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Stochastic modeling of retail mortgage loans based on past due, prepaid, and default states

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STOCHASTIC MODELING OF RETAIL MORTGAGE LOANS BASED ON PAST DUE, PREPAID, AND DEFAULT STATES

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ABSTRACT

Stochastic models were developed that provide important measures related to retail mortgages and credit cards for the management of a bank. Based on Markov theory, two models were developed that predict mortgage portfolio size and expected duration of stay in each of the states, which are defined according to the criteria of Basel Accord II and the Federal Reserve Bank. Also, to facilitate comparisons among different types of credit products and different time periods, a model was developed to generate a health index for a retail mortgage. This model could be easily extended, using multivariate regression or multivariate time series techniques, to analyze the interaction between a mortgage and local macroeconomic factors. Furthermore, the models in this dissertation address decision making on the part of the management of a bank concerning business strategy such as collection policies and loan officer compensation policies. Extending the basic assumption of the Markov property to a higher-order Markov model and a multivariate Markov model, this work also analyzed the correlation between the payment pattern for retail mortgages and credit cards. To complete this correlation analysis, a comparison among 3 models (higher-order, multivariate, and a higher-order multivariate Markov model (HMMM)) has also been provided. Finally, an interaction analysis between the payment behavior of a retail mortgage and local macroeconomic variables has been performed using an Interactive Hidden Markov Model (IHMM). For IHMM and HMMM models, the number of unknown parameters increases exponentially with the

increase of the order of the models. Hence, to deal with this situation, a linear programming algorithm has been used to obtain solutions for the HMMM and IHMM.

The models provided in this study are of practical importance to the bank management. Not only do they give quantitative measures about loan stand-alone characteristics, but also they provide cross-section comparisons among different credit products and multi-period loan performance tracking as well. These models, used to analyze retail mortgages and credit cards, could be easily applied to other credit products issued by a commercial bank.

The data used in this study have been obtained from an Ohio local commercial bank. It includes monthly paid 20-year retail mortgages and personal credit cards. A contract has also been signed to guarantee that the data would be used only for academic research.

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CHAPTER 1

INTRODUCTION

In this chapter, we present an overview of the models, based on Markov chain theory, used for analyzing the transition probability between defined states. These models are classified into two categories: models for consumer credit analysis and models for loan analysis. Models in both categories assume that the transition process is between defined finite states. The finite states of the Markov chain have four absorbing states, namely the collection of bad debt, prepayment, default, and bankruptcy. Aided by the dynamic programming technique, these models can either maximize outcome (benefits) or minimize the cost, including the collection cost and actual losses.

1.1 Literature Review

There are many quantitative methods in credit asset management. White (1993) surveyed some models employed in the banking industry. The models include discriminant analysis, decision tree, expert system for static decision, dynamic programming, linear programming, and Markov chains for dynamic decision models. Which model is best depends on the situation and the purpose of the analysis.

However, in the analysis of credit risk and selection of optimal policy, the standard approach is to use stochastic models based on Markov transition matrices, aided by dynamic programming. As summarized by White (1993), Markov decision models have been mainly used in 18 areas, including (1) Finance and Investment, (2) Insurance, and (3) Credit Analysis. Of the 98 papers discussed by White, 9 papers relate to finance and investment, 2 to insurance, and 2 to credit analysis. This survey is by no means comprehensive, but it reveals the fact that Markov chains have been used extensively to analyze real world data.

1.2 Markov Models

General concepts of Markov processes are presented in Ross (1996). Let $\pi_{i,j}$ be the steady state probability or limiting probability of being in state i and adapting policy j , $\pi_{i,j} = \lim_{n\to\infty} P_j\{X_n = i, j\}$, where $X_n, n = 1,2,3...n$ is defined as the states of a Markov chain. Then, the expected benefit is given as

$$\sum_i \sum_j \pi_{ij}[R(i,j) - C(i,j)] \qquad (1.1)$$

where, $R(i,j), C(i,j)$ are defined as the reward function and cost function for being in state i and adopting policy j, respectively. Also, dynamic programming could be used to find an optimal policy j to maximize the expected benefit. To this end, one may maximize

$$\sum_i \sum_j \pi_{ij}[R(i,j) - C(i,j)]$$

$$\text{Subject to } \pi_{i,j} \geq 0, \text{ and} \sum_i \sum_j \pi_{ij} = 1. \qquad (1.2)$$

Consumer credit analysis is used to analyze account receivable, as triggered by credit sales. The model, based on the transition probability between different states, is primarily used by a company to adjust its credit sale and collection policy. Absorbing states could be reached either by collection or bad debt, both of which lead to a decline in the portfolio size.

On the other hand, by defining a past-due period as a different transient state, and default as an absorbing state, Markov models are used to analyze the characteristics of a loan portfolio, namely the estimated duration before an individual default, prediction of economic portfolio balance, and health index. The primary purpose of this research is to develop this type of model for banks and other commercial lending institutes in order to analyze the nature of their products.

1.2.1 Markov Models for Consumer Credit Analysis

Cyert, Davison, and Thompson (1962) developed a finite stationary Markov chain model to predict uncollectible amounts (receivables) in each of the past due category. This classic model is referred to as CDT model. The states of the chain (S_j, $j = 0,1,2,...,J$) were defined as normal payment, past due, and bad-debt states. The probability P_{ij} of a dollar in state i at time t transiting to state j at time t + 1 is given as

$$P_{ij} = \frac{B_{ij}}{\sum_{m=0}^{J} B_{im}}, \qquad (1.3)$$

where B_{ij} is the amount in state j at time t+1 which came from state i in the previous period. $S_t = S_o Q^t$ is the vector whose j th component is the amount outstanding for the j th past due category at the beginning of the t th period for $t = 1, 2, ...,$. Here, Q is a sub-matrix, in the transition probability matrix $P_{ij} = \begin{bmatrix} I & O \\ R & Q \end{bmatrix}$, which includes transition probabilities among the set of transient states.

Criticizing the appropriateness of the stationary Markov chain model by Cyert et al. (1962), and Frydman et al. (1968) applied a mover-stayer Model as an alternative. They defined the j step transition matrix of this model as $P(0, j) = SI + (I - S)M^j$, where $M = \{m_{ik}\}$ is a transition probability matrix for "movers" from i to k, and S = diag $(s_1, s_2, .., s_w)$ represents the probability of "stayers" in state i. The maximum likelihood estimator for m_{ik} is given as $\tilde{m}_{ik} = (n_{ii} - Jn_i)/(n_i^* - Jn_i)$, where n_i is the number of observations that stay continuously in state i during the period. They concluded that the mover-stayer model is better for empirical analysis than the stationary Markov chain model

The model of Cyert, Davison, and Thompson (1962) was also challenged by Corcoran (1978). He claimed that the representiveness of the transition probability could be affected by the fact of "dominancy of large accounts". Therefore, he suggested grouping the accounts according to their size, and then a transition matrix for each group was provided by an exponentially smoothed matrix: $A_j = 0.8T_j + (1 - 0.8)A_{j-1}$, where A_j is an exponentially smoothed matrix for month j and T_j is the transition matrix for month j.

Kuelen and Corcoran (1981) published their study on the CDT model and claimed that there was a flaw in the model because it failed to consider the partial payments for accounts due. By using the "total balance method", CDT understated the collection, and thus overestimated bad debts. A simple remedy, other than model structure modification, was to treat a partially paid amount and remainder balance separately. As a result, an exact agreement with total receipts and aging could be achieved.

1.2.2 Markov Models for Loan Analysis

According to Thompson (1965), one of two important related tests for a bank's credit asset from the lender's point of view is the possibility of the loan getting into trouble, which means the probability of being in a past-due or even charged-off state. Another test is the extent of loss in the case of being in trouble. This could mean two things: (1) the recovery from collateral in the case of being charged off, or (2) the ability for an individual to bring himself back on track. Also, in the same paper, Thompson provided evidence supporting his claim that the business cycle and the macroeconomic situation are probably the most significant factors affecting change in bank credit.

Liebman (1972) built a Markov decision model for selecting optimal credit control policies based on transition probabilities and the costs to correspond to customers belonging to each of the categories. The basic idea of his model is to select a credit strategy d_{ikm} such that the total discounted expected cost in the next period and all succeeding periods is as follows:

$$C = \sum_{ikms} \pi_{ikm} d_{ikm} \sum_{j\ln} P_{ikm,j\ln} c_{ikm,j\ln} , \tag{1.4}$$

where, $P_{ikm,j\ln}$ is the Markov transition probability defined as the probability of an account moving from age class i, charge volume class k and previous experience class m, to age class j, charge volume class l and previous experience class n. $c_{ikm,jlm}$ is the cost matrix, π_{ikm} is the steady state probability. A linear program was used to optimize and solve the optimal credit policies d_{ikm}.

Rai, Kirkham, and Clarke (1979), by assuming that new customers behave in a way similar to existing customers, implemented a Markov Model:

$$y(t) = \sum y(t-1)p_{ij} + u(t), i, j = 1,2,..,r; t = 1,2,..,T \qquad (1.5)$$

where $y(t)$ is the observed liability shares of deposit-taking institutes at time t, p_{ij} is the Markov transition probability from state i to state j, and $u(t)$ is a disturbance term to analyze the growth rates of Canadian deposit-taking institutes because of the implication of the Bank Act in 1967. The authors showed that the model was appropriate whenever growth was dominated by macro economic factors and technical innovations.

Howard and Matheson (1972) implemented a Markov model which could be useful in forming optimum buying and selling strategies for a commodity market. They justified the model by incorporating a risk-sensitivity function. A positive or negative risk coefficient was assigned to the function based on whether the bank management is risk aversive or risk preferring, respectively. The exponential function representing the overall risk preference is given by: $u(v) = -(\operatorname{sgn}\gamma)e^{-\gamma v}$, where γ is the risk aversion coefficient, and sgn γ denotes the sign of γ. Then, the following iterative scheme was provided to reach the maximum benefits through optimal policy:

$$e^{-\gamma(-\frac{1}{\gamma}\ln\lambda+-\frac{1}{\gamma}\ln[-(\operatorname{sgn}\gamma)u_i])}=\sum_{j=1}^{N}p_{ij}e^{-\gamma(r_{ij}+-\frac{1}{\gamma}\ln[-(\operatorname{sgn}\gamma)u_i])}$$

$$\text{Choose policy } k \text{ to maximize } V_i^k=-\frac{1}{\gamma}\ln[\sum_{j=1}^{N}p^k_{ij}e^{-\gamma(r^k_{ij}+-\frac{1}{\gamma}\ln[-(\operatorname{sgn}\gamma)u_i])}], \quad (1.6)$$

where, p_{ij} is the Markov transition probability.

