f_{0i} + \sum_{i=r+1}^{n} \ln (1 - F_{0i}). \tag{5.1}$$

Using the Weibull distribution, and the transformation

$$t = t_0 e^{bx(t)}, \tag{5.2}$$

one obtains

$$\begin{aligned} L &= \sum_{i=1}^{r} \ln [e^{-bx(t_i)} \lambda\gamma (e^{-bx(t_i)} t_i)^{\gamma-1} e^{-\int_0^{t_i} e^{-bx(\tau_i)} \lambda\gamma (e^{-bx(\tau_i)} \tau_i)^{\gamma-1} d\tau_i}] \\ &+ \sum_{i=r+1}^{n} \ln [e^{-\int_0^{t_i} e^{-bx(\tau_i)} \lambda\gamma (e^{-bx(\tau_i)} \tau_i)^{\gamma-1} d\tau_i}] \\ &= \sum_{i=1}^{r} [-bx(t_i) + \ln\gamma + \ln\lambda + (\gamma-1)(-bx(t_i) + \ln t_i) \end{aligned}$$

$$
\begin{aligned}
&- \int_0^{t_i} \gamma\lambda e^{-bx(\tau_i)}(e^{-bx(\tau_i)}\tau_i)^{\gamma-1} d\tau_i] \\
&- \sum_{i=r+1}^{n} \int_0^{t_i} \gamma\lambda e^{-bx(\tau_i)}(e^{-bx(\tau_i)}\tau_i)^{\gamma-1} d\tau_i.
\end{aligned} \tag{5.3}
$$

Upon simplifying Eq. (5.3), one obtains that

$$
\begin{aligned}
L &= \sum_{i=1}^{r}[-bx(t_i) + \ln\gamma + \ln\lambda + (\gamma-1)(-bx(t_i) + \ln t_i) \\
&- \int_0^{t_i} \gamma\lambda\tau_i^{\gamma-1} e^{-\gamma bx(\tau_i)} d\tau_i] \\
&- \sum_{i=r+1}^{n} \int_0^{t_i} \gamma\lambda\tau_i^{\gamma-1} e^{-\gamma bx(\tau_i)} d\tau_i.
\end{aligned} \tag{5.4}
$$

Replace $x(t)$ by $c + a\sin\frac{\pi t}{6}$ gives

$$
\begin{aligned}
L &= \sum_{i=1}^{r}[-b(c + a\sin\frac{\pi t_i}{6}) + \ln\gamma + \ln\lambda + (\gamma-1)(-b(c + a\sin\frac{\pi t_i}{6}) + \ln t_i) \\
&- \int_0^{t_i} \gamma\lambda\tau_i^{\gamma-1} e^{-b\gamma(c+a\sin\frac{\pi\tau_i}{6})} d\tau_i] \\
&- \sum_{i=r+1}^{n} \int_0^{t_i} \gamma\lambda\tau_i^{\gamma-1} e^{-b\gamma(c+a\sin\frac{\pi\tau_i}{6})} d\tau_i].
\end{aligned} \tag{5.5}
$$

From Eq. (5.5), one obtains

$$
\begin{aligned}
\frac{\partial L}{\partial\gamma} &= \sum_{i=1}^{r}[\frac{1}{\gamma} - b(c + a\sin\frac{\pi t_i}{6}) + \ln(t_i) - \int_0^{t_i} \lambda\tau_i^{\gamma-1} e^{-b\gamma(c+a\sin\frac{\pi\tau_i}{6})} d\tau_i \\
&- \int_0^{t_i} \lambda\gamma\tau_i^{\gamma-1} \ln(\tau_i) e^{-b\gamma(c+a\sin\frac{\pi\tau_i}{6})} d\tau_i \\
&+ \int_0^{t_i} \lambda\gamma b(c + a\sin\frac{\pi\tau_i}{6})\tau_i^{\gamma-1} e^{-b\gamma(c+a\sin\frac{\pi\tau_i}{6})} d\tau_i] \\
&+ \sum_{i=r+1}^{n} [- \int_0^{t_i} \lambda\tau_i^{\gamma-1} e^{-b\gamma(c+a\sin\frac{\pi\tau_i}{6})} d\tau_i
\end{aligned}
$$

$$
\begin{aligned}
&- \int_0^{t_i} \lambda\gamma\tau_i^{\gamma-1}\ln(\tau_i)e^{-b\gamma(c+a\sin\frac{\pi\tau_i}{6})}d\tau_i \\
&+ \int_0^{t_i} \lambda\gamma b(c+a\sin\frac{\pi\tau_i}{6})\tau^{\gamma-1}e^{-b\gamma(c+a\sin\frac{\pi\tau_i}{6})}d\tau_i], \qquad (5.6)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial L}{\partial\lambda} &= \sum_{i=1}^{r}[\frac{1}{\lambda} - \int_0^{t_i}\gamma\tau_i^{\gamma-1}e^{-b\gamma(c+a\sin\frac{\pi\tau_i}{6})}d\tau_i] \\
&- \sum_{i=r+1}^{n}[\int_0^{t_i}\gamma\tau_i^{\gamma-1}e^{-b\gamma(c+a\sin\frac{\pi\tau_i}{6})}d\tau_i], \qquad (5.7)
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\partial L}{\partial b} &= \sum_{i=1}^{r}[-(c+a\sin\frac{\pi t_i}{6}) - (\gamma-1)(c+a\sin\frac{\pi t_i}{6}) \\
&+ \int_0^{t_i}\gamma^2\lambda(c+a\sin\frac{\pi\tau_i}{6})\tau_i^{\gamma-1}e^{-b\gamma(c+a\sin\frac{\pi\tau_i}{6})}d\tau_i] \\
&+ \sum_{i=r+1}^{n}[\int_0^{t_i}\gamma^2\lambda(c+a\sin\frac{\pi\tau_i}{6})\tau_i^{\gamma-1}e^{-b\gamma(c+a\sin\frac{\pi\tau_i}{6})}d\tau_i]. \qquad (5.8)
\end{aligned}
$$

Now, let $f_1 = \frac{\partial L}{\partial\gamma}$, $f_2 = \frac{\partial L}{\partial\lambda}$, and $f_3 = \frac{\partial L}{\partial b}$ and set $f_1 = f_2 = f_3 = 0$. Equations (5.6), (5.7), and (5.8) can be solved using the Newton-Raphson technique discussed in chapter 3. The Jacobian matrix is

$$
\mathbf{I}(t,\gamma,\lambda,b) = -\begin{bmatrix} \frac{\partial f_1}{\partial\gamma} & \frac{\partial f_1}{\partial\lambda} & \frac{\partial f_1}{\partial b} \\ \frac{\partial f_2}{\partial\gamma} & \frac{\partial f_2}{\partial\lambda} & \frac{\partial f_2}{\partial b} \\ \frac{\partial f_3}{\partial\gamma} & \frac{\partial f_3}{\partial\lambda} & \frac{\partial f_3}{\partial b} \end{bmatrix}, \qquad (5.9)
$$

where

$$\frac{\partial f_1}{\partial \lambda} = \frac{\partial f_2}{\partial \gamma}, \tag{5.10}$$

$$\frac{\partial f_1}{\partial b} = \frac{\partial f_3}{\partial \gamma}, \tag{5.11}$$

and

$$\frac{\partial f_2}{\partial b} = \frac{\partial f_3}{\partial \lambda}. \tag{5.12}$$

From Eq. (5.6), it is seen that

$$\begin{aligned}
\frac{\partial f_1}{\partial \gamma} &= \sum_{i=1}^{r}[\frac{-1}{\gamma^2} - \int_0^{t_i} \lambda\tau_i^{\gamma-1}\ln(\tau_i)e^{-b\gamma(c+a\sin\frac{\pi\tau_i}{6})}d\tau_i \\
&+ \int_0^{t_i} \lambda b(c+a\sin\frac{\pi\tau_i}{6})\tau_i^{\gamma-1}e^{-b\gamma(c+a\sin\frac{\pi\tau_i}{6})}d\tau_i \\
&- \int_0^{t_i} \lambda\tau_i^{\gamma-1}\ln(\tau_i)e^{-b\gamma(c+a\sin\frac{\pi\tau_i}{6})}d\tau_i \\
&- \int_0^{t_i} \gamma\lambda\tau_i^{\gamma-1}[\ln(\tau_i)]^2e^{-b\gamma(c+a\sin\frac{\pi\tau_i}{6})}d\tau_i \\
&+ \int_0^{t_i} \gamma\lambda b(c+a\sin\frac{\pi\tau_i}{6})\tau_i^{\gamma-1}\ln(\tau_i)e^{-b\gamma(c+a\sin\frac{\pi\tau_i}{6})}d\tau_i \\
&+ \int_0^{t_i} \lambda b(c+a\sin\frac{\pi\tau_i}{6})\tau_i^{\gamma-1}e^{-b\gamma(c+a\sin\frac{\pi\tau_i}{6})}d\tau_i \\
&+ \int_0^{t_i} \gamma\lambda b(c+a\sin\frac{\pi\tau_i}{6})\tau_i^{\gamma-1}\ln(\tau_i)e^{-b\gamma(c+a\sin\frac{\pi\tau_i}{6})}d\tau_i \\
&- \int_0^{t_i} \gamma\lambda[b(c+a\sin\frac{\pi\tau_i}{6})]^2\tau_i^{\gamma-1}e^{-b\gamma(c+a\sin\frac{\pi\tau_i}{6})}d\tau_i] \\
&+ \sum_{i=r+1}^{n}[-\int_0^{t_i} \lambda\tau_i^{\gamma-1}\ln(\tau_i)e^{-b\gamma(c+a\sin\frac{\pi\tau_i}{6})}d\tau_i \\
&+ \int_0^{t_i} \lambda b(c+a\sin\frac{\pi\tau_i}{6})\tau_i^{\gamma-1}e^{-b\gamma(c+a\sin\frac{\pi\tau_i}{6})}d\tau_i \\
&- \int_0^{t_i} \lambda\tau_i^{\gamma-1}\ln(\tau_i)e^{-b\gamma(c+a\sin\frac{\pi\tau_i}{6})}d\tau_i
\end{aligned}$$

$$
\begin{aligned}
&- \int_0^{t_i} \gamma\lambda\tau_i^{\gamma-1}[\ln(\tau_i)]^2 e^{-b\gamma(c+a\sin\frac{\pi\tau_i}{6})} d\tau_i \\
&+ \int_0^{t_i} \gamma\lambda b(c+a\sin\frac{\pi\tau_i}{6})\tau_i^{\gamma-1}\ln(\tau_i) e^{-b\gamma(c+a\sin\frac{\pi\tau_i}{6})} d\tau_i \\
&+ \int_0^{t_i} \lambda b(c+a\sin\frac{\pi\tau_i}{6})\tau_i^{\gamma-1} e^{-b\gamma(c+a\sin\frac{\pi\tau_i}{6})} d\tau_i \\
&+ \int_0^{t_i} \gamma\lambda b(c+a\sin\frac{\pi\tau_i}{6})\tau_i^{\gamma-1}\ln(\tau_i) e^{-b\gamma(c+a\sin\frac{\pi\tau_i}{6})} d\tau_i \\
&- \int_0^{t_i} \gamma\lambda[b(c+a\sin\frac{\pi\tau_i}{6})]^2\tau_i^{\gamma-1} e^{-b\gamma(c+a\sin\frac{\pi\tau_i}{6})} d\tau_i]. \qquad (5.13)
\end{aligned}
$$

Upon simplification, one has that

$$
\begin{aligned}
\frac{\partial f_1}{\partial\gamma} &= \sum_{i=1}^{r}\frac{-1}{\gamma^2} - \sum_{i=1}^{n}[\int_0^{t_i} 2\lambda\tau_i^{\gamma-1}\ln(\tau_i) e^{-b\gamma(c+a\sin\frac{\pi\tau_i}{6})} d\tau_i \\
&+ \int_0^{t_i} 2\lambda b(c+a\sin\frac{\pi\tau_i}{6})\tau_i^{\gamma-1} e^{-b\gamma(c+a\sin\frac{\pi\tau_i}{6})} d\tau_i \\
&+ \int_0^{t_i} 2\gamma\lambda b(c+a\sin\frac{\pi\tau_i}{6})\tau_i^{\gamma-1}\ln(\tau_i) e^{-b\gamma(c+a\sin\frac{\pi\tau_i}{6})} d\tau_i \\
&- \int_0^{t_i} \gamma\lambda\tau_i^{\gamma-1}[\ln(\tau_i)]^2 e^{-b\gamma(c+a\sin\frac{\pi\tau_i}{6})} d\tau_i \\
&- \int_0^{t_i} \gamma\lambda[b(c+a\sin\frac{\pi\tau_i}{6})]^2\tau_i^{\gamma-1} e^{-b\gamma(c+a\sin\frac{\pi\tau_i}{6})} d\tau_i]. \qquad (5.14)
\end{aligned}
$$

Also, from Eq. (5.6)

$$
\begin{aligned}
\frac{\partial f_1}{\partial\lambda} &= \sum_{i=1}^{r}[-\int_0^{t_i} \tau_i^{\gamma-1} e^{-b\gamma(c+a\sin\frac{\pi\tau_i}{6})} d\tau_i \\
&- \int_0^{t_i} \gamma\tau_i^{\gamma-1}\ln(\tau_i) e^{-b\gamma(c+a\sin\frac{\pi\tau_i}{6})} d\tau_i \\
&+ \int_0^{t_i} \gamma b(c+a\sin\frac{\pi\tau_i}{6})\tau_i^{\gamma-1} e^{-b\gamma(c+a\sin\frac{\pi\tau_i}{6})} d\tau_i] \\
&+ \sum_{i=r+1}^{n}[-\int_0^{t_i} \tau_i^{\gamma-1} e^{-b\gamma(c+a\sin\frac{\pi\tau_i}{6})} d\tau_i \\
&- \int_0^{t_i} \gamma\tau_i^{\gamma-1}\ln(\tau_i) e^{-b\gamma(c+a\sin\frac{\pi\tau_i}{6})} d\tau_i
\end{aligned}
$$

$$+ \int_0^{t_i} \gamma b(c + a\sin\frac{\pi\tau_i}{6})\tau_i^{\gamma-1} e^{-b\gamma(c+a\sin\frac{\pi\tau_i}{6})} d\tau_i]. \quad (5.15)$$

or

$$\begin{aligned} \frac{\partial f_1}{\partial\lambda} &= \sum_{i=1}^{n}[-\int_0^{t_i} \tau_i^{\gamma-1} e^{-b\gamma(c+a\sin\frac{\pi\tau_i}{6})} d\tau_i \\ &- \int_0^{t_i} \gamma\tau_i^{\gamma-1}\ln(\tau_i) e^{-b\gamma(c+a\sin\frac{\pi\tau_i}{6})} d\tau_i \\ &+ \int_0^{t_i} \gamma b(c + a\sin\frac{\pi\tau_i}{6})\tau_i^{\gamma-1} e^{-b\gamma(c+a\sin\frac{\pi\tau_i}{6})} d\tau_i]. \end{aligned} \quad (5.16)$$

Likewise,

$$\begin{aligned} \frac{\partial f_1}{\partial b} &= \sum_{i=1}^{r}[-(c + a\sin\frac{\pi t_i}{6}) + \int_0^{t_i} \gamma\lambda(c + a\sin\frac{\pi\tau_i}{6})\tau_i^{\gamma-1} e^{-b\gamma(c+a\sin\frac{\pi\tau_i}{6})} d\tau_i \\ &+ \int_0^{t_i} \gamma^2\lambda(c + a\sin\frac{\pi\tau_i}{6})\tau_i^{\gamma-1}\ln(\tau_i) e^{-b\gamma(c+a\sin\frac{\pi\tau_i}{6})} d\tau_i \\ &+ \int_0^{t_i} \gamma\lambda(c + a\sin\frac{\pi\tau_i}{6})\tau_i^{\gamma-1} e^{-b\gamma(c+a\sin\frac{\pi\tau_i}{6})} d\tau_i \\ &- \int_0^{t_i} \gamma^2\lambda b(c + a\sin\frac{\pi\tau_i}{6})^2\tau_i^{\gamma-1} e^{-b\gamma(c+a\sin\frac{\pi\tau_i}{6})} d\tau_i] \\ &+ \sum_{i=r+1}^{n}[\int_0^{t_i} \gamma\lambda(c + a\sin\frac{\pi\tau_i}{6})\tau_i^{\gamma-1} e^{-b\gamma(c+a\sin\frac{\pi\tau_i}{6})} d\tau_i \\ &+ \int_0^{t_i} \gamma^2\lambda(c + a\sin\frac{\pi\tau_i}{6})\tau_i^{\gamma-1}\ln(\tau_i) e^{-b\gamma(c+a\sin\frac{\pi\tau_i}{6})} d\tau_i \\ &+ \int_0^{t_i} \gamma\lambda(c + a\sin\frac{\pi\tau_i}{6})\tau_i^{\gamma-1} e^{-b\gamma(c+a\sin\frac{\pi\tau_i}{6})} d\tau_i \\ &- \int_0^{t_i} \gamma^2\lambda b(c + a\sin\frac{\pi\tau_i}{6})^2\tau_i^{\gamma-1} e^{-b\gamma(c+a\sin\frac{\pi\tau_i}{6})} d\tau_i]. \end{aligned} \quad (5.17)$$

Combing terms and simplifying, one sees that

$$\begin{aligned}
\frac{\partial f_1}{\partial b} &= \sum_{i=1}^{r}[-(c+a\sin\frac{\pi t_i}{6})] + \sum_{i=1}^{n}[\int_0^{t_i} 2\gamma\lambda(c+a\sin\frac{\pi\tau_i}{6})\tau_i^{\gamma-1}e^{-b\gamma(c+a\sin\frac{\pi\tau_i}{6})}d\tau_i \\
&+ \int_0^{t_i}\gamma^2\lambda(c+a\sin\frac{\pi\tau_i}{6})\tau_i^{\gamma-1}\ln(\tau_i)e^{-b\gamma(c+a\sin\frac{\pi\tau_i}{6})}d\tau_i \\
&- \int_0^{t_i}\gamma^2\lambda b(c+a\sin\frac{\pi\tau_i}{6})^2\tau_i^{\gamma-1}e^{-b\gamma(c+a\sin\frac{\pi\tau_i}{6})}d\tau_i]. \qquad (5.18)
\end{aligned}$$

Also,

$$\begin{aligned}
\frac{\partial f_2}{\partial \gamma} &= \sum_{i=1}^{r}[-\int_0^{t_i}\tau_i^{\gamma-1}e^{-b\gamma(c+a\sin\frac{\pi\tau_i}{6})}d\tau_i \\
&- \int_0^{t_i}\gamma\tau_i^{\gamma-1}\ln(\tau_i)e^{-b\gamma(c+a\sin\frac{\pi\tau_i}{6})}d\tau_i \\
&+ \int_0^{t_i}\gamma b(c+a\sin\frac{\pi\tau_i}{6})\tau_i^{\gamma-1}e^{-b\gamma(c+a\sin\frac{\pi\tau_i}{6})}d\tau_i] \\
&+ \sum_{i=r+1}^{n}[-\int_0^{t_i}\tau_i^{\gamma-1}e^{-b\gamma(c+a\sin\frac{\pi\tau_i}{6})}d\tau_i \\
&- \int_0^{t_i}\gamma\tau_i^{\gamma-1}\ln(\tau_i)e^{-b\gamma(c+a\sin\frac{\pi\tau_i}{6})}d\tau_i \\
&+ \int_0^{t_i}\gamma b(c+a\sin\frac{\pi\tau_i}{6})\tau_i^{\gamma-1}e^{-b\gamma(c+a\sin\frac{\pi\tau_i}{6})}d\tau_i], \qquad (5.19)
\end{aligned}$$

which reduces to

$$\begin{aligned}
\frac{\partial f_2}{\partial \gamma} &= \sum_{i=1}^{n}[-\int_0^{t_i}\tau_i^{\gamma-1}e^{-b\gamma(c+a\sin\frac{\pi\tau_i}{6})}d\tau_i \\
&- \int_0^{t_i}\gamma\tau_i^{\gamma-1}\ln(\tau_i)e^{-b\gamma(c+a\sin\frac{\pi\tau_i}{6})}d\tau_i \\
&+ \int_0^{t_i}\gamma b(c+a\sin\frac{\pi\tau_i}{6})\tau_i^{\gamma-1}e^{-b\gamma(c+a\sin\frac{\pi\tau_i}{6})}d\tau_i]. \qquad (5.20)
\end{aligned}$$

From Eqs. (5.20) and (5.16) one sees that

$$\frac{\partial f_2}{\partial \gamma} = \frac{\partial f_1}{\partial \lambda}. \tag{5.21}$$

For the rest of the derivatives in the Jacobian matrix (5.9), it is seen that

$$\frac{\partial f_2}{\partial \lambda} = \sum_{i=1}^{r} \frac{-1}{\lambda^2} \tag{5.22}$$

and,

$$\begin{aligned}
\frac{\partial f_2}{\partial b} &= \sum_{i=1}^{r}[\int_0^{t_i} \gamma^2(c + a\sin\frac{\pi\tau_i}{6})\tau_i^{\gamma-1}e^{-b\gamma(c+a\sin\frac{\pi\tau_i}{6})}d\tau_i] \\
&+ \sum_{i=r+1}^{n}[\int_0^{t_i} \gamma^2(c + a\sin\frac{\pi\tau_i}{6})\tau_i^{\gamma-1}e^{-b\gamma(c+a\sin\frac{\pi\tau_i}{6})}d\tau_i] \\
&= \sum_{i=1}^{n}[\int_0^{t_i} \gamma^2(c + a\sin\frac{\pi\tau_i}{6})\tau_i^{\gamma-1}e^{-b\gamma(c+a\sin\frac{\pi\tau_i}{6})}d\tau_i].
\end{aligned} \tag{5.23}$$

Finally, one needs to calculate the partial derivatives of f_3 with respect to γ, λ, and b. These derivatives are given below

$$\begin{aligned}
\frac{\partial f_3}{\partial \gamma} &= \sum_{i=1}^{r}[-(c + a\sin\frac{\pi t_i}{6}) + \int_0^{t_i} 2\gamma\lambda(c + a\sin\frac{\pi\tau_i}{6})\tau_i^{\gamma-1}e^{-b\gamma(c+a\sin\frac{\pi\tau_i}{6})}d\tau_i \\
&+ \int_0^{t_i} \gamma^2\lambda(c + a\sin\frac{\pi\tau_i}{6})\tau_i^{\gamma-1}\ln(\tau_i)e^{-b\gamma(c+a\sin\frac{\pi\tau_i}{6})}d\tau_i \\
&- \int_0^{t_i} \gamma^2\lambda b(c + a\sin\frac{\pi\tau_i}{6})^2\tau_i^{\gamma-1}e^{-b\gamma(c+a\sin\frac{\pi\tau_i}{6})}d\tau_i] \\
&+ \sum_{i=r+1}^{n}[\int_0^{t_i} 2\gamma\lambda(c + a\sin\frac{\pi\tau_i}{6})\tau_i^{\gamma-1}e^{-b\gamma(c+a\sin\frac{\pi\tau_i}{6})}d\tau_i \\
&+ \int_0^{t_i} \gamma^2\lambda(c + a\sin\frac{\pi\tau_i}{6})\tau_i^{\gamma-1}\ln(\tau_i)e^{-b\gamma(c+a\sin\frac{\pi\tau_i}{6})}d\tau_i \\
&- \int_0^{t_i} \gamma^2\lambda b(c + a\sin\frac{\pi\tau_i}{6})^2\tau_i^{\gamma-1}e^{-b\gamma(c+a\sin\frac{\pi\tau_i}{6})}d\tau_i],
\end{aligned} \tag{5.24}$$

which reduces to

$$\begin{aligned}\frac{\partial f_3}{\partial \gamma} &= \sum_{i=1}^{r}[-(c+a\sin\frac{\pi t_i}{6})] + \sum_{i=1}^{n}[\int_0^{t_i} 2\gamma\lambda(c+a\sin\frac{\pi\tau_i}{6})\tau_i^{\gamma-1}e^{-b\gamma(c+a\sin\frac{\pi\tau_i}{6})}d\tau_i \\ &+ \int_0^{t_i}\gamma^2\lambda(c+a\sin\frac{\pi\tau_i}{6})\tau_i^{\gamma-1}\ln(\tau_i)e^{-b\gamma(c+a\sin\frac{\pi\tau_i}{6})}d\tau_i \\ &- \int_0^{t_i}\gamma^2\lambda b(c+a\sin\frac{\pi\tau_i}{6})^2\tau_i^{\gamma-1}e^{-b\gamma(c+a\sin\frac{\pi\tau_i}{6})}d\tau_i]. \qquad (5.25)\end{aligned}$$

Hence, from Eqs. (5.25) and (5.18) it is seen that

$$\frac{\partial f_3}{\partial \gamma} = \frac{\partial f_1}{\partial b}. \qquad (5.26)$$

Also,

$$\begin{aligned}\frac{\partial f_3}{\partial \lambda} &= \sum_{i=1}^{r}[\int_0^{t_i}\gamma^2(c+a\sin\frac{\pi\tau_i}{6})\tau_i^{\gamma-1}e^{-b\gamma(c+a\sin\frac{\pi\tau_i}{6})}d\tau_i] \\ &+ \sum_{i=r+1}^{n}[\int_0^{t_i}\gamma^2(c+a\sin\frac{\pi\tau_i}{6})\tau_i^{\gamma-1}e^{-b\gamma(c+a\sin\frac{\pi\tau_i}{6})}d\tau_i] \\ &= \sum_{i=1}^{n}[\int_0^{t_i}\gamma^2(c+a\sin\frac{\pi\tau_i}{6})\tau_i^{\gamma-1}e^{-b\gamma(c+a\sin\frac{\pi\tau_i}{6})}d\tau_i]. \qquad (5.27)\end{aligned}$$

Hence, from Eqs. (5.27) and (5.23) one sees that

$$\frac{\partial f_3}{\partial \lambda} = \frac{\partial f_2}{\partial b}. \qquad (5.28)$$

Finally,

$$\begin{aligned}\frac{\partial f_3}{\partial b} &= \sum_{i=1}^{r}[-\int_0^{t_i}\gamma^3\lambda(c+a\sin\frac{\pi\tau_i}{6})^2\tau_i^{\gamma-1}e^{-b\gamma(c+a\sin\frac{\pi\tau_i}{6})}d\tau_i] \\ &+ \sum_{i=r+1}^{n}[-\int_0^{t_i}\gamma^3\lambda(c+a\sin\frac{\pi\tau_i}{6})^2\tau_i^{\gamma-1}e^{-b\gamma(c+a\sin\frac{\pi\tau_i}{6})}d\tau_i] \\ &= \sum_{i=1}^{n}[-\int_0^{t_i}\gamma^3\lambda(c+a\sin\frac{\pi\tau_i}{6})^2\tau_i^{\gamma-1}e^{-b\gamma(c+a\sin\frac{\pi\tau_i}{6})}d\tau_i]. \end{aligned} \quad (5.29)$$

5.2 Results From Simulation

Results in Tables 5.1, 5.2, and 5.3 for variable pressure (where pressure was expressed as a sine function $x(t) = c + a\sin\frac{\pi t}{6}$ with $c = 27.8$ and $a = 5$) and 10% censoring show that the ML estimate underestimates the parameter value of $\gamma = 1.5$. This bias, while significant, is not substantial (Mean estimate=1.34). It is clear however that unlike the constant pressure case the ML estimate does not have a normal distribution even for sample size 100. The estimates for lambda and b from Tables 5.4 to 5.9 are not biased in the sense that the mean estimate over 1000 replications is close to its parameter value. However, the ML estimates are not normally distributed. Figures 5.1 to 5.6 show the shape of the empirical distributions for the gamma, lambda, and b ML estimates and their deviations from normality for sample sizes 25 and 100.

For 20%, 30% and 40% censoring, it is clear from results in Tables 5.10 to 5.36 that the ML estimates for gamma, lambda and b are not normally distributed as assumed from theory. Also, it is seen that a sample size of even 100 is not sufficient for the asymptotic normal distribution of an ML estimate to hold. Figures 5.7 to 5.12 present the empirical distributions of these ML estimates which also show substantial deviations from normality. On the other hand, the ML estimate of gamma is slightly biased downward while the ML estimates of lambda and b are not biased.