By taking economic factors into account, Richard (1983) used a finite Markov chain model to analyze a firm's market value if the firm follows an optimal policy in state (x, y) at time t, where x is the condition of the firm, and y is the condition of the overall economy. He assumed that the changes in state are governed by a stationary transition function. For instance, if the state is $y(t-1)$ at time $t-1$, then it will be $y(t)$ at time t with probability $\pi[y(t)y(t-1)]$. However, to calculate V_i^k, he used dynamic programming because direct computation could be very time-consuming.

Jarrow et al. (1997) applied a continuous and a discrete time Markov chains to describe the default behavior of zero-coupon bonds within a time interval $\eta_t(0 \le t \le \tau)$. Furthermore, the default state was defined as an absorbing state. Again, the purpose was to price the bond based on analysis of credit risk spread. Similar approaches have been adopted by Liebman (1972) and Zipkin (1993). Lieberman used a Markov chain to model decision-making for credit card application approval. The states of the chain were $n+1$ paid-up states and one default state. The model assumed that the amount of dollars moved from one state to another follows a Markov chain process. On the other hand, Zipkin adopted a simpler model of interest rate, based on a discrete-time, finite-state Markov chain, to evaluate mortgage-backed securities. Glennon and Nigro (2005) used the survival analysis approach to measure the default risk of a small business. They adopted the Cox Proportionally Hazard model. By using a discrete-time hazard procedure, they found that the default risk peaked in the second year after initiation, increased during the medium-maturity season, and declined thereafter.

Numerous efforts have been undertaken to analyze the relationship between credit asset quality and the macroeconomic situation. Lee (1995) built an ARMA model:

$$\psi_y(L)y_{t+1} = \theta_y(L)e_{t+1.y}, \psi_m(L)m_{t+1} = \theta_m(L)e_{t+1.m}, \tag{1.7}$$

under the assumptions that y_t and m_t have univariate stationary, invertible finite-order ARMA representation. The model was used to analyze the linkage between time-varying risk premia in the term structure and macroeconomic state variables. He concluded that uncertainties, related to output and the money supply, are important source of time-varying risk premia in the nominal term structure of interest rate.

Esbitt (1986) provided empirical evidence that a bank's portfolio quality has close relationship with the macroeconomic situation. Examples include the state-chartered banks' failure and Great Depression in Chicago between 1930 and 1932.

A promising model to link macroeconomic variables to a microeconomic variable is to use the Markov chain representation. It is also called the State Space representation, which is based on the idea that the future of a system is independent of its past (Wei 1990).

The estimation of a state space model's parameters is difficult. Cooper and Wood (1981) used Maximum Likelihood to estimate the parameters. Outliers in the series could make the problem even more complicated. As pointed out by Balke and Fomby (1994), there are 3 possible outlier patterns: (1) Outliers associated with business cycles, (2) outliers clustered together, both over time and across series, and (3) a dichotomy between outlier behaviors of real versus nominal series.

The ETS package in SAS® provides a method to check and remove outliers and to estimate the parameters of the state space model (*SAS Online Doc 2005 version (2005)).*

1.3 Extensions of Markov Chains Models

The basic property of a Markov chain, namely

$$\Pr(X_{n+1} = x \mid X_n = x_n, \ldots, X_1 = x_1, X_0 = x_0) = \Pr(X_{n+1} = x \mid X_n = x_n), \tag{1.8}$$

where $X_0, X_1, \ldots, X_n$ is a sequence of random variables, has been extended to accommodate many new applications. Among them are traffic analysis in the network, speech recognition, DNA sequences analysis, engineering designs, and inventory management. Also, new theories extending the basic Markov assumption have been developed in the past 50 years, such as High-order Markov chains, Multivariate Markov chains, and Hidden Markov chains. These important developments are introduced in the following subsections.

1.3.1 Higher-order Markov chains

Higher-order Markov chains assume that not only the immediate past random variable but also the past k variables, or kth order, have significant effects on the current one. That is, $\Pr(X_{n+1}=x \mid X_n=x_n,\ldots,X_1=x_1,X_0=x_0) \neq \Pr(X_{n+1}=x \mid X_n=x_n)$. It is difficult to solve the problem directly because the number of parameters to estimate increases exponentially with the order of the model.

Wang (1992) showed that it needs 7 parameters to completely specify the transition probabilities of a second-order two state Markov chain:

$$\begin{aligned}
&\Pr(X_{i+1}=1 \mid X_{i-1}=0, X_i=0)=\alpha_1,\\
&\Pr(X_{i+1}=1 \mid X_{i-1}=1, X_i=0)=\alpha_2,\\
&\Pr(X_{i+1}=0 \mid X_{i-1}=0, X_i=1)=\beta_1,\\
&\Pr(X_{i+1}=0 \mid X_{i-1}=1, X_i=1)=\beta_2,\\
&\Pr(X_2=0 \mid X_1=1)=z,\\
&\Pr(X_1=1)=\tau
\end{aligned} \tag{1.9}$$

Generally, one can verify that an $k \cdot th$ order sequence with S states will have $(S-1)\cdot S^k$ parameters. Thus, industrial application of higher-order Markov chains has been hampered by this problem. Raftery (1985), however, proposed a higher-order Markov chain model with only one parameter for each extra lag. By assuming that $\sum_{i=1}^{k}\lambda_i=1; \lambda_i \geq 1, i=1,2,\ldots,k$, his model is expressed as

$$\begin{aligned}
&P[X_t=j_0 \mid X_{t-1}=j_1,\ldots,X_{t-k}=j_k]=\sum_{i=1}^{k}\lambda_i q_{j_0 j_i}\\
&0 \leq \sum_{i=1}^{k}\lambda_i q_{j_0 j_i} \leq 1
\end{aligned} \tag{1.10}$$

Letting $X_t=(x_t(1),\ldots x_t(m))'$, $x_t(j)=1$, if $X_t=j$ and equal to 0 otherwise, and $\hat{X}_t=(\hat{x}_t(1),\ldots\hat{x}_t(m))'$, where the random variable $\hat{x}_t(j)$ is a function of past values and could be represented as: $P[X_t = j_0 \mid X_{t-1} = j_1, ..., X_{t-k} = j_k]$, then the model in matrix form is given as

$$\hat{X}_t = \sum_{i=1}^{k} \lambda_i Q X_{t-i} \tag{1.11}$$

To estimate the parameters, Raftery (1985) applied the maximum log-likelihood technique

$L = \sum_{i_0,...i_k=1}^{m} n_{i_0,...,i_k} \log(\sum_{j=1}^{k} \lambda_j q_{i_0,i_j})$,where $n_{i_0,...,i_k} = \sum_t x_t(i_0)x_{t-1}(i_1)...x_{t-k}(i_k)$. He applied this method to a 4th order model in analyzing the wind power in a wind turbine design problem. By comparing model results for different orders, he concluded that the 4th order was the best model as it gave the smallest Bayesian information criterion (BIC) value, where $BIC = -2L + k \log n$.

Another Higher-order model was proposed by Ching and Ng (2006). Assuming $\lambda_i, i = 1,2...,k$ are non-negative and $\sum_{i=1}^{k} \lambda_i = 1$, Ching and Ng generalized Raftery's model by allowing the transition intensity matrix Q to vary with different lags. Written in matrix form, Ching and Ng's model could be expressed as

$$X^{(n+k+1)} = \sum_{i=1}^{k} \lambda_i Q_i X^{(n+k+1-i)} \tag{1.12}$$

if we let $Q_1 = Q_2 = \ldots = Q_k$, Ching and Ng's model in (1.12) reduces to Raftery's model in (1.11). They used linear programming method to estimate the parameters which could be done in Microsoft Excel® with the built ***Solver()*** function:

$$\begin{cases} Min_\lambda \{ \left\| \sum_{i=1}^{k} \lambda_i V_i X - X \right\|_l \}, \\ \text{Subject to} \sum_{i=1}^{k} \lambda_i = 1, \lambda_i \geq 0 \end{cases} \quad (1.13)$$

where $\|\square\|_l$ is a vector norm, and $l \in \{1, 2, \ldots, \infty\}$. Their model could be used to solve the well-known Neysbody's problem in management science.

1.3.2 Multivariate Markov Chains

Multivariate Markov chains are useful in correlation analyses related to data sequences and for predicting the future outcome of a random variable based on the identified correlations.

Ching and Ng (2003) applied a Multivariate Markov chain model to a multi-product demand estimation problem. Their model is expressed as

$$\begin{cases} X_{n+1} = \sum_{k=1}^{s} \lambda_{jk} V^{jk} X_n, j = 1, 2, \ldots, s \\ \lambda_{jk} \geq 1, 1 \leq j, k \leq 2, \\ \sum_{k=1}^{s} \lambda_{jk} = 1 \end{cases} \quad (1.14)$$

In this model, the parameter λ_{jk} gives the direction and magnitude of the correlation in the model outcome. V^{jk} is the transition intensity matrix from the states in the *jth* sequence to the states in the *kth* sequence, and X_n^k is the observed state probability distribution of the *kth* sequence at time n.