Tables 5.37 to 5.48 present the variance-covariance matrices for different variable pressures, censoring, and sample sizes. These are to be compared with their corresponding theoretical variance-covariance (Tables 5.49 to 5.60) from the Fisher information matrix. As for the fixed pressure case, the empirical variances and covariances are larger than their theoretical expected values. The discrepancy between theory and observed is striking for lambda and b where the estimates are not normally distributed. The variance for the gamma estimate is in closer agreement with the theoretical variance than that for the lambda and b estimates.

Table 5.1 Statistical properties of the ML estimate for $\gamma = 1.5$ over 1000 replications. Variable pressure, censoring =10%, sample size=25.

Mean	1.341455
Variance	1.287964×10^{-4}
Minimum	1.288587
Maximum	1.35703
Skewness	-2.311752
Kurtosis	7.535272
D'Agostino Skewness	Reject Normality with prob. level 0.0
D'Agostino Kurtosis	Reject Normality with prob. level 0.0
D'Agostino Omnibus	Reject Normality with prob. level 0.0

Table 5.2 Statistical properties of the ML estimate for $\gamma = 1.5$ over 1000 replications. Variable pressure, censoring =10%, sample size=50.

Mean	1.343592
Variance	5.626559×10^{-5}
Minimum	1.318285
Maximum	1.472254
Skewness	3.791434
Kurtosis	89.94577
D'Agostino Skewness	Reject Normality with prob. level 0.0
D'Agostino Kurtosis	Reject Normality with prob. level 0.0
D'Agostino Omnibus	Reject Normality with prob. level 0.0

Table 5.3 Statistical properties of the ML estimate for $\gamma = 1.5$ over 1000 replications. Variable pressure, censoring =10%, sample size=100.

Mean	1.344324
Variance	3.117618×10^{-5}
Minimum	1.332165
Maximum	1.472254
Skewness	11.53155
Kurtosis	277.0562
D'Agostino Skewness	Reject Normality with prob. level 0.0
D'Agostino Kurtosis	Reject Normality with prob. level 0.0
D'Agostino Omnibus	Reject Normality with prob. level 0.0

Table 5.4 Statistical properties of the ML estimate for $\lambda = 5.7 \times 10^{-5}$ over 1000 replications. Variable pressure, censoring =10%, sample size=25.

Mean	5.69743×10^{-5}
Variance	5.743639×10^{-13}
Minimum	0.0000331
Maximum	0.000057
Skewness	-31.32137
Kurtosis	0
D'Agostino Skewness	Reject Normality with prob. level 0.0
D'Agostino Kurtosis	Reject Normality with prob. level 0.0
D'Agostino Omnibus	Reject Normality with prob. level 0.0

Table 5.5 Statistical properties of the ML estimate for $\lambda = 5.7 \times 10^{-5}$ over 1000 replications. Variable pressure, censoring =10%, sample size=50.

Mean	5.69878×10^{-5}
Variance	1.4884×10^{-13}
Minimum	0.0000448
Maximum	0.000057
Skewness	-31.57532
Kurtosis	0
D'Agostino Skewness	Reject Normality with prob. level 0.0
D'Agostino Kurtosis	Reject Normality with prob. level 0.0
D'Agostino Omnibus	Reject Normality with prob. level 0.0

Table 5.6 Statistical properties of the ML estimate for $\lambda = 5.7 \times 10^{-5}$ over 1000 replications. Variable pressure, censoring =10%, sample size=100.

Mean	5.69878×10^{-5}
Variance	1.4884×10^{-13}
Minimum	0.0000448
Maximum	0.000057
Skewness	-31.57532
Kurtosis	0
D'Agostino Skewness	Reject Normality with prob. level 0.0
D'Agostino Kurtosis	Reject Normality with prob. level 0.0
D'Agostino Omnibus	Reject Normality with prob. level 0.0

Table 5.7 Statistical properties of the ML estimate for $b = -0.25$ over 1000 replications. Variable pressure, censoring =10%, sample size=25.

Mean	-0.2500117
Variance	1.315455×10^{-7}
Minimum	-0.2613897
Maximum	-0.2488243
Skewness	-30.88592
Kurtosis	970.5985
D'Agostino Skewness	Reject Normality with prob. level 0.0
D'Agostino Kurtosis	Reject Normality with prob. level 0.0
D'Agostino Omnibus	Reject Normality with prob. level 0.0

Table 5.8 Statistical properties of the ML estimate for $b = -0.25$ over 1000 replications. Variable pressure, censoring =10%, sample size=50.

Mean	-0.2499827
Variance	2.651471×10^{-7}
Minimum	-0.2501051
Maximum	-0.2337181
Skewness	31.56132
Kurtosis	997.413
D'Agostino Skewness	Reject Normality with prob. level 0.0
D'Agostino Kurtosis	Reject Normality with prob. level 0.0
D'Agostino Omnibus	Reject Normality with prob. level 0.0

Table 5.9 Statistical properties of the ML estimate for $b = -0.25$ over 1000 replications. Variable pressure, censoring =10%, sample size=100.

Mean	-0.249982
Variance	2.650804×10^{-7}
Minimum	-0.2500404
Maximum	-0.2337181
Skewness	31.5692
Kurtosis	997.7438
D'Agostino Skewness	Reject Normality with prob. level 0.0
D'Agostino Kurtosis	Reject Normality with prob. level 0.0
D'Agostino Omnibus	Reject Normality with prob. level 0.0

Table 5.10 Statistical properties of the ML estimate for $\gamma = 1.5$ over 1000 replications. Variable pressure, censoring =20%, sample size=25.

Mean	1.380845
Variance	4.338712×10^{-4}
Minimum	1.292952
Maximum	1.522883
Skewness	-1.203707
Kurtosis	6.453557
D'Agostino Skewness	Reject Normality with prob. level 0.0
D'Agostino Kurtosis	Reject Normality with prob. level 0.0
D'Agostino Omnibus	Reject Normality with prob. level 0.0

Table 5.11 Statistical properties of the ML estimate for $\gamma = 1.5$ over 1000 replications. Variable pressure, censoring =20%, sample size=50.

Mean	1.386477
Variance	1.332621×10^{-4}
Minimum	1.332556
Maximum	1.522883
Skewness	0.3944472
Kurtosis	22.92529
D'Agostino Skewness	Reject Normality with prob. level 0.0
D'Agostino Kurtosis	Reject Normality with prob. level 0.0
D'Agostino Omnibus	Reject Normality with prob. level 0.0

Table 5.12 Statistical properties of the ML estimate for $\gamma = 1.5$ over 1000 replications. Variable pressure, censoring =20%, sample size=100.

Mean	1.388601
Variance	4.178969×10^{-5}
Minimum	1.364758
Maximum	1.470404
Skewness	1.170251
Kurtosis	27.90537
D'Agostino Skewness	Reject Normality with prob. level 0.0
D'Agostino Kurtosis	Reject Normality with prob. level 0.0
D'Agostino Omnibus	Reject Normality with prob. level 0.0

Table 5.13 Statistical properties of the ML estimate for $\lambda = 5.7 \times 10^{-5}$ over 1000 replications. Variable pressure, censoring =20%, sample size=25.

Mean	5.69852×10^{-5}
Variance	1.839049×10^{-13}
Minimum	0.0000435
Maximum	0.000057
Skewness	-31.16803
Kurtosis	0
D'Agostino Skewness	Reject Normality with prob. level 0.0
D'Agostino Kurtosis	Reject Normality with prob. level 0.0
D'Agostino Omnibus	Reject Normality with prob. level 0.0

Table 5.14 Statistical properties of the ML estimate for $\lambda = 5.7 \times 10^{-5}$ over 1000 replications. Variable pressure, censoring =20%, sample size=50.

Mean	5.69864×10^{-5}
Variance	1.822573×10^{-13}
Minimum	0.0000435
Maximum	0.000057
Skewness	-31.57273
Kurtosis	0
D'Agostino Skewness	Reject Normality with prob. level 0.0
D'Agostino Kurtosis	Reject Normality with prob. level 0.0
D'Agostino Omnibus	Reject Normality with prob. level 0.0

Table 5.15 Statistical properties of the ML estimate for $\lambda = 5.7 \times 10^{-5}$ over 1000 replications. Variable pressure, censoring =20%, sample size=100.

Mean	5.69836×10^{-5}
Variance	2.6896×10^{-13}
Minimum	0.0000406
Maximum	0.000057
Skewness	-31.57532
Kurtosis	0
D'Agostino Skewness	Reject Normality with prob. level 0.0
D'Agostino Kurtosis	Reject Normality with prob. level 0.0
D'Agostino Omnibus	Reject Normality with prob. level 0.0

Table 5.16 Statistical properties of the ML estimate for $b = -0.25$ over 1000 replications. Variable pressure, censoring =20%, sample size=25.

Mean	-0.2499983
Variance	2.926913×10^{-7}
Minimum	-0.257061
Maximum	-0.2344469
Skewness	21.55839
Kurtosis	713.2076
D'Agostino Skewness	Reject Normality with prob. level 0.0
D'Agostino Kurtosis	Reject Normality with prob. level 0.0
D'Agostino Omnibus	Reject Normality with prob. level 0.0

Table 5.17 Statistical properties of the ML estimate for $b = -0.25$ over 1000 replications. Variable pressure, censoring =20%, sample size=50.

Mean	-0.2499864
Variance	2.421619×10^{-7}
Minimum	-0.2501766
Maximum	-0.2344469
Skewness	31.53584
Kurtosis	996.3461
D'Agostino Skewness	Reject Normality with prob. level 0.0
D'Agostino Kurtosis	Reject Normality with prob. level 0.0
D'Agostino Omnibus	Reject Normality with prob. level 0.0

Table 5.18 Statistical properties of the ML estimate for $b = -0.25$ over 1000 replications. Variable pressure, censoring =20%, sample size=100.

Mean	-0.2499931
Variance	4.061849×10^{-8}
Minimum	-0.2500975
Maximum	-0.2436325
Skewness	31.48286
Kurtosis	994.1259
D'Agostino Skewness	Reject Normality with prob. level 0.0
D'Agostino Kurtosis	Reject Normality with prob. level 0.0
D'Agostino Omnibus	Reject Normality with prob. level 0.0

Table 5.19 Statistical properties of the ML estimate for $\gamma = 1.5$ over 1000 replications. Variable pressure, censoring =30%, sample size=25.

Mean	1.414562
Variance	6.638744×10^{-4}
Minimum	1.309188
Maximum	1.467707
Skewness	-1.250334
Kurtosis	3.400071
D'Agostino Skewness	Reject Normality with prob. level 0.0
D'Agostino Kurtosis	Reject Normality with prob. level 0.02
D'Agostino Omnibus	Reject Normality with prob. level 0.0

Table 5.20 Statistical properties of the ML estimate for $\gamma = 1.5$ over 1000 replications. Variable pressure, censoring =30%, sample size=50.

Mean	1.420859
Variance	2.482535×10^{-4}
Minimum	1.35258
Maximum	1.558679
Skewness	-0.3329671
Kurtosis	8.630065
D'Agostino Skewness	Reject Normality with prob. level 0.000024
D'Agostino Kurtosis	Reject Normality with prob. level 0.0
D'Agostino Omnibus	Reject Normality with prob. level 0.0

Table 5.21 Statistical properties of the ML estimate for $\gamma = 1.5$ over 1000 replications. Variable pressure, censoring =30%, sample size=100.

Mean	1.424861
Variance	1.108091×10^{-4}
Minimum	1.38102
Maximum	1.558679
Skewness	1.010989
Kurtosis	29.32565
D'Agostino Skewness	Reject Normality with prob. level 0.0
D'Agostino Kurtosis	Reject Normality with prob. level 0.0
D'Agostino Omnibus	Reject Normality with prob. level 0.0

Table 5.22 Statistical properties of the ML estimate for $\lambda = 5.7 \times 10^{-5}$ over 1000 replications. Variable pressure, censoring =30%, sample size=25.

Mean	5.69728×10^{-5}
Variance	4.510112×10^{-13}
Minimum	0.0000362
Maximum	0.000057
Skewness	-29.82707
Kurtosis	0
D'Agostino Skewness	Reject Normality with prob. level 0.0
D'Agostino Kurtosis	Reject Normality with prob. level 0.0
D'Agostino Omnibus	Reject Normality with prob. level 0.0

Table 5.23 Statistical properties of the ML estimate for $\lambda = 5.7 \times 10^{-5}$ over 1000 replications. Variable pressure, censoring =30%, sample size=50.

Mean	5.69859×10^{-5}
Variance	1.877189×10^{-13}
Minimum	0.0000433
Maximum	0.000057
Skewness	-31.56529
Kurtosis	0
D'Agostino Skewness	Reject Normality with prob. level 0.0
D'Agostino Kurtosis	Reject Normality with prob. level 0.0
D'Agostino Omnibus	Reject Normality with prob. level 0.0

Table 5.24 Statistical properties of the ML estimate for $\lambda = 5.7 \times 10^{-5}$ over 1000 replications. Variable pressure, censoring =30%, sample size=100.

Mean	5.69863×10^{-5}
Variance	1.8769×10^{-13}
Minimum	0.0000433
Maximum	0.000057
Skewness	-31.57532
Kurtosis	0
D'Agostino Skewness	Reject Normality with prob. level 0.0
D'Agostino Kurtosis	Reject Normality with prob. level 0.0
D'Agostino Omnibus	Reject Normality with prob. level 0.0

Table 5.25 Statistical properties of the ML estimate for $b = -0.25$ over 1000 replications. Variable pressure, censoring =30%, sample size=25.

Mean	-0.2500268
Variance	1.852176×10^{-7}
Minimum	-0.2625799
Maximum	-0.2477871
Skewness	-25.55183
Kurtosis	732.313
D'Agostino Skewness	Reject Normality with prob. level 0.0
D'Agostino Kurtosis	Reject Normality with prob. level 0.0
D'Agostino Omnibus	Reject Normality with prob. level 0.0

Table 5.26 Statistical properties of the ML estimate for $b = -0.25$ over 1000 replications. Variable pressure, censoring =30%, sample size=50.