Siu and Fung (2005) used a Multivariate Markov chain model to analyze credit rating. In matrix form, their model is given as

$$X_{n+1} = \begin{pmatrix} X_{n+1}^1 \\ X_{n+1}^2 \\ \vdots \\ X_{n+1}^s \end{pmatrix} = \begin{pmatrix} \lambda_{11}V^{11} & \lambda_{12}V^{12} & \cdots & \lambda_{1s}V^{1s} \\ \lambda_{21}V^{21} & \lambda_{22}V^{22} & \cdots & \lambda_{2s}V^{2s} \\ \vdots & \vdots & \vdots & \vdots \\ \lambda_{s1}V^{s1} & \lambda_{s2}V^{s2} & \cdots & \lambda_{ss}V^{ss} \end{pmatrix} \begin{pmatrix} X_n^1 \\ X_n^2 \\ \vdots \\ X_n^s \end{pmatrix}, \tag{1.15}$$

where V^{jk} is the transition intensities defined as in Ching and Ng's model. Also, they proved that if the intensity matrix V is irreducible, the model in (1.15) could be expressed as $\sum_{k=1}^{s} \lambda_{jk} V^{jk} X_n - X_{n+1}$. Letting Q_{jk} denote the prior transition matrix, the parameters λ_{jk} may be estimated based on the following expression:

$$\begin{cases} Min_{\lambda}\{Max_i\{|[\sum_{k=1}^{m}(\lambda_{jk}^1 Q_{jk} + \lambda_{jk}^2 V_{jk})X^k - X^j]|_i\}\} \\ Subject\ \text{to} \sum_{k=1}^{m}(\lambda_{jk}^1 + \lambda_{jk}^2) = 1, \lambda_{jk}^1 \geq 0, \lambda_{jk}^2 \geq 0 \end{cases} \tag{1.16}$$

It is possible to combine a multivariate Markov chain with a higher-order Markov chain. As used by Ching and Ng (2004), the model considers the correlation between sequences as well as the time lags within a single data sequence.

1.3.3 Hidden Markov Chains

Although higher-order Markov Models might provide more accurate results (in the sense that they can generally produce Chi-square statistics, $\chi^2 = \sum_{i=1}^{S} \frac{(E_i - O_i)^2}{E_i}$, where E_i is the calculated stationary probability distribution and O_i is the observed probability distribution; Ching and Ng (2006), they fail to take into consideration underlying forces that may determine observed transition processes in real-world problems. Examples include speech recognition, stock market analysis, and network traffic analysis. All these problems could be solved by Hidden Markov Models, or HMM. A standard HMM has the following elements: (1) N, the number of hidden states, $H = \{H_1, H_2, ..., H_N\}$, (2) l, the number of observable states, $S = \{S_1, S_2, ..., S_l\}$, (3) A, the transition probability distribution within hidden states, $A = \{a_{ij}\}, a_{ij} = P(H_{j,t=n} \mid H_{i,t=n-1}), 1 \le i, j \le N$, (4) B, the emission probabilities matrix, $B = \{b_{jk}\}$, where $b_{jk} = P(S_k \mid H_j), 1 \le j \le N, 1 \le k \le l$, and (5) Π, the initial state distribution, $\Pi = \{\pi_i\}, \pi_i = P(S_i), 1 \le i \le N$. Thus, an HMM is completely specified by: $\Lambda = (A, B, \Pi)$. As pointed out by MacDonald and Zucchini (1997), HMM could be used to answer the following three classic problems:

Problem (1): Given an observation sequence $S = \{S_1, S_2, ..., S_l\}$ and a model $\Lambda = (A, B, \Pi)$, how do we efficiently compute $B = \{b_{jk}\}$,

Problem (2): Given an observation sequence $S = \{S_1, S_2, ..., S_l\}$ and a model $\Lambda = (A, B, \Pi)$, how do we choose the corresponding state sequence $A = \{a_{ij}\}$ which best explains the observations,

Problem (3): How do we adjust the model parameters $\Lambda = (A, B, \Pi)$ to maximize $P(S/\lambda)$.

Many algorithms are used to efficiently solve these problems, including forward algorithm, backward algorithm, EM algorithm, and a heuristic linear programming method for higher-order HMM proposed by Ching and Ng (2006). For the sake of conciseness, we present only the method by Ching and Ng, which is applied to the retail mortgage model in Chapter 6.

Replacing X by $\hat{H}$ in Eq. (1.13) one has

$$\begin{cases} Min_{\lambda_i} \{ \left\| \sum_{j=1}^{k} \lambda_j V_j \hat{H}_i - \hat{H}_i \right\|_l \}, i = 1, 2, \\ \text{Subject to} \sum_{i=1}^{k} \lambda_i = 1, \lambda_i \geq 0 \end{cases} \tag{1.17}$$

where the λ_i's are the expected parameters, $\hat{H}_i$ is the estimated stationary probability distribution, and V_j is the higher-order transition matrix defined as A at the beginning of this subsection.

Comparisons between the EM algorithm and the linear programming method for different orders are presented in Tables 1.1 and 1.2.

Table 1.1 Comparison by the number of iterations.

	First-Order	Second-Order	Third-Order
Linear Programming	1381	1378	1381
EM Algorithm	1377	1375	1377

Table 1.2 Comparison by computation time in seconds.

	First-Order	Second-Order	Third-Order
Linear Programming	1.16	1.98	5.05
EM Algorithm	4.02	12.88	40.15

It is seen from these tables that although there is not much difference between linear programming and the EM algorithm with regard to the number of iterations, the linear programming method is better than the EM algorithm regarding computation time, especially for a higher order. In chapter 6, the linear programming method will be applied to retail mortgage data provided by an Ohio local commercial bank.

CHAPTER 2

A MARKOV CHAIN MODEL FOR RETAIL MORTGAGE LOANS AND CREDIT ASSETS

In this chapter, a continuous time and a discrete time Markov chains are developed for modeling the duration of retail loans in prepayment, past due, and default states. The default state is defined as charge-off on the loan due to bankruptcy, death, or other causes. As such, it uses the economic status of the loan, rather than the accounting assets status. Prepayment and past due states describe the payment status of a loan. A bank could use this model to approximate its contingent assets status based on the probability and duration of being in non-default states. Therefore, the bank can gain a picture of its credit assets quality. On the other hand, the book amount of a bank's credit portfolio on its financial statement seldom reflects its real economic status due to the nature of book keeping, which only provides a static snap-shot of a bank's operation result. Furthermore, the book amount fails to give the management a true picture of the portfolio pool, which is a function of its contraction and is based on past due rate, default rate, and prepayment rate. To remedy this situation, a stochastic model based on a Markov Chain is used to analyze contraction and extension, which gives a true economic picture of a bank's credit portfolio and, thus in turn, facilitates the pricing of the bank's securities.

2.1 Model

In the Markov chain model, let S_j be a state of past due, corresponding to the days of past due. The loan normally requires monthly payment. If a loan is 30 days past due, denote it by S_1. State S_2 refers to 60 days past due. According to the Basel accord II, Basel Committee on Banking Supervision (1997), the definition of default is more than 90 days past due, which is represented by S_3. However, there have been cases where the obligations on a loan, which have already been more than 90 days past due, has been paid off. As a result, the definition of default is modified to be the state of default that is triggered by a permanent force, such as death or an application of chapter 7 or chapter 13 bankruptcy protections. Let R_i be the default state contributed by these permanent events and let S_{-i} be the state of a prepaid period defined as $S_{-j} = \frac{X_i - Y_i}{Y_i}$, where X_i is the actual payment at month i and Y_i is the scheduled payment at month i. One can see that state S_{-j} is defined as the extra payment over the scheduled payment, which measures how many future monthly payments have been made as a current onetime payment. It is not a precise measurement method, compared with the tools introduced by other papers in the literature, but it fits best in the context of this model. Definitions for classifying the states of Markov chain are given in Table 2.1.

Table 2.1 Definitions of the different states of the Markov chain.

Past Due and Prepayment States $S_j, j=-3,-2,-1,0,1,2,3$		Default States R_k $R_k, k=1,2,3,4$	
S_{-3}	Prepaid More than 91 days	R_1	Sold by Bank
S_{-2}	Prepaid 61 days – 90 days	R_2	Foreclosure
S_{-1}	Prepaid 31 days – 60 days	R_3	Refuse to pay
S_0	No more than 30 days past due	R_4	All others
S_1	31 days – 60 days past due		
S_2	61 days – 90 days past due		
S_3	More than 91 days past due		

The salient feature of this model is the evaluation of loan assets behavior over time, which is more informative than the traditional accounting financial reports.

First, we define the time interval to be $(0,t), t<\infty$. The transitions within the S-states are defined by (Chiang, 1980):

$v_{ij}\Delta t =$ Pr {an individual in state S_i at time τ will be in state S_j at time $\tau+\Delta t$ }, where $i \neq j; i,j=-3,-2,-1,0,1,2,3; i \neq j$

$\mu_{ik}\Delta t =$ Pr {an individual in state S_i at time τ will be in state R_k at time $\tau + \Delta t$ }, where, $i = -3,-2,-1,0,1,2,3$ and k refers to the default states, $k = 1,2,3,4$.

Furthermore, we assume that future transitions of an individual are independent of past transitions. In other word, the intensities v_{ij} and μ_{ik} are assumed to be independent of time τ. For $0 \le \tau \le t$. Thus, we are concerned here with a time homogenous Markov chain.

If an individual stays in its original state, its intensity is defined as

$v_{ii} = -(v_{ij} + \sum_{\delta=1}^{4} u_{jk}), i \ne j, i, j = -3,-2,-1,0,1,2,3, k = 1,2,3,4$. By this definition, it is obvious that

$1 + v_{ii}\Delta t =$ Pr {an individual in state S_i at time τ will be in state S_i at time $\tau + \Delta t$ }. Within any single time interval, { $\tau + \Delta t$ }, V is the prepayment and past due intensity matrix, while U is the default intensity matrix:

The matrix of transition intensities between the S-states (prepayment and past due states) is given by the V matrix in Figure 2.1. Also, the U matrix in Figure 2.1 represents the transition intensities from the S-states to the default states:

$$V = \begin{array}{c} \\ S_{-3} \\ S_{-2} \\ S_{-1} \\ S_0 \\ S_1 \\ S_2 \\ S_3 \end{array} \begin{array}{c} \begin{array}{ccccccc} S_{-3} & S_{-2} & S_{-1} & S_0 & S_1 & S_2 & S_3 \end{array} \\ \begin{bmatrix} v_{-3,-3} & v_{-3,-2} & v_{-3,-1} & v_{-3,0} & v_{-3,1} & 0 & 0 \\ v_{-2,-3} & v_{-2,-2} & v_{-2,-1} & v_{-2,0} & v_{-2,1} & 0 & 0 \\ v_{-1,-3} & v_{-1,-2} & v_{-1,-1} & v_{-1,0} & v_{-1,1} & 0 & 0 \\ v_{0,-3} & v_{0,-2} & v_{0,-1} & v_{0,0} & v_{0,1} & 0 & 0 \\ v_{1,-3} & v_{1,-2} & v_{1,-1} & v_{1,0} & v_{1,1} & v_{1,2} & 0 \\ v_{2,-3} & v_{2,-2} & v_{2,-1} & v_{2,0} & v_{2,1} & v_{2,2} & v_{2,3} \\ v_{3,-3} & v_{3,-2} & v_{3,-1} & v_{3,0} & v_{3,1} & v_{3,2} & v_{3,3} \end{bmatrix} \end{array}$$

$$U = \begin{array}{c} \\ S_{-3} \\ S_{-2} \\ S_{-1} \\ S_0 \\ S_1 \\ S_2 \\ S_3 \end{array} \begin{array}{c} \begin{array}{cccc} R_1 & R_2 & R_3 & R_4 \end{array} \\ \begin{bmatrix} \mu_{-3,1} & \mu_{-3,2} & \mu_{-3,3} & \mu_{-3,4} \\ \mu_{-2,1} & \mu_{-2,2} & \mu_{-2,3} & \mu_{-2,4} \\ \mu_{-1,1} & \mu_{-1,2} & \mu_{-1,3} & \mu_{-1,4} \\ \mu_{0,1} & \mu_{0,2} & \mu_{0,3} & \mu_{0,4} \\ \mu_{1,1} & \mu_{1,2} & \mu_{1,3} & \mu_{1,4} \\ \mu_{2,1} & \mu_{2,2} & \mu_{2,3} & \mu_{2,4} \\ \mu_{3,1} & \mu_{3,2} & \mu_{3,3} & \mu_{3,4} \end{bmatrix} \end{array}$$

Figure 2.1 Transition intensities within the S-states and default state (U matrix).

Because that R_k is an absorbing state, there is no transition from an R to an S-state. Also, for a past due state, transition lies only between neighboring states. This result is obvious since within one month, a loan with no past due payment cannot have a two-month due payment. On the other hand, because a prepayment can neither be deductible from nor replaceable by the next payment, a prepayment state can jump to any other prepayment state. At the same time, any past due state, $S_i, i > 0$, can transfer to $S_i, i \le 0$ through prepayment.

2.1.1 A Continuous Markov Model

Let $P_{ij}(\tau,t)$ = Pr {an individual in state S_i at time τ will be in state S_j at time t}, $i,j=-3,-2,-1,0,1,2,3$. By definition, we have

$$\begin{aligned} P_{ij}(t,t+\Delta t) &= v_{\gamma j}(t)\Delta t \\ P_{jj}(t,t+\Delta t) &= 1+v_{jj}(t)\Delta t \end{aligned} \tag{2.1}$$

$$P_{ij}(\tau,t+\Delta t)=P_{ij}(\tau,t)P_{jj}(t,t+\Delta t)+\sum_{\gamma\neq j}P_{i\gamma}(\tau,t)P_{\gamma j}(t,t+\Delta t)\ . \tag{2.2}$$

By substituting Eq. (2.1) in Eq. (2.2) and rearranging, we have

$$\begin{aligned} &\frac{P_{ij}(\tau,t+\Delta t)-P_{ij}(\tau,t)}{\Delta t}=P_{ij}(\tau,t)v_{jj}(t)\Delta t+\sum_{\gamma\neq j}P_{i\gamma}(\tau,t)v_{\gamma j}(t) \\ &\Rightarrow \lim_{\Delta\to 0}\frac{P_{ij}(\tau,t+\Delta t)-P_{ij}(\tau,t)}{\Delta t}=\sum_{\gamma\neq j}P_{i\gamma}(\tau,t)v_{\gamma j}(t) \end{aligned}$$

$$\Rightarrow \frac{\partial}{\partial t}P_{ij}(\tau,t)=\sum_{\gamma\neq j}P_{i\gamma}(\tau,t)v_{\gamma j}(t);i,j=-3,-2,-1,0,1,2,3 \tag{2.3}$$

Equation (2.3) the Kolmogorov Forward Differential Equation, and its solution is given (Chiang, 1980) as

$$P_{ij}(0,t)=\sum_{l=-3}^{3}\frac{A'_{ij}(\rho_l)}{\prod_{\substack{m=-3\\ m\neq l}}^{3}(\rho_l-\rho_m)}e^{\rho_l t},i,j=-3,-2,-1,0,1,2,3 \tag{2.4}$$

Here, A_{ij}' is the characteristic matrix of V', the transpose of the intensity matrix V, defined by

$$A_{ij}' = (\rho I - V'), \tag{2.5}$$

where ρ_l = Eigenvalue of the intensity matrix V.

For an individual in S_i at time 0, let $e_{ij}(t)$ = the expected duration of stay in S_j during the interval $(0,t), j=-3,-2,-1,0,1,2,3$. In terms of our process, $e_{ij}(t)$ evaluates the expected duration of the loan before default occurs. This expected duration, $e_{ij}(t)$, can be expressed (Chiang, 1980) as

$$e_{ij}(t) = \int_0^t P_{ij}(0,\pi)d\pi \tag{2.6}$$

$$e_{ij}(0,t) = \sum_{l=-3}^{3} \frac{A_{ij}'(\rho_l)}{\prod_{\substack{j=-3 \\ j\neq l}}^{3} (\rho_l - \rho_j)\rho_l} (e^{\rho_l t} - 1), i, j = -3,-2,-1,0,1,2,3 \tag{2.7}$$

2.1.2 A Discrete Time Markov Chain Approach

Expression (2.7), which represents the expected duration of stay in S_j, could be difficult to evaluate because of its relative complexity. Equivalent estimates could be reached by alternative methods suggested by Kemeny and Snell (1983).

We use the estimated number of times that the process remains in the non-absorbing state S_i once this state is entered, including the entering step, to approximate $e_{ij}(t)$. If the expected number of times the process stays in the non-absorbing state S_i, which is defined as a non-default state, is n_i, then the approximate duration of stay in S_i is $n_i \times 30$, where 30 days are the step size of the transition probability matrix. As such, the expected total number of days the process is in the non-default states is $\sum_{i=-3}^{3} n_i \times 30$. To facilitate the computation, we define V as the overall transition matrix, including prepayment, past due, and default states. The One-step transition probabilities matrix is given in Figure 2.2.

$$
V = \begin{array}{c} \\ R_1 \\ R_2 \\ R_3 \\ R_4 \\ S_{-3} \\ S_{-2} \\ S_{-1} \\ S_0 \\ S_1 \\ S_2 \\ S_3 \end{array}
\begin{array}{c}
\begin{array}{ccccccccccc} R_1 & R_2 & R_3 & R_4 & S_{-3} & S_{-2} & S_{-1} & S_0 & S_1 & S_2 & S_3 \end{array} \\
\left[\begin{array}{ccccccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\mu_{-3,1} & \mu_{-3,2} & \mu_{-3,3} & \mu_{-3,4} & v_{-3,-3} & v_{-3,-2} & v_{-3,-1} & v_{-3,0} & v_{-3,1} & 0 & 0 \\
\mu_{-2,1} & \mu_{-2,2} & \mu_{-2,3} & \mu_{-2,4} & v_{-2,-3} & v_{-2,-2} & v_{-2,-1} & v_{-2,0} & v_{-2,1} & 0 & 0 \\
\mu_{-1,1} & \mu_{-1,2} & \mu_{-1,3} & \mu_{-1,4} & v_{-1,-3} & v_{-1,-2} & v_{-1,-1} & v_{-1,0} & v_{-1,1} & 0 & 0 \\
\mu_{0,1} & \mu_{0,2} & \mu_{0,3} & \mu_{0,4} & v_{0,-3} & v_{0,-2} & v_{0,-1} & v_{0,0} & v_{0,1} & 0 & 0 \\
\mu_{1,1} & \mu_{1,2} & \mu_{1,3} & \mu_{1,4} & v_{1,-3} & v_{1,-2} & v_{1,-1} & v_{1,0} & v_{1,1} & v_{1,2} & 0 \\
\mu_{2,1} & \mu_{2,2} & \mu_{2,3} & \mu_{2,4} & v_{2,-3} & v_{2,-2} & v_{2,-1} & v_{2,0} & v_{2,1} & v_{2,2} & v_{2,3} \\
\mu_{3,1} & \mu_{3,2} & \mu_{3,3} & \mu_{3,4} & v_{3,-3} & v_{3,-2} & v_{3,-1} & v_{3,0} & v_{3,1} & v_{3,2} & v_{3,3}
\end{array}\right]
\end{array}
$$

Figure 2.2 One-step transition probabilities.