Mean	-0.2499907
Variance	2.058879×10^{-7}
Minimum	-0.2505908
Maximum	-0.2356774
Skewness	31.4331
Kurtosis	992.128
D'Agostino Skewness	Reject Normality with prob. level 0.0
D'Agostino Kurtosis	Reject Normality with prob. level 0.0
D'Agostino Omnibus	Reject Normality with prob. level 0.0

Table 5.27 Statistical properties of the ML estimate for $b = -0.25$ over 1000 replications. Variable pressure, censoring =30%, sample size=100.

Mean	-0.2499869
Variance	2.052673×10^{-7}
Minimum	-0.250088
Maximum	-0.2356774
Skewness	31.55361
Kurtosis	997.0893
D'Agostino Skewness	Reject Normality with prob. level 0.0
D'Agostino Kurtosis	Reject Normality with prob. level 0.0
D'Agostino Omnibus	Reject Normality with prob. level 0.0

Table 5.28 Statistical properties of the ML estimate for $\gamma = 1.5$ over 1000 replications. Variable pressure, censoring =40%, sample size=25.

Mean	1.433256
Variance	2.332259×10^{-3}
Minimum	0.9271806
Maximum	1.579988
Skewness	-2.573467
Kurtosis	19.60515
D'Agostino Skewness	Reject Normality with prob. level 0.0
D'Agostino Kurtosis	Reject Normality with prob. level 0.0
D'Agostino Omnibus	Reject Normality with prob. level 0.0

Table 5.29 Statistical properties of the ML estimate for $\gamma = 1.5$ over 1000 replications. Variable pressure, censoring =40%, sample size=50.

Mean	1.452074
Variance	5.952188×10^{-4}
Minimum	1.263457
Maximum	1.579988
Skewness	-2.107651
Kurtosis	12.26067
D'Agostino Skewness	Reject Normality with prob. level 0.0
D'Agostino Kurtosis	Reject Normality with prob. level 0.0
D'Agostino Omnibus	Reject Normality with prob. level 0.0

Table 5.30 Statistical properties of the ML estimate for $\gamma = 1.5$ over 1000 replications. Variable pressure, censoring =40%, sample size=100.

Mean	1.451247
Variance	3.678155×10^{-4}
Minimum	1.386606
Maximum	1.579988
Skewness	-0.6873839
Kurtosis	5.095651
D'Agostino Skewness	Reject Normality with prob. level 0.0
D'Agostino Kurtosis	Reject Normality with prob. level 0.0
D'Agostino Omnibus	Reject Normality with prob. level 0.0

Table 5.31 Statistical properties of the ML estimate for $\lambda = 5.7 \times 10^{-5}$ over 1000 replications. Variable pressure, censoring =40%, sample size=25.

Mean	5.69835×10^{-5}
Variance	4.545123×10^{-13}
Minimum	0.0000464
Maximum	0.0000667
Skewness	-7.228319
Kurtosis	0
D'Agostino Skewness	Reject Normality with prob. level 0.0
D'Agostino Kurtosis	Reject Normality with prob. level 0.0
D'Agostino Omnibus	Reject Normality with prob. level 0.0

Table 5.32 Statistical properties of the ML estimate for $\lambda = 5.7 \times 10^{-5}$ over 1000 replications. Variable pressure, censoring =40%, sample size=50.

Mean	5.69926×10^{-5}
Variance	1.164417×10^{-13}
Minimum	0.0000464
Maximum	0.0000586
Skewness	-29.82013
Kurtosis	0
D'Agostino Skewness	Reject Normality with prob. level 0.0
D'Agostino Kurtosis	Reject Normality with prob. level 0.0
D'Agostino Omnibus	Reject Normality with prob. level 0.0

Table 5.33 Statistical properties of the ML estimate for $\lambda = 5.7 \times 10^{-5}$ over 1000 replications. Variable pressure, censoring =40%, sample size=100.

Mean	5.69894×10^{-5}
Variance	1.1236×10^{-13}
Minimum	0.0000464
Maximum	0.000057
Skewness	-31.57532
Kurtosis	0
D'Agostino Skewness	Reject Normality with prob. level 0.0
D'Agostino Kurtosis	Reject Normality with prob. level 0.0
D'Agostino Omnibus	Reject Normality with prob. level 0.0

Table 5.34 Statistical properties of the ML estimate for $b = -0.25$ over 1000 replications. Variable pressure, censoring =40%, sample size=25.

Mean	-0.2504066
Variance	2.358565×10^{-5}
Minimum	-0.3735061
Maximum	-0.2368246
Skewness	-18.89583
Kurtosis	435.9554
D'Agostino Skewness	Reject Normality with prob. level 0.0
D'Agostino Kurtosis	Reject Normality with prob. level 0.0
D'Agostino Omnibus	Reject Normality with prob. level 0.0

Table 5.35 Statistical properties of the ML estimate for $b = -0.25$ over 1000 replications. Variable pressure, censoring =40%, sample size=50.

Mean	-0.2500563
Variance	1.149322×10^{-6}
Minimum	-0.2694506
Maximum	-0.2368246
Skewness	-12.26173
Kurtosis	266.4028
D'Agostino Skewness	Reject Normality with prob. level 0.0
D'Agostino Kurtosis	Reject Normality with prob. level 0.0
D'Agostino Omnibus	Reject Normality with prob. level 0.0

Table 5.36 Statistical properties of the ML estimate for $b = -0.25$ over 1000 replications. Variable pressure, censoring =40%, sample size=100.

Mean	-0.2499968
Variance	1.745895×10^{-7}
Minimum	-0.2504289
Maximum	-0.2368246
Skewness	31.37351
Kurtosis	989.5951
D'Agostino Skewness	Reject Normality with prob. level 0.0
D'Agostino Kurtosis	Reject Normality with prob. level 0.0
D'Agostino Omnibus	Reject Normality with prob. level 0.0

Table 5.37 The Empirical variance-covariance matrix of the ML estimates over 1000 replications. Variable pressure, censoring =10%, sample size=25.

$$\begin{bmatrix} 1.29\times10^{-4} & -2.01\times10^{-11} & 1.60\times10^{-7} \\ -2.01\times10^{-11} & 5.74\times10^{-13} & 3.00\times10^{-10} \\ 1.60\times10^{-7} & 3.00\times10^{-10} & 1.32\times10^{-7} \end{bmatrix}$$

Table 5.38 The Empirical variance-covariance matrix of the ML estimates over 1000 replications. Variable pressure, censoring =20%, sample size=25.

$$\begin{bmatrix} 4.34\times10^{-4} & -1.80\times10^{-9} & 3.20\times10^{-6} \\ -1.80\times10^{-9} & 1.84\times10^{-13} & -2.00\times10^{-10} \\ 3.20\times10^{-6} & -2.00\times10^{-10} & 2.93\times10^{-7} \end{bmatrix}$$

Table 5.39 The Empirical variance-covariance matrix of the ML estimates over 1000 replications. Variable pressure, censoring =30%, sample size=25.

$$\begin{bmatrix} 6.64\times10^{-4} & -1.00\times10^{-9} & 2.02\times10^{-6} \\ -1.00\times10^{-9} & 4.51\times10^{-13} & 1.00\times10^{-10} \\ 2.02\times10^{-6} & 1.00\times10^{-10} & 1.85\times10^{-7} \end{bmatrix}$$

Table 5.40 The Empirical variance-covariance matrix of the ML estimates over 1000 replications. Variable pressure, censoring =40%, sample size=25.

$$\begin{bmatrix} 2.33 \times 10^{-3} & -7.30 \times 10^{-9} & 1.29 \times 10^{-6} \\ -7.30 \times 10^{-9} & 4.55 \times 10^{-13} & -1.00 \times 10^{-9} \\ 1.29 \times 10^{-6} & -1.00 \times 10^{-9} & 2.36 \times 10^{-7} \end{bmatrix}$$

Table 5.41 The Empirical variance-covariance matrix of the ML estimates over 1000 replications. Variable pressure, censoring =10%, sample size=50.

$$\begin{bmatrix} 5.63 \times 10^{-5} & -1.60 \times 10^{-9} & 2.14 \times 10^{-6} \\ -1.6 \times 10^{-9} & 1.49 \times 10^{-13} & -2.00 \times 10^{-10} \\ 2.14 \times 10^{-6} & -2.00 \times 10^{-10} & 2.65 \times 10^{-7} \end{bmatrix}$$

Table 5.42 The Empirical variance-covariance matrix of the ML estimates over 1000 replications. Variable pressure, censoring =20%, sample size=50.

$$\begin{bmatrix} 1.33 \times 10^{-4} & -1.80 \times 10^{-9} & 2.23 \times 10^{-6} \\ -1.80 \times 10^{-9} & 1.82 \times 10^{-13} & -2.00 \times 10^{-10} \\ 2.23 \times 10^{-6} & -2.00 \times 10^{-10} & 2.42 \times 10^{-7} \end{bmatrix}$$

Table 5.43 The Empirical variance-covariance matrix of the ML estimates over 1000 replications. Variable pressure, censoring =30%, sample size=50.

$$\begin{bmatrix} 2.48 \times 10^{-4} & -1.90 \times 10^{-9} & 2.15 \times 10^{-6} \\ -1.90 \times 10^{-9} & 1.88 \times 10^{-13} & -2.00 \times 10^{-10} \\ 2.15 \times 10^{-6} & -2.00 \times 10^{-10} & 2.06 \times 10^{-7} \end{bmatrix}$$

Table 5.44 The Empirical variance-covariance matrix of the ML estimates over 1000 replications. Variable pressure, censoring =40%, sample size=50.

$$\begin{bmatrix} 5.95 \times 10^{-4} & -1.90 \times 10^{-9} & 1.21 \times 10^{-5} \\ -1.90 \times 10^{-9} & 1.16 \times 10^{-13} & -2.00 \times 10^{-10} \\ 1.21 \times 10^{-5} & -2.00 \times 10^{-10} & 1.15 \times 10^{-6} \end{bmatrix}$$

Table 5.45 The Empirical variance-covariance matrix of the ML estimates over 1000 replications. Variable pressure, censoring =10%, sample size=100.

$$\begin{bmatrix} 3.12 \times 10^{-5} & -1.60 \times 10^{-9} & 2.1 \times 10^{-6} \\ -1.60 \times 10^{-9} & 1.49 \times 10^{-13} & -2.00 \times 10^{-10} \\ 2.10 \times 10^{-6} & -2.00 \times 10^{-10} & 2.65 \times 10^{-7} \end{bmatrix}$$

Table 5.46 The Empirical variance-covariance matrix of the ML estimates over 1000 replications. Variable pressure, censoring =20%, sample size=100.

$$\begin{bmatrix} 4.18 \times 10^{-5} & -1.30 \times 10^{-9} & 5.55 \times 10^{-7} \\ -1.30 \times 10^{-9} & 2.69 \times 10^{-13} & -1.00 \times 10^{-10} \\ 5.55 \times 10^{-7} & -1.00 \times 10^{-10} & 4.06 \times 10^{-8} \end{bmatrix}$$

Table 5.47 The Empirical variance-covariance matrix of the ML estimates over 1000 replications. Variable pressure, censoring =30%, sample size=100.

$$\begin{bmatrix} 1.11 \times 10^{-4} & -1.80 \times 10^{-8} & 1.99 \times 10^{-6} \\ -1.80 \times 10^{-8} & 1.88 \times 10^{-13} & -2.00 \times 10^{-10} \\ 1.99 \times 10^{-6} & -2.00 \times 10^{-10} & 2.05 \times 10^{-7} \end{bmatrix}$$

Table 5.48 The Empirical variance-covariance matrix of the ML estimates over 1000 replications. Variable pressure, censoring =40%, sample size=100.

$$\begin{bmatrix} 3.68 \times 10^{-4} & -1.40 \times 10^{-8} & 2.05 \times 10^{-6} \\ -1.40 \times 10^{-8} & 1.12 \times 10^{-13} & -1.00 \times 10^{-10} \\ 2.05 \times 10^{-6} & -1.00 \times 10^{-10} & 1.75 \times 10^{-7} \end{bmatrix}$$

Table 5.49 The Fisher variance-covariance matrix of the ML estimates over 1000 replications. Variable pressure, censoring =10%, sample size=25.

$$\begin{bmatrix} 6.60 \times 10^{-4} & -1.31 \times 10^{-10} & 5.36 \times 10^{-8} \\ -1.31 \times 10^{-10} & 1.25 \times 10^{-17} & 2.29 \times 10^{-11} \\ 5.36 \times 10^{-8} & 2.29 \times 10^{-11} & 9.91 \times 10^{-13} \end{bmatrix}$$

Table 5.50 The Fisher variance-covariance matrix of the ML estimates over 1000 replications. Variable pressure, censoring =20%, sample size=25.

$$\begin{bmatrix} 7.44 \times 10^{-4} & -1.33 \times 10^{-10} & 5.40 \times 10^{-8} \\ -1.33 \times 10^{-10} & 7.16 \times 10^{-18} & -2.25 \times 10^{-11} \\ 5.40 \times 10^{-8} & -2.25 \times 10^{-11} & 4.44 \times 10^{-13} \end{bmatrix}$$

Table 5.51 The Fisher variance-covariance matrix of the ML estimates over 1000 replications. Variable pressure, censoring =30%, sample size=25.

$$\begin{bmatrix} 8.14 \times 10^{-4} & -1.42 \times 10^{-10} & 5.84 \times 10^{-8} \\ -1.42 \times 10^{-10} & 9.09 \times 10^{-18} & 2.35 \times 10^{-11} \\ 5.84 \times 10^{-8} & 2.35 \times 10^{-11} & 6.99 \times 10^{-13} \end{bmatrix}$$

Table 5.52 The Fisher variance-covariance matrix of the ML estimates over 1000 replications. Variable pressure, censoring =40%, sample size=25.

$$\begin{bmatrix} 9.75 \times 10^{-4} & -2.49 \times 10^{-10} & 1.03 \times 10^{-7} \\ -2.49 \times 10^{-10} & 1.82 \times 10^{-17} & -4.08 \times 10^{-11} \\ 1.03 \times 10^{-7} & -4.08 \times 10^{-11} & 7.70 \times 10^{-14} \end{bmatrix}$$

Table 5.53 The Fisher variance-covariance matrix of the ML estimates over 1000 replications. Variable pressure, censoring =10%, sample size=50.