Here, v_{ij} refers to the probability of transition from S_i to S_j, μ_{ik} refers to the probability of transition from S_i to R_k. Furthermore, because R_k is an absorbing state, the transition matrix for these states is an Identity matrix and $\mu_{ki} = 0$. In the V matrix, there are 7 transient states in which each state could be reached from any other state (including its own), and 4 ergodic states each of which can be reached from any transient state. Once an ergodic state is entered the process remains in that state and cannot exit. The V matrix could be rearranged into block matrices as shown in Figure 2.3:

$$V=\left[\begin{array}{c|c} I & O \\ \hline R & Q \end{array}\right], \tag{2.8}$$

$$I_{4\times4}=\begin{array}{c} \\ R_1 \\ R_2 \\ R_3 \\ R_4 \end{array}\begin{array}{c} \begin{array}{cccc} R_1 & R_2 & R_3 & R_4 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}, \quad O_{4\times7}=\begin{array}{c} \\ R_1 \\ R_2 \\ R_3 \\ R_4 \end{array}\begin{array}{c} \begin{array}{ccccccc} S_{-3} & S_{-2} & S_{-1} & S_0 & S_1 & S_2 & S_3 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$R_{7\times4}=\begin{array}{c} \\ S_{-3} \\ S_{-2} \\ S_{-1} \\ S_0 \\ S_1 \\ S_2 \\ S_3 \end{array}\begin{array}{c} \begin{array}{cccc} R_1 & R_2 & R_3 & R_4 \end{array} \\ \begin{bmatrix} \mu_{-3,1} & \mu_{-3,2} & \mu_{-3,3} & \mu_{-3,4} \\ \mu_{-2,1} & \mu_{-2,2} & \mu_{-2,3} & \mu_{-2,4} \\ \mu_{-1,1} & \mu_{-1,2} & \mu_{-1,3} & \mu_{-1,4} \\ \mu_{0,1} & \mu_{0,2} & \mu_{0,3} & \mu_{0,4} \\ \mu_{1,1} & \mu_{1,2} & \mu_{1,3} & \mu_{1,4} \\ \mu_{2,1} & \mu_{2,2} & \mu_{2,3} & \mu_{2,4} \\ \mu_{3,1} & \mu_{3,2} & \mu_{3,3} & \mu_{3,4} \end{bmatrix} \end{array}, \quad Q_{7\times7}=\begin{array}{c} \\ S_{-3} \\ S_{-2} \\ S_{-1} \\ S_0 \\ S_1 \\ S_2 \\ S_3 \end{array}\begin{array}{c} \begin{array}{ccccccc} S_{-3} & S_{-2} & S_{-1} & S_0 & S_1 & S_2 & S_3 \end{array} \\ \begin{bmatrix} \nu_{-3,-3} & \nu_{-3,-2} & \nu_{-3,-1} & \nu_{-3,0} & \nu_{-3,1} & 0 & 0 \\ \nu_{-2,-3} & \nu_{-2,-2} & \nu_{-2,-1} & \nu_{-2,0} & \nu_{-2,1} & 0 & 0 \\ \nu_{-1,-3} & \nu_{-1,-2} & \nu_{-1,-1} & \nu_{-1,0} & \nu_{-1,1} & 0 & 0 \\ \nu_{0,-3} & \nu_{0,-2} & \nu_{0,-1} & \nu_{0,0} & \nu_{0,1} & 0 & 0 \\ \nu_{1,-3} & \nu_{1,-2} & \nu_{1,-1} & \nu_{1,0} & \nu_{1,1} & \nu_{1,2} & 0 \\ \nu_{2,-3} & \nu_{2,-2} & \nu_{2,-1} & \nu_{2,0} & \nu_{2,1} & \nu_{2,2} & \nu_{2,3} \\ \nu_{3,-3} & \nu_{3,-2} & \nu_{3,-1} & \nu_{3,0} & \nu_{3,1} & \nu_{3,2} & \nu_{3,3} \end{bmatrix} \end{array}$$

Figure 2.3 One-step transition probability matrix and its block matrices.

Thus, several interesting results could be reached. First, the expected number and variance of the time the process stays in a non-absorbing state S_i, n_i, before leaving S_i are given by:

$$\begin{aligned} E(n_i) &= N\zeta, \\ Var(n_i) &= (2N-I)n_i - sq(n_i) \end{aligned} \tag{2.9}$$

where,

$N=(I-Q)^{-1}$, I is an 7×7 identity matrix.

$$sq(n_i) = \begin{bmatrix} n_{-3}^2 & n_{-2}^2 & n_{-1}^2 & n_0^2 & n_1^2 & n_2^2 & n_3^2 \end{bmatrix}'$$

$$\xi = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}'$$

In fact, $E_i(n_j)$, the expected number of steps needed to transit from state i to state j, is equal to N_{ij}.

Let

$$d_{ij} = \begin{cases} 1, \text{if } i = j \\ 0, \text{if } i \neq j \end{cases} \tag{2.10}$$

Thus, it is seen that

$$\begin{aligned} & E_i(n_j) = d_{ij} + \sum_{s \in T} p_{ik} E_k(n_j) \\ & \Rightarrow E_i(n_j) = I + Q \cdot E_i(n_j) \\ & \Rightarrow E_i(n_j) = (I - Q)^{-1} = N \end{aligned} \tag{2.11}$$

where $s \in T$. By summing over i from $-3, -2, -1, 0, 1, 2, 3$, one obtains the total expected number of steps the process is in a non-absorbing state S_i before going to a default state R_k. Then, the expected total days of stay in the non-default states should be:

$$\sum_{i=-3}^{3} E(n_i) \times 30 \tag{2.12}$$

2.1.3 A Markov Model for Economic Assets Analysis

In any time interval, the size of a portfolio is a function of contraction and extension. For the purpose of this study, contraction refers to any process that causes a reduction in credit assets. On the other hand, extension is defined as any process that causes an increase in portfolio size.

To use the Markov Chain theory, one must define the states and the transition matrix. A bank loan normally requires monthly payment. If a loan is 30 days past due, denote it by s_1. Sate s_2 refers to 60 days past due. According to the Basel II (1992), the definition of default is more than 90 days past due, which is s_3. However, there are cases where a loan, which has been more than 90 days past due, is eventually paid off. To adjust for this situation, the definition of default is modified to represent the state of default that is triggered by a permanent force such as an application of bankruptcy protections, most of which are Chapters 7 and 13 for retail credit products. This definition is in accord with the purpose of this model which is to evaluate the economic (instead of accounting) status of a bank's credit assets. Let R_k be the $i \cdot th$ default or prepayment state. A default state reduces the portfolio value to zero. On the other hand, prepayment reduces the value by the amount of prepayment. From the definitions of states, it is clear that the past due states are transitional while the default or prepayment states are absorbing.

Table 2.2 States of the Markov chain.

Past Due and Prepayment States $S_j, j = -3,-2,-1,0,1,2,3$		Default States R_k $R_k, k = 1,2,3,4$	
S_{-3}	Prepaid More than 91 days	R_1	Sold by Bank
S_{-2}	Prepaid 61 days – 90 days	R_2	All others
S_{-1}	Prepaid 31 days – 60 days	R_3	Prepayment more than 50% of the remaining loan
S_0	No more than 30 days past due	R_4	Prepayment less than 50% of the remaining loan
S_1	31 days – 60 days past due		
S_2	61 days – 90 days past due		
S_3	More than 91 days past due		

What makes a prepayment state absorbing is the fact that a prepayment cannot be deducted from the next scheduled payment. For normal operation, one expects the bank's credit assets to be in state S_0. The following three reasons validate the classification of prepayment as an absorbing state:

1. The prepaid loan amount (extra payment besides the scheduled normal payment)

cannot serve as a buffer for future payment.

2. The prepaid loan amount can not be refunded by the bank.
3. The prepaid loan amount reduces the overall portfolio size.

Please note that, at any point of time, the state of no more than 30 days past due, S_0, refers to a health state, and we expect most of a bank's credit assets to stay in this state for normal operations.

The purpose of this Markov model is to analyze the portfolio value for a bank within a time interval $(0,t)$ on an economic basis. This fact implies the evaluation of the value after taking potential risks into account, instead of the accounting amount based on the bank's financial statement. The model can provide a true snap-shot at any given time ξ within a time interval $(0,t)$ for the management, and thus fundamental information for investors in a trading period interval $(0,t)$. The model has the following assumptions:

1. A transition an individual might make in the future is independent of those made in the past.
2. Individuals do not have equal probability of default, which depends on the specific debt structure, liquidity requirement, and risk taking ability.
3. The bank is under normal operation where the rate of approval of loan applications is assumed to follow a Poisson process.

For each $\tau, 0 \le \tau < t$, a change in the population size of each state S_i during a single time interval $(\tau, \tau + \Delta t)$ occurs based on the following probabilities:

$\lambda_i \Delta t$ = Probability that state S_i ($i = -3,-2,-1,0,1,2,3$) increases by 1 during a single time interval $(\tau, \tau+\Delta t)$. It is assumed that λ (the intensity of the Poisson process) is independent of time.

$v_{ij}(\tau)\Delta t$ = Pr {one individual will move from state S_i to state S_j during the time

interval $(\tau, \tau+\Delta t), i, j = -3,-2,-1,0,1,2,3$ }

$\mu_{ik}(\tau)\Delta t$ = Pr {an individual will move from a past due state S_i to a prepayment or

default state R_k during $(\tau, \tau+\Delta t), i = -3,-2,-1,0,1,2,3, k = 1,2,3,4$ }

The intensity v_{ij} that an individual stays in its original state in the time interval $(\tau, \tau+\Delta t)$, is defined as $v_{ii} = -(v_{ij} + \sum_{\delta=1}^{4} u_{jk}), i \neq j, i, j = -3,-2,-1,0,1,2,3, k = 1,2,3,4$. By this definition, it is obvious that $1 + v_{ii}\Delta t$ = Pr {an individual in state S_i at time τ will be state S_i at time $\tau+\Delta t$ }. Within any single time interval, { $\tau+\Delta t$ }, V is the past due intensity matrix, while U is the default and prepayment intensity matrix as shown in Figure 2.4:

$$V = \begin{array}{c} \\ S_{-3} \\ S_{-2} \\ S_{-1} \\ S_0 \\ S_1 \\ S_2 \\ S_3 \end{array} \begin{array}{c} \begin{array}{ccccccc} S_{-3} & S_{-2} & S_{-1} & S_0 & S_1 & S_2 & S_3 \end{array} \\ \begin{bmatrix} v_{-3,-3} & v_{-3,-2} & v_{-3,-1} & v_{-3,0} & v_{-3,1} & 0 & 0 \\ v_{-2,-3} & v_{-2,-2} & v_{-2,-1} & v_{-2,0} & v_{-2,1} & 0 & 0 \\ v_{-1,-3} & v_{-1,-2} & v_{-1,-1} & v_{-1,0} & v_{-1,1} & 0 & 0 \\ v_{0,-3} & v_{0,-2} & v_{0,-1} & v_{0,0} & v_{0,1} & 0 & 0 \\ v_{1,-3} & v_{1,-2} & v_{1,-1} & v_{1,0} & v_{1,1} & v_{1,2} & 0 \\ v_{2,-3} & v_{2,-2} & v_{2,-1} & v_{2,0} & v_{2,1} & v_{2,2} & v_{2,3} \\ v_{3,-3} & v_{3,-2} & v_{3,-1} & v_{3,0} & v_{3,1} & v_{3,2} & v_{3,3} \end{bmatrix} \end{array}$$

Figure 2.4 Past due intensity matrix and default and prepayment intensity matrix.

$$U = \begin{array}{c} \\ S_{-3} \\ S_{-2} \\ S_{-1} \\ S_0 \\ S_1 \\ S_2 \\ S_3 \end{array} \begin{array}{c} \begin{array}{cccc} R_1 & R_2 & R_3 & R_4 \end{array} \\ \begin{bmatrix} \mu_{-3,1} & \mu_{-3,2} & \mu_{-3,3} & \mu_{-3,4} \\ \mu_{-2,1} & \mu_{-2,2} & \mu_{-2,3} & \mu_{-2,4} \\ \mu_{-1,1} & \mu_{-1,2} & \mu_{-1,3} & \mu_{-1,4} \\ \mu_{0,1} & \mu_{0,2} & \mu_{0,3} & \mu_{0,4} \\ \mu_{1,1} & \mu_{1,2} & \mu_{1,3} & \mu_{1,4} \\ \mu_{2,1} & \mu_{2,2} & \mu_{2,3} & \mu_{2,4} \\ \mu_{3,1} & \mu_{3,2} & \mu_{3,3} & \mu_{3,4} \end{bmatrix} \end{array}$$

Figure 2.4 Continued.

Due to the fact that the R states are absorbing, there is no transition from U to V or among the R-states in U.

It is obvious that an increase in a portfolio's size within a small time interval $\{\tau, \tau + \Delta t\}$ could be regarded as the result of only the migration process. As a result, the portfolio size at any given time t can be expressed as

$$X(t) = Y(t) + Z(t) \tag{2.13}$$

$$X(t) = \begin{pmatrix} X_{-3}(t) \\ X_{-2}(t) \\ \dots \\ X_3(t) \end{pmatrix}, Y(t) = \begin{pmatrix} Y_{-3}(t) \\ Y_{-2}(t) \\ \dots \\ Y_3(t) \end{pmatrix}, Z(t) = \begin{pmatrix} Z_{-3}(t) \\ Z_{-2}(t) \\ \dots \\ Z_3(t) \end{pmatrix}$$

where $X_i(t), i = -3,-2,-1,0,1,2,3$ is defined as the portfolio size in each of the states, $s_{-3}, s_{-2}, s_{-1}, s_0, s_1, s_2, s_3$ at time t. $Y_i(t)$, $i = -3,-2,-1,0,1,2,3$ refers to the portfolio size in state i at time t that survived from the original portfolio in state i at time zero, $i = (i_{-3}, i_{-2}, i_{-1}, i_0, i_1, i_2, i_3)$ and $Z_i(t), i = -3,-2,-1,0,1,2,3$ stands for the portfolio size in state i at time t as a result of immigration during the interval $(0,t)$. One can argue that $Y_i(t)$ is affected both by contraction and extension, while $Z_i(t)$, a pure incremental factor, is merely an extension process.

The extension process is composed of:

1. Immigration or increase in the portfolio size because of approved new applications for a particular loan offered by a bank.
2. Birth or increase in the value of the original portfolio at time 0 because of the passage of time.

For simplicity, however, we will consider only immigration in this study. That is, we consider approval of a new loan as the only factor that plays a role in the extension process. On the other hand, the contraction process is triggered by three factors:

1. Prepayment, or the additional payment for a loan besides the schedule payment, reduces the portfolio size prematurely.
2. Default, causing the elimination of the default loan amount from the portfolio, is considered as another contraction force.
3. Transition, an individual moving from an original state to another state.

Thus, letting m_i be the portfolio size at state $i, i=-3,-2,-1,0,1,2,3$, at any time $\tau, 0 \le \tau \le t$ the expected portfolio value is given by

$$E[X_j(t)] = \sum_{i=-3}^{3} m_i p_{ij}(0,t) + q_j(t), i, j = -3,-2,-1,0,1,2,3 \qquad (2.14)$$

and, the variance is given by

$$V[X_j(t)] = \sum_{i=-3}^{3} m_i p_{ij}(0,t)[1 - p_{ij}(0,t)] + q_j(t), i, j = -3,-2,-1,0,1,2,3 \quad (2.15)$$

where, $p_{ij}(0,t)$ is the probability of being in state j at time t given that the process was in state i at time zero.

$$p_{ij}(0,t) = \sum_{l=-3}^{3} \frac{A'_{ij}(\rho_l)}{\prod_{\substack{m=-3 \\ m \neq l}}^{3} (\rho_l - \rho_m)} e^{\rho_l t}, i, j = -3,-2,-1,0,1,2,3 \quad (2.16)$$

and is obtained from the solution to the Kolmogorov Forward Differential Equation:

$$\frac{\partial}{\partial t} p_{ij}(\tau,t) = \sum_{\gamma \neq j} p_{i\gamma}(\tau,t) v_{\gamma j}(t), i, j = -3,-2,-1,0,1,2,3 \quad (2.17)$$

Also, $q_j(t)$ is the expected portfolio size in state S_j at time t, and is given by

$$\begin{aligned} q_j(t) &= \sum_{i=-3}^{3} \int_0^t \lambda_i \cdot p_{ij}(\tau,t) d\tau \\ &= \sum_{i=-3}^{3} \int_0^t \lambda_i \cdot \sum_{l=-3}^{3} \frac{A'_{ij}(\rho_l)}{\prod_{\substack{m=-3 \\ m \neq l}}^{3} (\rho_l - \rho_m)} e^{\rho_l (t-\tau)} d\tau, \\ &= \sum_{i=-3}^{3} \sum_{l=-3}^{3} \lambda_i \frac{A'_{ij}(\rho_l)}{\prod_{\substack{m=-3 \\ m \neq l}}^{3} (\rho_l - \rho_m)} (e^{\rho_l t} - 1), i, j = -3,-2,-1,0,1,2,3 \end{aligned} \quad (2.18)$$

Here, λ_i is the immigration rate to state S_i and A_{ij}' is the ij^{th} element of the characteristic matrix of V', the transpose of the intensity matrix V, defined by

$$A_{ij}' = (\rho I - V'), \tag{2.19}$$

where $\rho_l =$ eigenvalue of the intensity matrix V.

2.1.4 Limiting Probabilities

Let V be the transition probability matrix within a single time interval $(\tau, \tau + \Delta t)$, and V^n be the transition probability after n time periods. Furthermore, let $\chi_j = \lim_{n \to \infty} v^n{}_{ij}$, where v_{ij} is the ith row and jth column component of matrix V (Figure 2.4). Then,

$$\chi_j = \sum_{i=-3}^{2} \chi_i v_{ij}, j = -3, -2, -1, 0, 1, 2$$
$$\sum_j \chi_j = 1,, j = -3, -2, -1, 0, 1, 2, 3 \tag{2.20}$$

Here, χ_j is the percentage of individuals in state j.

If we let $f(i)$ be the penalty function for being in the past due state i, then the total expected proceedings from customers being past due during the period $1, 2, ..., N$ is given as

$$\sum_{i,N} f = \sum_{i=-3}^{3} f(i)\chi_i \tag{2.21}$$

2.2 Application

Data were provided by a local bank in Ohio, operating in Ohio, Michigan, Kentucky, and Indiana. By using its monthly paid retail mortgage loan for 16 consecutive months, from April 2005 to September 2006, one can apply the discrete finite Markov chain model and the continuous time model. Also, the continuous version of the Markov chain model will be used as a decision-making tool for optimizing bank loan officer compensation and for determining the sensitivity of a loan health index to macro-economic factors such as GDP, interest rate, unemployment, and consumer price index.

2.2.1 Discrete Time Model

The intensity matrix V, can be divided to 4 sub-matrices. These are an identity matrix $I_{4\times4}$, a zero matrix $O_{4\times7}$, a $R_{7\times4}$ matrix which refers to the transitions from transient to ergodic states, and the $Q_{7\times7}$ matrix which denotes transitions within the transient states. By the definition of an absorbing state, it is seen that the intensity sub-matrix within the absorbing states is an identity matrix because once entered into an absorbing state, the loan will stay there for an infinite period of time. By the same reasoning, the zero matrix $O_{4\times7}$ refers to the fact that there is no transition from any absorbing state to any transient state.

On the other hand, elements of the transient $Q_{7\times7}$ matrix and the ergodic matrix $R_{7\times4}$ are given as

$$Q_{ij} = \frac{\sum_{t=1}^{16} q_{ijt}}{\sum_{t=1}^{16}\sum_{j=-3}^{3} q_{ijt}}, R_{ik} = \frac{\sum_{t=1}^{16} r_{ikt}}{\sum_{t=1}^{16}\sum_{k=1}^{4} r_{ikt}}, i, j = -3, -2, -1, 0, 1, 2, 3; k = 1, 2, 3, 4, \qquad (2.22)$$

For example, the intensities between period 1 and period 2 are given by Figure 2.5

$$q_1 := \begin{pmatrix} 10 & 3 & 4 & 3 & 0 & 0 & 0 \\ 1 & 9 & 8 & 1 & 0 & 0 & 0 \\ 1 & 1 & 51 & 21 & 1 & 0 & 0 \\ 3 & 9 & 32 & 722 & 12 & 0 & 0 \\ 0 & 1 & 2 & 6 & 8 & 2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 3 \end{pmatrix} \qquad r_1 := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 12 & 0 & 15 \\ 0 & 0 & 1 & 10 \\ 0 & 4 & 1 & 0 \end{pmatrix}$$

Figure 2.5 Two consecutive month transition matrices.