$$\begin{bmatrix} 3.25 \times 10^{-4} & -5.95 \times 10^{-11} & 2.40 \times 10^{-8} \\ -5.95 \times 10^{-11} & 4.17 \times 10^{-18} & -1.03 \times 10^{-11} \\ 2.40 \times 10^{-8} & -1.03 \times 10^{-11} & 3.31 \times 10^{-13} \end{bmatrix}$$

Table 5.54 The Fisher variance-covariance matrix of the ML estimates over 1000 replications. Variable pressure, censoring =20%, sample size=50.

$$\begin{bmatrix} 3.56 \times 10^{-4} & -7.55 \times 10^{-11} & 3.05 \times 10^{-8} \\ -7.55 \times 10^{-11} & 5.08 \times 10^{-18} & -1.27 \times 10^{-11} \\ 3.05 \times 10^{-8} & -1.27 \times 10^{-11} & 3.96 \times 10^{-13} \end{bmatrix}$$

Table 5.55 The Fisher variance-covariance matrix of the ML estimates over 1000 replications. Variable pressure, censoring =30%, sample size=50.

$$\begin{bmatrix} 3.88 \times 10^{-4} & -6.56 \times 10^{-11} & 2.66 \times 10^{-8} \\ -6.56 \times 10^{-11} & 3.89 \times 10^{-18} & -1.08 \times 10^{-11} \\ 2.66 \times 10^{-8} & -1.08 \times 10^{-11} & 3.67 \times 10^{-13} \end{bmatrix}$$

Table 5.56 The Fisher variance-covariance matrix of the ML estimates over 1000 replications. Variable pressure, censoring =40%, sample size=50.

$$\begin{bmatrix} 4.20 \times 10^{-4} & -7.79 \times 10^{-11} & 3.17 \times 10^{-8} \\ -7.79 \times 10^{-11} & 6.08 \times 10^{-18} & -1.26 \times 10^{-11} \\ 3.17 \times 10^{-8} & -1.26 \times 10^{-11} & 2.78 \times 10^{-13} \end{bmatrix}$$

Table 5.57 The Fisher variance-covariance matrix of the ML estimates over 1000 replications. Variable pressure, censoring =10%, sample size=100.

$$\begin{bmatrix} 1.61 \times 10^{-4} & -2.96 \times 10^{-11} & 1.20 \times 10^{-8} \\ -2.96 \times 10^{-11} & 2.09 \times 10^{-18} & -5.15 \times 10^{-12} \\ 1.20 \times 10^{-8} & -5.15 \times 10^{-12} & 1.68 \times 10^{-13} \end{bmatrix}$$

Table 5.58 The Fisher variance-covariance matrix of the ML estimates over 1000 replications. Variable pressure, censoring =20%, sample size=100.

$$\begin{bmatrix} 1.75 \times 10^{-4} & -4.56 \times 10^{-11} & 1.84 \times 10^{-8} \\ -4.56 \times 10^{-11} & 4.80 \times 10^{-18} & -7.67 \times 10^{-12} \\ 1.84 \times 10^{-8} & -7.67 \times 10^{-12} & 3.51 \times 10^{-13} \end{bmatrix}$$

Table 5.59 The Fisher variance-covariance matrix of the ML estimates over 1000 replications. Variable pressure, censoring =30%, sample size=100.

$$\begin{bmatrix} 1.89 \times 10^{-4} & -4.36 \times 10^{-11} & 1.76 \times 10^{-8} \\ -4.36 \times 10^{-11} & 3.07 \times 10^{-18} & -7.15 \times 10^{-12} \\ 1.76 \times 10^{-8} & -7.15 \times 10^{-12} & 1.97 \times 10^{-13} \end{bmatrix}$$

Table 5.60 The Fisher variance-covariance matrix of the ML estimates over 1000 replications. Variable pressure, censoring =40%, sample size=100.

$$\begin{bmatrix} 2.12 \times 10^{-4} & -5.31 \times 10^{-11} & 2.16 \times 10^{-8} \\ -5.31 \times 10^{-11} & 3.81 \times 10^{-18} & -8.56 \times 10^{-12} \\ 2.16 \times 10^{-8} & -8.56 \times 10^{-12} & 8.85 \times 10^{-14} \end{bmatrix}$$

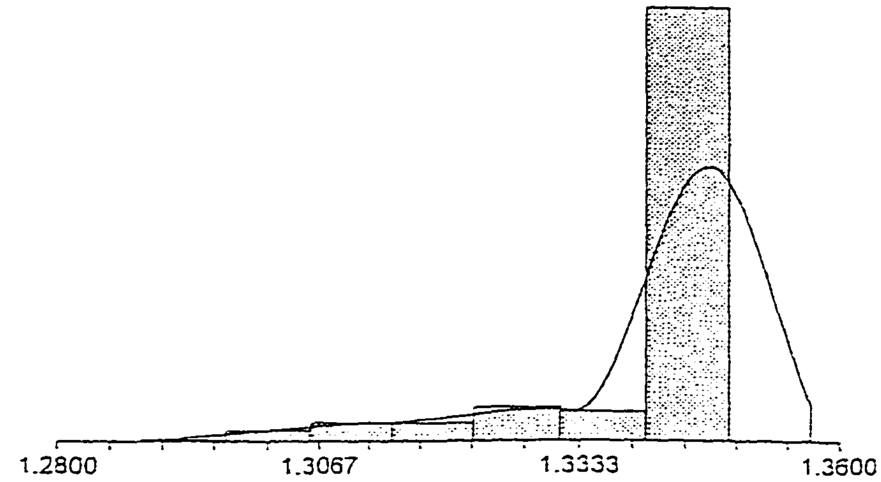

Fig. 5.1 Relative frequency histogram of the ML estimate for gamma=1.5, variable pressure, censoring=10%, sample size=25.

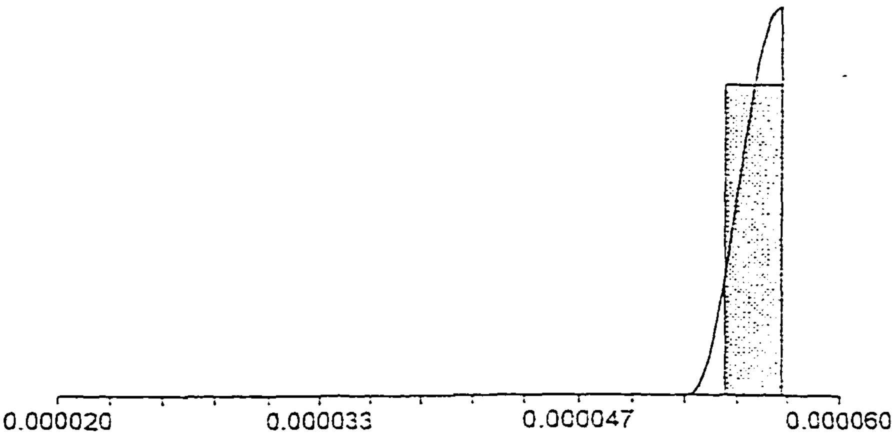

Fig. 5.2 Relative frequency histogram of the ML estimate for lambda=0.000057, variable pressure, censoring=10%, sample size=25.

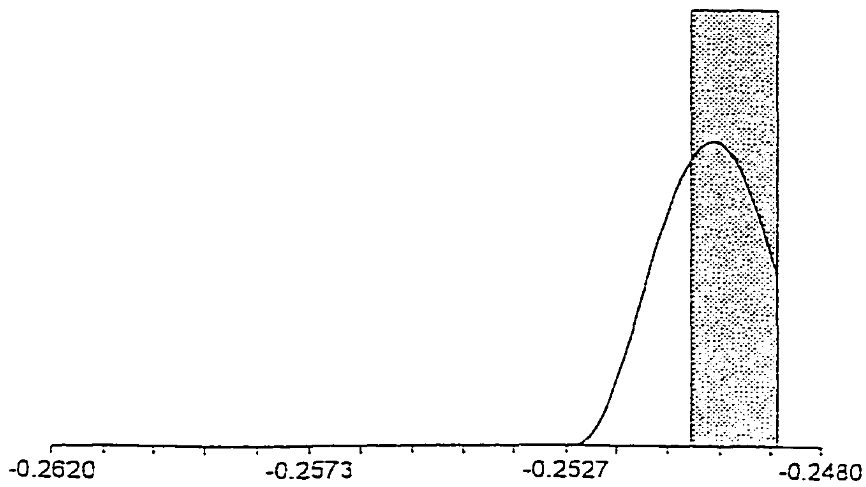

Fig. 5.3 Relative frequency histogram of the ML estimate for b=-0.25, variable pressure, censoring=10%, sample size=25.

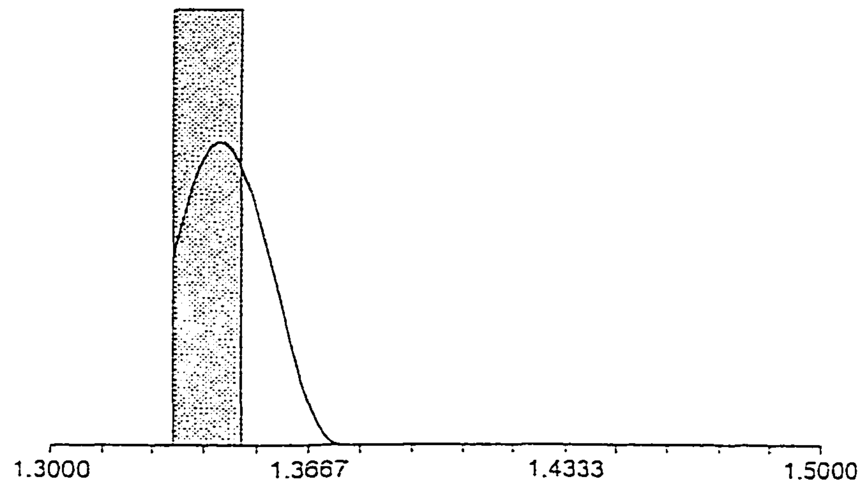

Fig. 5.4 Relative frequency histogram of the ML estimate for gamma=1.5, variable pressure, censoring=10%, sample size=100.

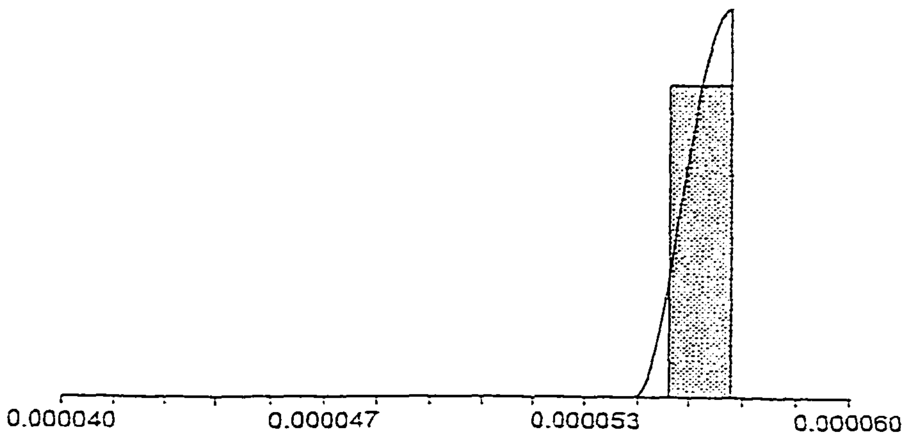

Fig. 5.5 Relative frequency histogram of the ML estimate for lambda=0.000057, variable pressure, censoring=10%, sample size=100.

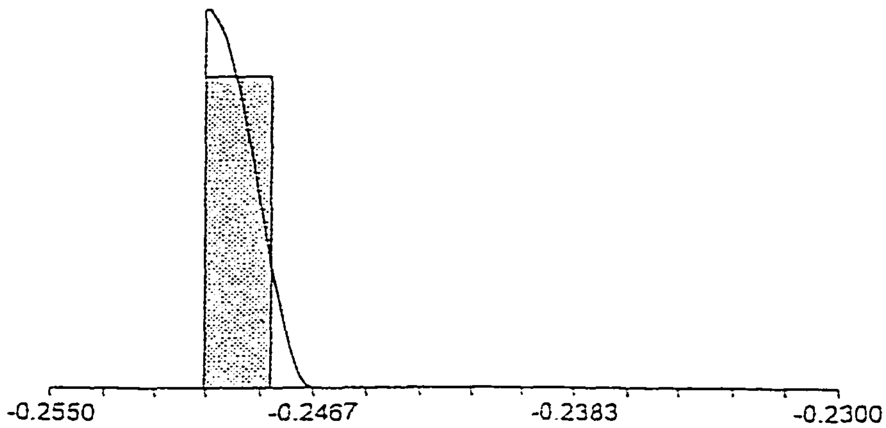

Fig. 5.6 Relative frequency histogram of the ML estimate for b=-0.25, variable pressure, censoring=10%, sample size=100.

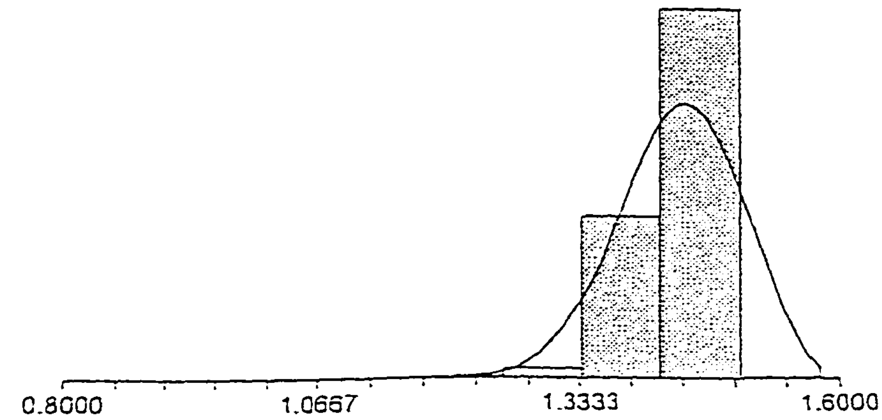

Fig. 5.7 Relative frequency histogram of the ML estimate for gamma=1.5, variable pressure, censoring=40%, sample size=25.