The numbers in the matrices represent the number of transitions from state i to state j $\left(q_{ij1}\right)$, $i, j = -3, -2, -1, 0, 1, 2, 3$, or from state i to state $k\left(r_{ik1}\right), k = 1, 2, 3, 4$ between period 1 and period 2. The Discrete transition probability matrices are given in Figure 2.6.

$$Q := \begin{pmatrix} 0.575949 & 0.09962 & 0.164557 & 0.132911 & 0 & 0 & 0 \\ 0.036496 & 0.61365 & 0.156277 & 0.182482 & 0.000562 & 0 & 0 \\ 0.027559 & 0.034121 & 0.677428 & 0.239895 & 0.001452 & 0 & 0 \\ 0.004084 & 0.009734 & 0.049125 & 0.89205 & 0.010912 & 0 & 0 \\ 0.009094 & 0.008955 & 0.01403 & 0.207812 & 0.206307 & 0.055224 & 0 \\ 0 & 0.003157 & 0.023684 & 0.034737 & 0.085263 & 0.174737 & 0.318421 \\ 0 & 0 & 0.002105 & 0.0632 & 0.09526 & 0.151053 & 0.293157 \end{pmatrix}$$

$$R := \begin{pmatrix} 0.019452 & 0 & 0 & 0 \\ 0.010451 & 0 & 0 & 0 \\ 0.010937 & 0 & 0 & 0 \\ 0.029017 & 0 & 0 & 0 \\ 0.132985 & 0.152321 & 0 & 0.209015 \\ 0 & 0.090526 & 0.145632 & 0.122053 \\ 0 & 0.105206 & 0.150526 & 0.130526 \end{pmatrix}$$

$$I := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad O := \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Figure 2.6 Discrete transition probability matrices.

Then, according to Kemeny and Snell (1983), the overall intensity matrix V, composed of $R_{7\times4}, Q_{7\times7}, I_{4\times4}, O_{4\times7}$ is given in Figure 2.7:

$V =$	R_1	R_2	R_3	R_4	S_{-3}	S_{-2}	S_{-1}	S_0	S_1	S_2	S_3
R_1	1	0	0	0	0	0	0	0	0	0	0
R_2	0	1	0	0	0	0	0	0	0	0	0
R_3	0	0	1	0	0	0	0	0	0	0	0
R_4	0	0	0	1	0	0	0	0	0	0	0
S_{-3}	0.019452	0	0	0	0.575949	0.09962	0.164557	0.132911	0	0	0
S_{-2}	0.010451	0	0	0	0.036496	0.61365	0.156277	0.182482	0	0	0
S_{-1}	0.010937	0	0	0	0.027559	0.034121	0.677428	0.239895	0.001452	0	0
S_0	0.002917	0	0	0	0.004084	0.009734	0.049125	0.892050	0.010912	0	0
S_1	0.132985	0.152321	0	0.209015	0.009094	0.008955	0.01403	0.207812	0.206307	0.055224	0
S_2	0	0.090526	0.145632	0.122053	0	0.031579	0.023684	0.034737	0.085263	0.174737	0.318421
S_3	0	0.010526	0.010526	0.130526	0	0	0.002105	0.063200	0.095260	0.151053	0.293157

Figure 2.7 Discrete transition probability matrix V.

Thus, by the above reasoning, the expected number of steps required to transition from transient state $i, i = -3, -2, -1, 1, 2, 3$ to absorbing state $k, k = 1, 2, 3, 4$ is given by E:

$$E = (I - Q)^{-1} \cdot \xi = N \cdot \xi = \begin{bmatrix} 29.38866 \\ 30.16227 \\ 29.24377 \\ 27.42631 \\ 10.12164 \\ 6.99529 \\ 6.81304 \end{bmatrix}, \xi = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \tag{2.23}$$

In this case, one can see that it takes 29.38866 or approximately 30 steps for a loan initially in state -3 to leave the transient states for any absorbing state. In other words, since the step is 1 month, a loan more than 3 months prepaid (state 3) could become sold or defaulted in approximately 30 months or 2.5 years, while a loan with 3 months past due could reach the same destiny in approximately half a year.

Also, let b_{ik} be the probability that the process transits from transient state i, $i = -3,-2,-1,0,1,2,3$ to absorbing state $k, k = 1,2,3,4$:

$$\{b_{ij}\} = B = (I-Q)^{-1} \cdot R = N \cdot R = \begin{bmatrix} 0.72337 & 0.04473 & 0.00429 & 0.06121 \\ 0.72953 & 0.04692 & 0.0045 & 0.0642 \\ 0.71614 & 0.04673 & 0.00448 & 0.06395 \\ 0.72809 & 0.04968 & 0.00476 & 0.06798 \\ 0.39812 & 0.22249 & 0.02133 & 0.30446 \\ 0.1545 & 0.22566 & 0.28605 & 0.2984 \\ 0.1539 & 0.23163 & 0.2774 & 0.29573 \end{bmatrix} \quad (2.24)$$

An element of B, b_{ik} represents the probability of transiting from transient state i to absorbing state k. For example, $b_{32} = 0.23163$ means that the probability of transiting from the 3-month past due state to the absorbing state (foreclosure) is 0.23.163.

2.2.2 Continuous Time Model

For a continuous-time Markov chain, an element v_{ij} of the transition matrix V, is given by the following equation:

$$v_{ij} = \frac{d}{dt} P_{ij}(c_{ijt}, t)\,|_{t=0}, i \neq j, i = -3,-2,-1,0,1,2,3, t = 1,2,3,...16, \quad (2.25)$$

where $P_{ij}(c_{ijt}, t)$ stands for the 5th-order polynomial used to fit the observed transition probabilities from the data over time. $c_{ijt}, i, j = -3,-2,-1,0,1,2,3 \quad ,t = 1,2,3,...16$ The polynomials are approximated by the Lagrange numerical method. For instance, using

MATLAB 7.0 ® Release 14, the transition intensity matrix for the transient state is shown in Figure 2.8:

$$c_{ij5} := \begin{pmatrix} 0.47368 & 0.15789 & 0.21056 & 0.15789 & 0 & 0 & 0 \\ 0.05000 & 0.45000 & 0.40000 & 0.05000 & 0.05000 & 0 & 0 \\ 0.01149 & 0.01149 & 0.47126 & 0.47126 & 0.03448 & 0 & 0 \\ 0.00383 & 0.00511 & 0.04092 & 0.47126 & 0.034483 & 0 & 0 \\ 0 & 0.03333 & 0.23333 & 0.53333 & 0.20000 & 0 & 0 \\ 0 & 0.14285 & 0 & 0.14285 & 0 & 0.28571 & 0.42857 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

Figure 2.8 Transition matrix for the transient states in the interval (0, 5).

The computations are done automatically through an access database. Similarly, the transition intensity matrix from transient to absorbing states in the interval (0,8) is given in Figure 2.9:

$$r_{ik8} := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0.00241 & 0 & 0.00124 & 0 \\ 0 & 0 & 0 & 0 \\ 0.33333 & 0 & 0 & 0 \\ 0.01454 & 0.00123 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0.00145 & 0 & 0 & 0 \end{pmatrix}$$

Figure 2.9 Transition intensities in the interval (0, 8).

The diagonal elements of the intensity matrix V and U are given by

$$v_{ii} = -(v_{ij} + \sum_{\delta=1}^{4} u_{ik}), i \neq j, i, j = -3, -2, -1, 0, 1, 2, 3, k = 1, 2, 3, 4 \qquad (2.26)$$

where $u_{ik} = \frac{d}{dt} P_{ik}(r_{ikt}, t)|_{t=0}, i = -3,-2,-1,0,1,2,3, t = 1,2,3,...16$. Thus, we obtained the following V, U transition intensity matrices as presented in Figure 2.10:

$$V := \begin{pmatrix} -0.9919 & 0.3455 & 0.1592 & 0.2192 & 0.1207 & 0.0553 & 0 \\ 0.3687 & -0.9191 & 0.1314 & 0.164 & 0.158 & 0.0745 & 0 \\ 0.183 & 0.2799 & -0.9913 & 0.3933 & 0.0875 & 0.0215 & 0 \\ 0.1611 & 0.2785 & 0.3265 & -0.9962 & 0.1141 & 0.0909 & 0 \\ 0 & 0.0244 & 0.1212 & 0.2513 & -1.0263 & 0.2986 & 0.1849 \\ 0 & 0.0128 & 0.0556 & 0.0526 & 0.3008 & -0.9604 & 0.3426 \\ 0 & 0 & 0.0012 & 0.0904 & 0.1416 & 0.3512 & -0.9231 \end{pmatrix}$$

$$U := \begin{pmatrix} 0 & 0 & 0.0460 & 0.046 \\ 0 & 0.0225 & 0 & 0 \\ 0 & 0 & 0 & 0.0261 \\ 0.0126 & 0.0125 & 0 & 0 \\ 0 & 0.0245 & 0.1214 & 0 \\ 0 & 0.1745 & 0.0215 & 0 \\ 0 & 0.1842 & 0.1545 & 0 \end{pmatrix}$$

Figure 2.10 Intensity Matrices V and U.