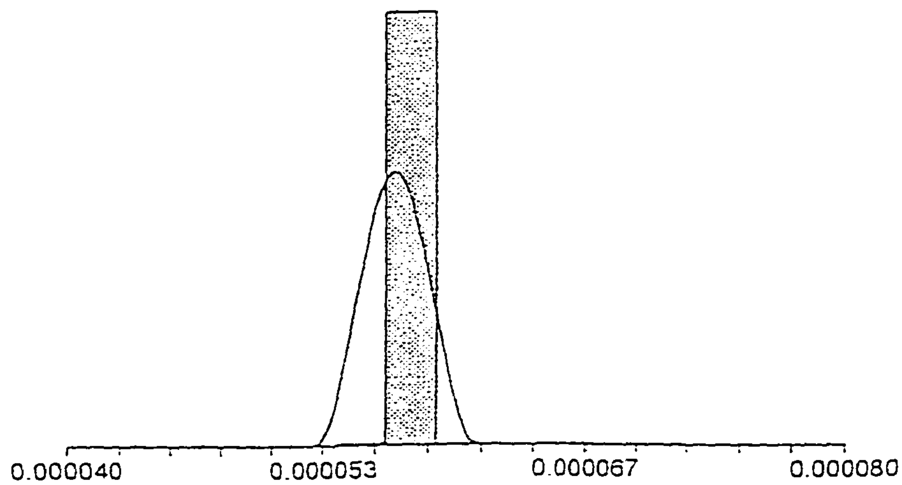

Fig. 5.8 Relative frequency histogram of the ML estimate for lambda=0.000057, variable pressure, censoring=40%, sample size=25.

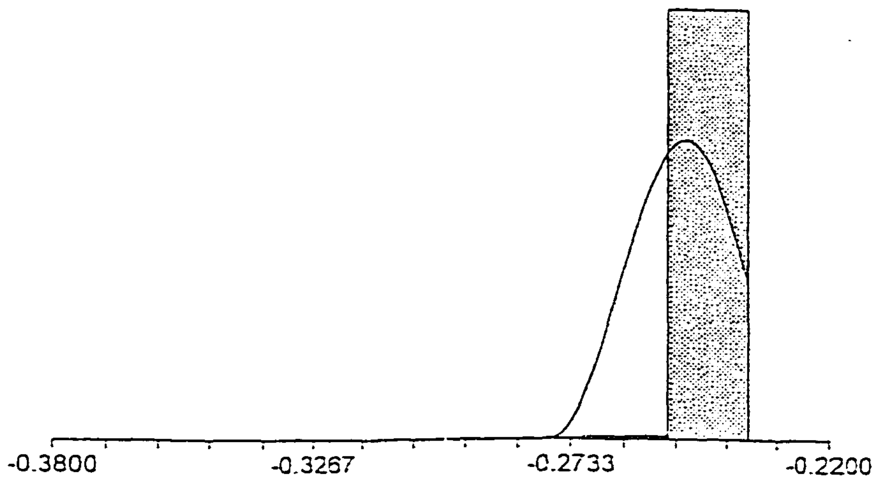

Fig. 5.9 Relative frequency histogram of the ML estimate for b=-0.25, variable pressure, censoring=40%, sample size=25.

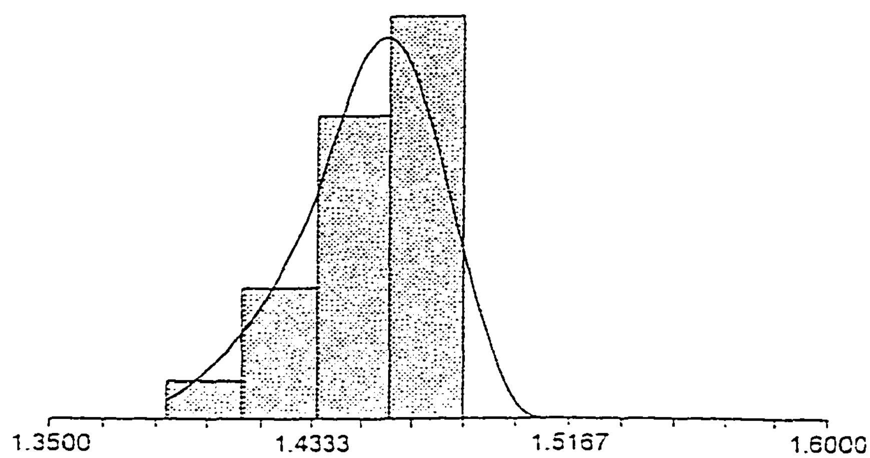

Fig. 5.10 Relative frequency histogram of the ML estimate for gamma=1.5, variable pressure, censoring=40%, sample size=100.

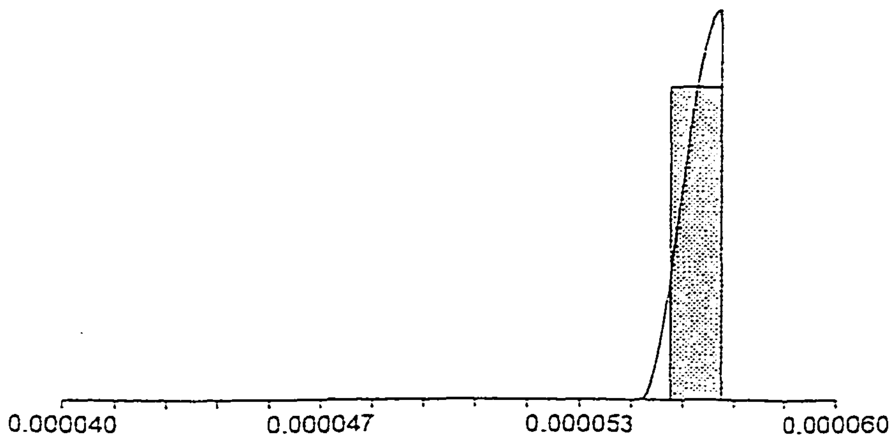

Fig. 5.11 Relative frequency histogram of the ML estimate for lambda=0.000057, variable pressure, censoring=40%, sample size=100.

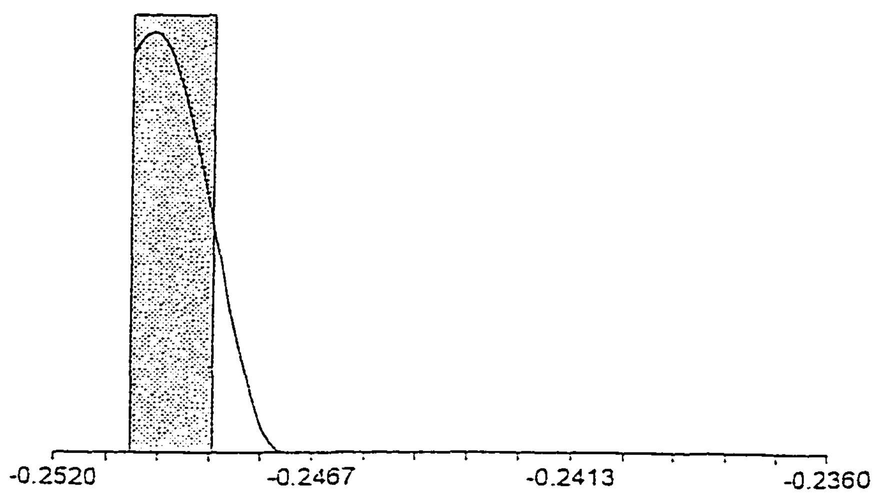

Fig. 5.12 Relative frequency histogram of the ML estimate for b=-0.25, variable pressure, censoring=40%, sample size=100.

CHAPTER 6

CONCLUSION AND FUTURE RESEARCH

6.1 Conclusion

The maximum likelihood estimation procedure is to be recommended for parameter estimation when there are censored observations. From statistical theory, it is known that a maximum likelihood estimator is asymptotically (large samples) unbiased and has a normal distribution with a variance-covariance matrix given by the inverse of the Fisher information matrix. According to the simulation results, it is clear that the maximum likelihood estimator under fixed or variable pressure, and for different sample sizes with different levels of censoring, is unbiased or close to being unbiased. On the other hand, the estimator is not normally distributed especially when the parameter value being estimated is small (-0.25 for b and 5.7×10^{-5} for lambda). This lack of normality is also manifested in lack of agreement between the observed variance-covariance matrix and the theoretical variance-covariance matrix. These results imply that for regular sample sizes of 100 or less, one may not use normal theory and the Fisher variance-covariance matrix to test hypothesis or set confidence limits for the maximum likelihood estimates. While the maximum likelihood estimates are reliable, statistical inference drawn from hypotheses testing and confidence intervals using the Fisher variance-covariance matrix can be in serious error and therefore misleading since the variance-covariance values in the Fisher matrix underestimate the observed variance-covariance matrix from simulation, confidence intervals will tend to be narrower than expected and test statistics larger than expected. The lack of normality of the maximum likelihood estimator is alleviated for parameter estimates large in value such as that for $\gamma = 1.5$, but only under the constant pressure assumption. For estimates of small value parameters, normal theory seems to require large sample sizes beyond what is normally encountered in practice.

Although the error term associated with a maximum likelihood estimate of a Weibull parameter can be inaccurate (in the sense of underestimation the true error), one may still use the estimator itself (which is unbiased or close to being unbiased) in the Weibull distribution to model creep induced failure times of cured-in-place rehabilitation liners. For instance, from the accelerated lifetime model

$$t = t_0 e^{bx}, \tag{6.1}$$

where t_0 has the Weibull distribution and x represents pressure in psi, one may predict the probability of survival of a liner beyond age t to be

$$e^{-\hat{\lambda}(te^{-\hat{b}x})^{\hat{\gamma}}}. \tag{6.2}$$

Also, the predicted pressure x under which a liner survives for a given time t (say 50 years) with probability $1 - p$ is given by

$$x = \frac{\ln \hat{\lambda} - \hat{\gamma} \ln t - \ln(-\ln(1-p))}{\hat{\gamma}\hat{b}}. \tag{6.3}$$

Likewise, the predicted probability of survival beyond age t for variable pressure from the accelerated lifetime model

$$t = t_0 e^{bx(t)}, \tag{6.4}$$

is given by

$$e^{-\int_0^t \hat{\gamma}\hat{\lambda}\tau^{\hat{\gamma}-1} e^{-\hat{\gamma}\hat{b}x(\tau)} d\tau}. \tag{6.5}$$

From Eq. (6.5) one can predict the pressure x under which the pipe liner survives to age t (say 50 years) with probability $1 - p$.

6.2 Future Work

Future work could include the following points:

1. Change the pressure function $x(t) = c + a \sin \frac{\pi t}{6}$ to $x(t) = c + a \cos \frac{\pi t}{6}$ in the accelerated lifetime testing under variable pressure.
2. Change the amplitude "a" for both functions, and study the effect it may have on the estimates as well as on statistical inference.
3. Consider a constant step-wise function instead of a continuous function for pressure over time.
4. Consider a variable piecewise seasonal function for pressure over time.
5. Compare the statistical properties of estimates from the various models.

APPENDIX A

FORTRAN PROGRAM FOR THE NEWTON-RAPHSON METHOD UNDER CONSTANT PRESSURE

APPENDIX A

FORTRAN PROGRAM FOR THE NEWTON-RAPHSON METHOD UNDER CONSTANT PRESSURE

```
c       This program uses the Newton-Raphson method to find the
c       maximum likelihood estimates under constant pressure

        Implicit Real*8 (a-h,o-z)
        character*80 title
        common/in/  gamma, lambda, b, x, m
        common/out/ t(500)
        l1 = 5
        l2 = 6
        l3 = 7
        Call stdio(l1,l2)
        Read(l1,'(a90)') title
        write(l2,'(a80)') title
        read(l1,*) gamma
        write(l2,*) 'gamma = ',gamma
        read(l1,*) lambda
        write(l2,*) 'lambda = ',lambda
        read(l1,*) b
        write(l2,*) 'b = ',b
        read(l1,*) x
        write(l2,*) 'x = ',x
        read(l1,*) m
        write(l2,*) 'm = ',m
        if (m .gt. 500) then
            write(*,*) 'Error: Sample size (m) must not exceed 500.'
            goto 1000
```

```
      endif
      read(l1,*) n
      write(l2,*) 'n = ',n
      close (l1)
      write(l2,'(8x,a1,10x,a3,1x,6(17x,a7))') 'K','ERR','x(1)',
     $         'x(2)','x(3)','aa(i,1)','aa(i,2)','aa(i,3)'

      open(l3,file='ran.dmp',status='unknown')
      do i = 1, n
          write(*,*) 'Iteration ',i,' of ',n
          call consrep(l3)
          call newtonr(l2)
      enddo
1000  close(l2)
      stop
      end

      subroutine consrep(l3)
      Implicit real*8 (a-h,o-z)
      integer  ISEED, M
      common/in/  gamma, lambda, b, x, m
      common/out/ t(500)
      dimension r(500)
      external DRNUN, RNSET
      ISEED = ITIME()
      Call RNSET(ISEED)
      write(l3,*) ' iseed = ', iseed
      Call DRNUN(m,R)
      write(l3,*) '    Random #           T                        R'
      do 6 J=1,M
          T(j)=((-dlog(1.0d0-R(j))/LAMBDA)**(1.0d0/GAMMA))*dexp(B*X)
          write(l3,*) j,t(j),r(j)
6     continue
      return
```

```
      end

      Subroutine newtonr(l2)
      implicit real*8(a-h,o-z)
      dimension aa(3,4),Y(3),X(3),F(12),PSI(500)
      common/in/  gamma, lambda, b, z, m
      common/out/ t(500)
      TOL=0.00001d0
       X(1)=gamma
       X(2)=lambda
       X(3)=b
       do I=1,m
          PSI(I)=26.5d0
       enddo

    K=1
100 SUM1=0.0d0
    SUM2=0.0d0
    SUM3=0.0d0
    SUM11=0.0d0
    SUM22=0.0d0
    SUM33=0.0d0

    do 105 I=1,M
    if(T(I).LE.1.14) then
    F1=1.0/X(1)+dlog(T(I))+X(2)*T(I)**X(1)*dexp(-X(1)*X(3)*PSI(I) )
   $     *(X(3)*PSI(I)-dlog(T(I)))-X(3)*PSI(I)
        SUM1=SUM1+F1

        F2=1.0/X(2)-T(I)**X(1)*dexp(-X(1)*X(3)*PSI(I))
        SUM2=SUM2+F2

        F3=X(1)*X(2)*PSI(I)*T(I)**X(1)*dexp(-X(1)*X(3)*PSI(I))-
   $       X(1)*PSI(I)
```

```
      SUM3=SUM3+F3

     else

    F11=X(2)*T(I)**X(1)*dexp(-X(1)*X(3)*PSI(I))*(X(3)*PSI(I)-
   $          dlog(T(I)))
    SUM11=SUM11+F11

    F22=T(I)**X(1)*dexp(-X(1)*X(3)*PSI(I))
    SUM22=SUM22+F22

    F33=X(1)*X(2)*PSI(I)*T(I)**X(1)*dexp(-X(1)*X(3)*PSI(I))
    SUM33=SUM33+F33

             endif

105      continue

   F(1)=SUM1+SUM11
   F(2)=SUM2-SUM22
   F(3)=SUM3+SUM33

   DER11=0.0d0
   DER111=0.0d0
   DER12=0.0d0
   DER112=0.0d0
   DER13=0.0d0
   DER113=0.0d0
   DER22=0.0d0
   DER33=0.0d0
   DER333=0.0d0

   do 110 I=1,M
```

```
 if(T(I).LE.1.14 )then
        F111=-1.0/X(1)**2+X(2)*T(I)**X(1)*dexp(-X(1)*X(3)*PSI(I))*
$                  dlog(T(I))
$                *(X(3)*PSI(I)-dlog(T(I)))-X(2)*X(3)*PSI(I)*T(I)**X(1)*
$            dexp(-X(1)*X(3)*PSI(I))*(X(3)*PSI(I)-dlog(T(I)))
      DER11=DER11+F111

      F112=T(I)**X(1)*dexp(-X(1)*X(3)*PSI(I))*(X(3)*PSI(I)-
$             dlog(T(I)))
        DER12=DER12+F112

     F113=-X(1)*X(2)*PSI(I)*T(I)**X(1)*dexp(-X(1)*X(3)*PSI(I))*
$       X(3)*PSI(I)-dlog(T(I)))+X(2)*PSI(I)*T(I)**X(1)*dexp(-X(1)*
$       X(3)*PSI(I))-PSI(I)
    DER13=DER13+F113

    F222=-1.0/X(2)**2
    DER22=DER22+F222

 F223=X(1)*PSI(I)*T(I)**X(1)*dexp(-X(1)*X(3)*PSI(I))
 DER23=DER23+F223

 F333=-X(1)**2*X(2)*PSI(I)**2*T(I)**X(1)*dexp(-X(1)*X(3)*PSI(I))
 DER33=DER33+F333

          else

 F1111=X(2)*T(I)**X(1)*dexp(-X(1)*X(3)*PSI(I))*dlog(T(I))*
$        (X(3)*PSI(I)-dlog(T(I)))-X(2)*X(3)*PSI(I)*T(I)**X(1)*
$        dexp(-X(1)*X(3)*PSI(I))*(X(3)*PSI(I)-dlog(T(I)))
 DER111=DER111+F1111

 F1112=T(I)**X(1)*dexp(-X(1)*X(3)*PSI(I))*(X(3)*PSI(I)-
$        dlog(T(I)))
```

```
      DER112=DER112+F1112

      F1113=-X(1)*X(2)*PSI(I)*T(I)**X(1)*dexp(-X(1)*X(3)*PSI(I))*
     $       (X(3)*PSI(I)-dlog(T(I)))+X(2)*PSI(I)*T(I)**X(1)*
     $       dexp(-X(1)*X(3)*PSI(I))
      DER113=DER113+F1113

      F2223=X(1)*PSI(I)*T(I)**X(1)*dexp(-X(1)*X(3)*PSI(I))
      DER223=DER223+F2223

      F3333=-X(1)**2*X(2)*PSI(I)**2*T(I)**X(1)*dexp(-X(1)*X(3)*
     $           PSI(I))
      DER333=DER333+F3333

        endif

 110    continue

       F(4)=DER11+DER111
       F(5)=DER12+DER112
       F(6)=DER13+DER113
       F(7)=F(5)
       F(8)=DER22
       F(9)=DER23+DER223
       F(10)=F(6)
       F(11)=F(9)
       F(12)=DER33+DER333

c         Compute the Jacobian matrix

          aa(1,1)=F(4)
          aa(1,2)=F(5)
          aa(1,3)=F(6)
          aa(2,1)=F(7)
```

```
          aa(2,2)=F(8)
          aa(2,3)=F(9)
          aa(3,1)=F(10)
          aa(3,2)=F(11)
          aa(3,3)=F(12)

c        Compute -F(X)

         aa(1,4)=-F(1)
         aa(2,4)=-F(2)
         aa(3,4)=-F(3)

c        Solves the n X n linear system J(X) Y = -F(X)

          det=aa(1,1)*(aa(2,2)*aa(3,3)-aa(2,3)*aa(3,2))-
     $       aa(1,2)*(aa(2,1)*aa(3,3)-aa(2,3)*aa(3,1))+
     $       aa(1,3)*(aa(2,1)*aa(3,2)-aa(2,2)*aa(3,1))


        det1=aa(1,4)*(aa(2,2)*aa(3,3)-aa(2,3)*aa(3,2))-
     $      aa(1,2)*(aa(2,4)*aa(3,3)-aa(2,3)*aa(3,4))+
     $      aa(1,3)*(aa(2,4)*aa(3,2)-aa(2,2)*aa(3,4))


        det2=aa(1,1)*(aa(2,4)*aa(3,3)-aa(2,3)*aa(3,4))-
     $      aa(1,4)*(aa(2,1)*aa(3,3)-aa(2,3)*aa(3,1))+
     $      aa(1,3)*(aa(2,1)*aa(3,4)-aa(2,4)*aa(3,1))


        det3=aa(1,1)*(aa(2,2)*aa(3,4)-aa(2,4)*aa(2,3))-
     $      aa(2,1)*(aa(2,1)*aa(3,4)-aa(2,4)*aa(3,1))+
     $      aa(1,4)*(aa(2,1)*aa(2,3)-aa(2,2)*aa(3,1))
```

```
      Y(1)=det1/det
      Y(2)=det2/det
      Y(3)=det3/det

      ERR=0.0
            do 9 I=1,3
            if(abs(Y(I)).GT.ERR) ERR=abs(Y(I))
9        continue
      if(ERR.LE.0.00001) goto 11
      do 10 I=1,3
            X(I)=X(I)+Y(I)
10          continue
            K=K+1
      goto 100
11          continue
            Write(12,'(i10,4(1x,1pg23.16),1x,3(1x,1pg23.16))')
   $               K,ERR, (x(i),i=1,3),(aa(1,j),j=1,3)
            Write(12,'(106x,1x,1pg23.16,1x,1pg23.16,1x,1pg23.16)')
   $                ((aa(i,j),j=1,3),i=2,3)
            Return
            end
```

APPENDIX B

FORTRAN PROGRAM FOR THE NEWTON-RAPHSON METHOD UNDER VARIABLE PRESSURE

APPENDIX B

FORTRAN PROGRAM FOR THE NEWTON-RAPHSON METHOD UNDER VARIABLE PRESSURE

```
c     This program uses the Newton-Raphson method to find the maximum
c     likelihood estimates under variable pressure

      Implicit real*8 (a-h,o-z)
      dimension aa(3,4),Y(3),F(12),T(500)
      common /one/ a,b,c,ab,pi,x(3),n
      external z1, z2, z3 ,z4,z5,z6,z7,z8,
     $        z9,z10,z11,z12,z13,z14,z15,z16
      integer N, J, M
      M=3
      N=20
      TOL=.00001
      PI = 4*atan(1.0)
      l1 = 5
      l2 = 6
      Call stdio(l1,l2)
      Read(l1,'(a80)') title
      write(l2,'(a80)') title
      read(l1,*) alpha
      write(l2,*) 'alpha = ',alpha
      read(l1,*) beta
      write(l2,*) 'beta = ',beta
      read(l1,*) gamma
      write(l2,*) 'gamma = ',gamma
      read(l1,*) nel
      write(l2,*) 'nel = ',nel
```

```
      read(l1,*) a
      write(l2,*) 'a = ',a
      read(l1,*) c
      write(l2,*) 'c = ',c
      read(l1,*) ab
      write(l2,*) 'ab = ',ab
      read(l1,*) nsim
      write(l2,*) 'nsim = ',nsim
      close (l1)
      write(l2,'(8x,a1,10x,a3,1x,6(17x,a7))') 'K','ERR','x(1)',
     $         'x(2)','x(3)','aa(i,1)','aa(i,2)','aa(i,3)'
          do 1000 ij = 1, nsim
          write(*,*) 'Iteration ',ij,' of ',nsim

         call gnerate(alpha,beta,gamma,a,ab,c,nel,n,t)
          K=1
             x(1) = alpha
             x(2) = beta
             x(3) = gamma
100          SUM1=0.0
             SUM2=0.0
             SUM3=0.0
             SUM11=0.0
             SUM22=0.0
             SUM33=0.0

       do 105 I=1,NEL
                B=T(I)
                if(T(I).LE.1.14)then
              wj = anteg(z1)
           F1=1.0/X(1)-X(3)*(C+AB*dsin(PI*T(I)/6.0))+dlog(T(I))+WJ
              SUM1=SUM1+F1

              wj = anteg(z2)
```

```
       F2=1.0/X(2)+WJ
          SUM2=SUM2+F2
          wj = anteg(z3)
       F3=-(C+AB*dsin(PI*T(I)/6.0))-(X(1)-1)*(C+AB*dsin(PI*T(I)/6.0))+WJ
          SUM3=SUM3+F3

     else
          wj = anteg(z4)
       F11=WJ
          SUM11=SUM11+F11

          wj = anteg(z5)
       F22=WJ
          SUM22=SUM22+F22
          wj = anteg(z6)

       F33=WJ
          SUM33=SUM33+F33

     endif

105 continue

      F(1)=SUM1+SUM11
      F(2)=SUM2-SUM22
      F(3)=SUM3+SUM33

      DER11=0.0
      DER111=0.0
      DER12=0.0
      DER112=0.0
      DER13=0.0
      DER113=0.0
      DER22=0.0
```

```
DER33=0.0
DER333=0.0

do 110 I=1,NEL

if(T(I).LE.1.14 )then
        wj = anteg(z7)
  F111=-1.0/X(1)**2+WJ
      DER11=DER11+F111

          wj = anteg(z8)
    F112=WJ
        DER12=DER12+F112

          wj = anteg(z9)
    F113=WJ
        DER13=DER13+F113

  F222=-1.0/X(2)**2
       DER22=DER22+F222

          wj = anteg(z10)
  F223=WJ
       DER23=DER23+F223
          wj = anteg(z11)

  F333=WJ
       DER33=DER33+F333

else

          wj = anteg(z12)
  F1111=WJ
        DER111=DER111+F1111
```

```
          wj = anteg(z13)
      F1112=WJ
            DER112=DER112+F1112

          wj = anteg(z14)
      F1113=WJ
            DER113=DER113+F1113

          wj = anteg(z15)
      F2223=WJ
            DER223=DER223+F2223

          wj = anteg(z16)
      F3333=WJ
            DER333=DER333+F3333

      endif

110 continue

      F(4)=DER11+DER111
      F(5)=DER12+DER112
      F(6)=DER13+DER113
      F(7)=F(5)
      F(8)=DER22
      F(9)=DER23+DER223
      F(10)=F(6)
      F(11)=F(9)
      F(12)=DER33+DER333

c     Compute the Jacobian matrix

      aa(1,1)=F(4)
```

```
      aa(1,2)=F(5)
      aa(1,3)=F(6)
      aa(2,1)=F(7)
      aa(2,2)=F(8)
      aa(2,3)=F(9)
      aa(3,1)=F(10)
      aa(3,2)=F(11)
      aa(3,3)=F(12)

c     Compute -F(X)

      aa(1,4)=-F(1)
      aa(2,4)=-F(2)
      aa(3,4)=-F(3)

c     Solves the n X n linear system J(X) Y = -F(X)

        det=aa(1,1)*(aa(2,2)*aa(3,3)-aa(2,3)*aa(3,2))-
     $              aa(1,2)*(aa(2,1)*aa(3,3)-aa(2,3)*aa(3,1))+
     $              aa(1,3)*(aa(2,1)*aa(3,2)-aa(2,2)*aa(3,1))


        det1=aa(1,4)*(aa(2,2)*aa(3,3)-aa(2,3)*aa(3,2))-
     $              aa(1,2)*(aa(2,4)*aa(3,3)-aa(2,3)*aa(3,4))+
     $              aa(1,3)*(aa(2,4)*aa(3,2)-aa(2,2)*aa(3,4))


        det2=aa(1,1)*(aa(2,4)*aa(3,3)-aa(2,3)*aa(3,4))-
     $              aa(1,4)*(aa(2,1)*aa(3,3)-aa(2,3)*aa(3,1))+
     $              aa(1,3)*(aa(2,1)*aa(3,4)-aa(2,4)*aa(3,1))


        det3=aa(1,1)*(aa(2,2)*aa(3,4)-aa(2,4)*aa(2,3))-
     $              aa(2,1)*(aa(2,1)*aa(3,4)-aa(2,4)*aa(3,1))+
```

```
     $                    aa(1,4)*(aa(2,1)*aa(2,3)-aa(2,2)*aa(3,1))


        Y(1)=det1/det
        Y(2)=det2/det
        Y(3)=det3/det


        ERR=0.0
              do 9 I=1,M
              if(abs(Y(I)).LE.ERR) goto 9
        ERR=abs(Y(I))
9       continue

      if(ERR.LE.0.00001) goto 11
      do 10 I=1, M
             X(I)=X(I)+Y(I)
10       continue

      K=K+1
      goto 100
11    continue
      Write(12,'(i10,4(1x,1pg23.16),1x,3(1x,1pg23.16))')
     $          K,ERR, (x(i),i=1,3),(aa(1,j),j=1,3)
      Write(12,'(106x,1x,1pg23.16,1x,1pg23.16,1x,1pg23.16)')
     $          ((aa(i,j),j=1,3),i=2,3)
1000  continue
      stop
      end

      function z1(w)
      implicit real*8 (a-h,o-z)
      common /one/ a,b,c,ab,pi,x(3),n
      Z1=-X(2)*W**(X(1)-1)*dexp(-X(1)*X(3)*(C+AB*dsin(PI*W/6.0)))
```

```
$     -X(1)*X(2)*W**(X(1)-1)*dlog(W)*dexp(-X(1)*X(3)*
$     (C+AB*dsin(PI*W/6.0)))
$      +X(1)*X(2)*X(3)*(C+AB*dsin(PI*W/6.0))*W**(X(1)-1)*
$   dexp(-X(1)*X(3)*(C+AB*dsin(PI*W/6.0)))
 return
 end

 function z2(w)
 implicit real*8 (a-h,o-z)
 common /one/ a,b,c,ab,pi,x(3),n
 Z2=-X(1)*W**(X(1)-1)*dexp(-X(1)*X(3)*(C+AB*dsin(PI*W/6.0)))
 return
 end

 function z3(w)
 implicit real*8 (a-h,o-z)
 common /one/ a,b,c,ab,pi,x(3),n
 Z3=X(1)**2*X(2)*(C+AB*dsin(PI*W/6.0))*W**(X(1)-1)*
$ dexp(-X(1)*X(3)*(C+AB*dsin(PI*W/6.0)))
 return
 end

 function z4(w)
 implicit real*8 (a-h,o-z)
 common /one/ a,b,c,ab,pi,x(3),n
 Z4=-X(2)*W**(X(1)-1)*dexp(-X(1)*X(3)*(C+AB*dsin(PI*W/6.0)))
$     -X(1)*X(2)*W**(X(1)-1)*dlog(W)*dexp(-X(1)*X(3)*
$     (C+AB*dsin(PI*W/6.0)))
$      +X(1)*X(2)*X(3)*(C+AB*dsin(PI*W/6.0))*W**(X(1)-1)*
$   dexp(-X(1)*X(3)*(C+AB*dsin(PI*W/6.0)))
 return
 end

 function z5(w)
```

```
      implicit real*8 (a-h,o-z)
      common /one/ a,b,c,ab,pi,x(3),n
      Z5=-X(1)*W**(X(1)-1)*dexp(-X(1)*X(3)*(C+AB*dsin(PI*W/6.0)))
      return
      end

      function z6(w)
      implicit real*8 (a-h,o-z)
      common /one/ a,b,c,ab,pi,x(3),n
      Z6=X(1)**2*X(2)*(C+AB*dsin(PI*W/6.0))*W**(X(1)-1)*
     $ dexp(-X(1)*X(3)*(C+AB*dsin(PI*W/6.0)))
      return
      end

      function z7(w)
      implicit real*8 (a-h,o-z)
      common /one/ a,b,c,ab,pi,x(3),n
      Z7=-2*X(2)*W**(X(1)-1)*dlog(W)*
     $ dexp(-X(1)*X(3)*(C+AB*dsin(PI*W/6.0)))
     $    +2*X(2)*X(3)*(C+AB*dsin(PI*W/6.0))*W**(X(1)-1)*
     $ dexp(-X(1)*X(3)*(C+AB*dsin(PI*W/6.0)))
     $    +2*X(1)*X(2)*X(3)*(C+AB*dsin(PI*W/6.0))*W**(X(1)-1)*
     $ dlog(W)*dexp(-X(1)*X(3)*(C+AB*dsin(PI*W/6.0)))
     $ -X(1)*X(2)*W**(X(1)-1)*(dlog(W))**2*
     $  dexp(-X(1)*X(3)*(C+AB*dsin(PI*W/6.0)))
     $ -X(1)*X(2)*X(3)**2*(C+AB*dsin(PI*W/6.0))**2*
     $ W**(X(1)-1)*dexp(-X(1)*X(3)*(C+AB*dsin(PI*W/6.0)))
      return
      end

      function z8(w)
      implicit real*8 (a-h,o-z)
      common /one/ a,b,c,ab,pi,x(3),n
      Z8=-W**(X(1)-1)*dexp(-X(1)*X(3)*(C+AB*dsin(PI*W/6.0)))
```

```
$  -X(1)*W**(X(1)-1)*dlog(W)*
$   dexp(-X(1)*X(3)*(C+AB*dsin(PI*W/6.0)))
$       +X(1)*X(3)*(C+AB*dsin(PI*W/6.0))*W**(X(1)-1)*
$   dexp(-X(1)*X(3)*(C+AB*dsin(PI*W/6.0)))
 return
 end

 function z9(w)
 implicit real*8 (a-h,o-z)
 common /one/ a,b,c,ab,pi,x(3),n
 Z9=2*X(1)*X(2)*(C+AB*dsin(PI*W/6.0))*W**(X(1)-1)*
$ dexp(-X(1)*X(3)*(C+AB*dsin(PI*W/6.0)))
$    +X(1)**2*X(2)*(C+AB*dsin(PI*W/6.0))*W**(X(1)-1)*
$ dlog(W)*dexp(-X(1)*X(3)*(C+AB*dsin(PI*W/6.0)))
$ -X(1)**2*X(2)*X(3)*(C+AB*dsin(PI*W/6.0))**2*
$ W**(X(1)-1)*dexp(-X(1)*X(3)*(C+AB*dsin(PI*W/6.0)))
 return
 end

 function z10(w)
 implicit real*8 (a-h,o-z)
 common /one/ a,b,c,ab,pi,x(3),n
 Z10=X(1)**2*(C+AB*dsin(PI*W/6.0))*W**(X(1)-1)*
$ dexp(-X(1)*X(3)*(C+AB*dsin(PI*W/6.0)))
 return
 end

 function z11(w)
 implicit real*8 (a-h,o-z)
 common /one/ a,b,c,ab,pi,x(3),n
 Z11=-X(1)**3*X(2)*(C+AB*dsin(PI*W/6.0))**2*
$   W**(X(1)-1)*dexp(-X(1)*X(3)*(C+AB*dsin(PI*W/6.0)))
 return
 end
```

```
      function z12(w)
      implicit real*8 (a-h,o-z)
      common /one/ a,b,c,ab,pi,x(3),n
      Z12=-2*X(2)*W**(X(1)-1)*dlog(W)*
     $ dexp(-X(1)*X(3)*(C+AB*dsin(PI*W/6.0)))
     $    +2*X(2)*X(3)*(C+AB*dsin(PI*W/6.0))*W**(X(1)-1)*
     $ dexp(-X(1)*X(3)*(C+AB*dsin(PI*W/6.0)))
     $    +2*X(1)*X(2)*X(3)*(C+AB*dsin(PI*W/6.0))*W**(X(1)-1)*
     $ dlog(W)*dexp(-X(1)*X(3)*(C+AB*dsin(PI*W/6.0)))
     $ -X(1)*X(2)*W**(X(1)-1)*(dlog(W))**2*
     $  dexp(-X(1)*X(3)*(C+AB*dsin(PI*W/6.0)))
     $ -X(1)*X(2)*X(3)**2*(C+AB*dsin(PI*W/6.0))**2*
     $ W**(X(1)-1)*dexp(-X(1)*X(3)*(C+AB*dsin(PI*W/6.0)))
      return
      end

      function z13(w)
      implicit real*8 (a-h,o-z)
      common /one/ a,b,c,ab,pi,x(3),n
      Z13=-W**(X(1)-1)*dexp(-X(1)*X(3)*(C+AB*dsin(PI*W/6.0)))
     $  -X(1)*W**(X(1)-1)*dlog(W)*
     $   dexp(-X(1)*X(3)*(C+AB*dsin(PI*W/6.0)))
     $      +X(1)*X(3)*(C+AB*dsin(PI*W/6.0))*W**(X(1)-1)*
     $   dexp(-X(1)*X(3)*(C+AB*dsin(PI*W/6.0)))
      return
      end

      function z14(w)
      implicit real*8 (a-h,o-z)
      common /one/ a,b,c,ab,pi,x(3),n
      Z14=2*X(1)*X(2)*(C+AB*dsin(PI*W/6.0))*W**(X(1)-1)*
     $ dexp(-X(1)*X(3)*(C+AB*dsin(PI*W/6.0)))
     $    +X(1)**2*X(2)*(C+AB*dsin(PI*W/6.0))*W**(X(1)-1)*
```

```
$ dlog(W)*dexp(-X(1)*X(3)*(C+AB*dsin(PI*W/6.0)))
$ -X(1)**2*X(2)*X(3)*(C+AB*dsin(PI*W/6.0))**2*
$ W**(X(1)-1)*dexp(-X(1)*X(3)*(C+AB*dsin(PI*W/6.0)))
 return
 end

 function z15(w)
 implicit real*8 (a-h,o-z)
 common /one/ a,b,c,ab,pi,x(3),n
 Z15=X(1)**2*(C+AB*dsin(PI*W/6.0))*W**(X(1)-1)*
$ dexp(-X(1)*X(3)*(C+AB*dsin(PI*W/6.0)))
 return
 end

 function z16(w)
 implicit real*8 (a-h,o-z)
 common /one/ a,b,c,ab,pi,x(3),n
 Z16=-X(1)**3*X(2)*(C+AB*dsin(PI*W/6.0))**2*
$   W**(X(1)-1)*dexp(-X(1)*X(3)*(C+AB*dsin(PI*W/6.0)))
 return
 end

 function anteg(z)
 implicit real*8 (a-h,o-z)
 common /one/ a,b,c,ab,pi,x(3),n
 external z
 H = (B-A)/N
   WJ0 = Z(A) + Z(B)
 WJ1 = 0.0
 WJ2 = 0.0
 MM=N-1
 do 20 J=1,MM
      W = A+J*H
        if (J.EQ.2*(J/2)) then
```

```
              WJ2 = WJ2+Z(W)
             else
              WJ1 = WJ1+Z(W)
           endif
 20    continue
       WJ = WJ0+2*WJ2+4*WJ1
       anteg = WJ*H/3
       return
       end

       subroutine gnerate(alpha,beta,gamma,a,ab,c,nn,n,tj)
        implicit real*8 (a-h,o-z)
        external RNSET,DRNUN
        dimension tj(nn)
        F(X,c,ab,d,bb)=DEXP(-bb*d*(c+ab*SIN(PI*X/6.0)))*X**(d-1)
        HT=.01
        k =1
100     continue
          ISEED = ITIME()
          Call RNSET(ISEED)
          Call DRNUN(1,R)
          R1=-dlog(1-R)/(BETA*ALPHA)
          T0=0.0
          do 4 AJJ=1,NN,HT
             TJJ=T0+AJJ
             H=(TJJ-A)/N
             XI0=F(A,c,ab,alpha,gamma)+F(TJJ,c,ab,alpha,gamma)
             XI1=0.0
             XI2=0.0
             MM=N-1
             do 20 I=1, MM
                X=A+I*H
                if (I.EQ.2*(I/2)) then
                   XI2=XI2+F(X,c,ab,alpha,gamma)
```

```
             else
                XI1=XI1+F(X,c,ab,alpha,gamma)
             endif
20         continue
           XI=XI0+2*XI2+4*XI1
           XI=XI*H/3.0
      if (XI.Gt.R1) goto 5
4     continue
      goto 100
5     tj(k) = tjj
      k = k + 1
6     if (k .le. nn) goto 100
      return
      end
```

APPENDIX C

FORTRAN PROGRAM FOR FINDING THE INVERSE OF A MATRIX

APPENDIX C

FORTRAN PROGRAM FOR FINDING THE INVERSE OF A MATRIX

```
c     This program finds the inverse of a matrix

      double precision A(3,3),AINV(3,3),DET
      integer IPASS
      ND=3
      open(UNIT=4,FILE='matrix.txt',STATUS='OLD')
      do II=1, 334
      read (4, *)A(1,1),A(1,2),A(1,3),
     $           A(2,1),A(2,2),A(2,3),
     $      A(3,1),A(3,2),A(3,3)
       FACTOR=0.0
       DET=1.0
        do 1 I=1,ND
         do 1 J=1,ND
          if(I.EQ.J)then
          AINV(I,I)=1.0
                else
          AINV(I,J)=0.0
         endif

 1    continue
         do 9 IPASS=1,ND
         IMX=IPASS
           do 2 IROW=IPASS,ND
           if(dabs(A(IROW,IPASS)).GT.dabs(A(IMX,IPASS)))then
       IMX=IROW
```

```
          endif
2     continue
      if(IMX.NE.IPASS)then
        do 3 ICOL=1,ND
          TEMP=AINV(IPASS,ICOL)
            AINV(IPASS,ICOL)=AINV(IMX,ICOL)
            AINV(IMX,ICOL)=TEMP
          if(ICOL.GE.IPASS)then
          TEMP=A(IPASS,ICOL)
          A(IPASS,ICOL)=A(IMX,ICOL)
          A(IMX,ICOL)=TEMP
          endif
3     continue
           endif
           PIVOT=A(IPASS,IPASS)
           DET=DET*PIVOT
            if(DET.EQ.0.0)then
            write(*,10)
            stop
            endif
         do 6 ICOL=1,ND
         AINV(IPASS,ICOL)=AINV(IPASS,ICOL)/PIVOT
          if(ICOL.GE.IPASS)then
          A(IPASS,ICOL)=A(IPASS,ICOL)/PIVOT
          endif
6      continue
          do 8 IROW=1,ND
            if(IROW.NE.IPASS)then
            FACTOR=A(IROW,IPASS)
            endif
          do 7 ICOL=1,ND
          if(IROW.NE.IPASS)then
          AINV(IROW,ICOL)=AINV(IROW,ICOL)-FACTOR*AINV(IPASS,ICOL)
          A(IROW,ICOL)=A(IROW,ICOL)-FACTOR*A(IPASS,ICOL)
```

```
            endif
7         continue
8            continue
9            continue

        open(UNIT=5,FILE='matrix_inverse.txt',STATUS='new')
        write(5,*)AINV(1,1),AINV(1,2),AINV(1,3),
      $            AINV(2,1),AINV(2,2),AINV(2,3),
      $            AINV(3,1),AINV(3,2),AINV(3,3)
        enddo
        close(4)
        close(5)

10         format(5X,'--ERROR IN INVERSE--THE MATRIX IS SINGULAR')
           stop
           end
```

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