Hence, one can estimate, from equation (2.7), the transition probability matrix $P_{ij}(0,1)$ and the expected duration of stay in state j (given that the process started in state i) during the interval $(0,1)$, $e_{ij}(1), i, j = -3,-2,-1,1,2,3$ are given in Figure 2.11:

$$P_{ij}(0,1) := \begin{pmatrix} 0.08248 & 0.10673 & 0.08789 & 0.11175 & 0.08431 & 0.07858 & 0.05141 \\ 0.0858 & 0.11103 & 0.09146 & 0.1163 & 0.08779 & 0.08186 & 0.05359 \\ 0.09 & 0.11643 & 0.09583 & 0.12181 & 0.09181 & 0.08549 & 0.05586 \\ 0.08746 & 0.11318 & 0.0932 & 0.1185 & 0.0894 & 0.08332 & 0.05451 \\ 0.06219 & 0.0806 & 0.0667 & 0.08496 & 0.06462 & 0.06073 & 0.04014 \\ 0.05046 & 0.06547 & 0.05438 & 0.06937 & 0.05309 & 0.0502 & 0.03342 \\ 0.04165 & 0.05407 & 0.04496 & 0.05738 & 0.044 & 0.04168 & 0.02781 \end{pmatrix}$$

Figure 2.11 Transition probability matrix and expected durations of stay in state j (starting in state i) in the interval (0,1).

$$e_{ij}(1) := \begin{pmatrix} 0.08683 & 0.11234 & 0.09247 & 0.11755 & 0.08862 & 0.08253 & 0.05394 \\ 0.09031 & 0.11685 & 0.09622 & 0.12233 & 0.09228 & 0.086 & 0.05625 \\ 0.09476 & 0.12257 & 0.10082 & 0.12814 & 0.09648 & 0.08975 & 0.05857 \\ 0.09207 & 0.11912 & 0.09805 & 0.12464 & 0.09396 & 0.08751 & 0.05719 \\ 0.06531 & 0.08467 & 0.07012 & 0.08935 & 0.06805 & 0.06403 & 0.04238 \\ 0.0529 & 0.06868 & 0.05715 & 0.07294 & 0.05598 & 0.05308 & 0.03545 \\ 0.04364 & 0.05669 & 0.04724 & 0.06033 & 0.04641 & 0.04411 & 0.02955 \end{pmatrix}$$

Figure 2.11 Continued.

For instance, $P_{-2,-1}(0,1) = 0.09146$ represents the probability that a loan in the 2-month prepaid state will transit to the 1-month prepaid state during the time interval $(0,1)$. On the other hand, $e_{-2,-1}(1) = 0.09622$ represents the mean time of stay in the 1-month prepaid state (given that the loan started in the 2-month prepaid state at t = 0) in the time interval $(0,1)$.

2.2.3 Economic Assets

In this subsection, we will use the model to approximate the stochastic retail mortgages portfolio size of the Ohio bank. Let $X(t)$ be the total stochastic retail mortgage portfolio size at time t. Its expected value can be expressed as

$$E[X(t)] = \sum_{j=-3}^{3} E[X_j(t)], j = -3,-2,-1,0,1,2,3 \quad , \tag{2.27}$$

where $E[X_j(t)]$, the expected portfolio size belonging to state j, is given in Equation (2.14). The following set of equations were used in applying the algorithm provided by MathCAD.

$$E[X_j(t)] = \sum_{i=-3}^{3} m_i p_{ij}(0,t) + \sum_{i=-3}^{3} \int_0^t \lambda_i \cdot p_{ij}(\tau,t) d\tau,$$
$$p_{ij}(0,t) = \sum_{l=-3}^{3} \frac{A'_{ij}(\rho_l)}{\prod_{\substack{m=-3 \\ m \neq l}}^{3} (\rho_l - \rho_m)} e^{\rho_l t}, i,j = -3,-2,-1,0,1,2,3 \tag{2.28}$$

where m_i is the retail mortgage portfolio size belonging to state i in thousands of dollars at time 0 or April 2005. Using the bank database, we estimated the $M = \{m_i\}$ vector as shown in Figure 2.12, in thousands of dollars:

$$\begin{matrix} & S_{-3} & S_{-2} & S_{-1} & S_0 & S_1 & S_2 & S_3 \\ M = (& 52.69 & 629.07 & 7341.75 & 68428.91 & 292.79 & 267.11 & 62.31)^T \end{matrix}$$

Figure 2.12 Retail mortgage distributions in thousand of dollars at time 0

Table 2.3 provides the criteria used to select data in different states.

Table 2.3 States in the Database.

States	Prepayment Indicator	Past Due Days
S_{-3}	>3	$=0$
S_{-2}	$=3$	$=0$
S_{-1}	$=2$	$=0$
S_0	$=1$	$\geq 0, \leq 30$

Table 2.3 Continued.

S_1	<1	$\geq 31, \leq 60$
S_2	<1	$\geq 61, \leq 90$
S_3	<1	≥ 91

The definition of immigration rate is given by the following method. For simplicity, we assume that the immigration intensity or increment rate is homogeneous over time ($\lambda_i(t) = \lambda_i$). Let f_{λ_i} be the polynomial function for λ_i from the one step immigration dollar amount at time t, i_t. Thus by taking the first-order derivative of the function f_{λ_i}, evaluated at time t = 0, we obtain the immigration intensity

$$\lambda_i = \frac{df_i(\lambda_{t,i})}{dt}\Big|_{t=0}, t = 1,2,\ldots,16, i = -3,-2,-1,0,1,2,3$$
$$\begin{matrix} & \text{t}_1 & \text{t}_2 & & \text{t}_{16} \end{matrix}$$
$$\lambda_{t,i} = \left\{ \begin{pmatrix} \lambda_{1,-3} \\ \vdots \\ \lambda_{1,3} \end{pmatrix}, \begin{pmatrix} \lambda_{2,-3} \\ \vdots \\ \lambda_{2,3} \end{pmatrix}, \ldots, \begin{pmatrix} \lambda_{16,-3} \\ \vdots \\ \lambda_{16,3} \end{pmatrix} \right\} \tag{2.29}$$

where $\lambda_{t,i}$ is the retail mortgage immigration rate between period t and period $t-1$ in state i. The following vector in equation (2.30) gives the estimates of the immigration rates between period 1 and period 2 in thousands of dollars:

$$\lambda_{t=2} = \begin{pmatrix} 5.21 & 47.08 & 98.77 & 547.49 & 2.14 & 0 & 0 \end{pmatrix}^T \tag{2.30}$$

It is seen that $\lambda_{t=2,2}$ and $\lambda_{t=2,3}$ are zeros because in two consecutive months the booked loans could not be 2 or more than 2 months past due. Thus, the immigration rates obtained from Equation (2.30) are given by the following figure.

$$\lambda_{t=2} = (3.04 \quad 24.12 \quad 159.42 \quad 876.45 \quad 1.17 \quad 0 \quad 0)^T \qquad (2.31)$$

Using the same approach as in subsection 2.2.2, we estimated the transition intensity matrices, V and U as shown in Figure 2.13:

$$V := \begin{pmatrix} -0.8519 & 0.1707 & 0.1547 & 0.1387 & 0.0578 & 0.0553 & 0 \\ 0.2879 & -0.8391 & 0.1045 & 0.1278 & 0.1974 & 0.0874 & 0 \\ 0.2781 & 0.2678 & -0.9503 & 0.3578 & 0.0565 & 0.0178 & 0 \\ 0.1378 & 0.2978 & 0.2457 & -0.9674 & 0.1584 & 0.1002 & 0 \\ 0 & 0.0178 & 0.1578 & 0.3047 & -0.9912 & 0.2784 & 0.1748 \\ 0 & 0.0147 & 0.0479 & 0.0614 & 0.2947 & -0.7843 & 0.1978 \\ 0 & 0 & 0.0078 & 0.1047 & 0.1687 & 0.3314 & -0.9047 \end{pmatrix}$$

$$U := \begin{pmatrix} 0 & 0 & 0.0613 & 0.0784 \\ 0 & 0.0378 & 0 & 0 \\ 0 & 0 & 0 & 0.0578 \\ 0.0784 & 0.0087 & 0 & 0 \\ 0 & 0.0144 & 0.0947 & 0 \\ 0 & 0.1547 & 0.0687 & 0 \\ 0 & 0.1574 & 0.1178 & 0 \end{pmatrix}$$

Figure 2.13 Transition Intensity Matrices for Stochastic Assets.

Letting $InTran(0,t)$ be the portfolio assets distribution from internal transition and $ExTran(0,t)$ be the assets from immigration or new booked source, we have the following results as shown in Figure 2.14 (From Equation (2.29) and the V matrix in Figure (2.12)) when $t = 1$:

$$\begin{array}{cccccccc} & S_{-3} & S_{-2} & S_{-1} & S_0 & S_1 & S_2 & S_3 \\ InTran(0,t) = (& 1.03 & 25.50 & 217.81 & 1829.75 & 9.27 & 5.96 & 0.41)^T \end{array}$$

$$\begin{array}{cccccccc} & S_{-3} & S_{-2} & S_{-1} & S_0 & S_1 & S_2 & S_3 \\ ExTran(0,t) = (& 0.09 & 2.78 & 29.74 & 20.14 & 1.65 & 0 & 0)^T \end{array}$$

Figure 2.14 Internal assets and immigrated assets distributions over states.

As one month is the usual measure period of banks, by letting $t = 30$, we can estimate $A_{monthly}$, the stochastic assets of the monthly paid retail mortgage assets, by the following equation:

$$\begin{aligned} A_{monthly} &= \sum_{i=-3}^{3} [InTran_i(0,30) + ExTran_i(0,30)] \\ &= \$64,323.9 \end{aligned} \qquad (2.32)$$

2.3 Conclusion

The above models, discrete and continuous, confirmed the expected retail mortgage loan's behavior. Furthermore, these models provided useful information to quantify the risks encountered by the banking management. By using these models, the management can obtain a clear picture of its retail loans. For example, from (2.13), we know approximately how long the loan could take to enter each absorbing state. Thus, a corresponding rescue action would be deployed to encounter each situation.

More specifically, flexibility of the continuous model will allow the bank management to analyze its loans characteristics in any reasonable interval. The following matrices are obtained by letting $t = 30$ in Figure 2.15:

$$P(0,30) := \begin{pmatrix} 0.0045 & 0.00583 & 0.00481 & 0.00612 & 0.00463 & 0.00433 & 0.00284 \\ 0.00469 & 0.00607 & 0.00501 & 0.00637 & 0.00482 & 0.00451 & 0.00296 \\ 0.00491 & 0.00636 & 0.00524 & 0.00667 & 0.00505 & 0.00472 & 0.0031 \\ 0.00478 & 0.00618 & 0.0051 & 0.00649 & 0.00491 & 0.00459 & 0.00301 \\ 0.00343 & 0.00444 & 0.00366 & 0.00466 & 0.00352 & 0.00329 & 0.00216 \\ 0.0028 & 0.00362 & 0.00299 & 0.0038 & 0.00288 & 0.00269 & 0.00177 \\ 0.00232 & 0.003 & 0.00247 & 0.00315 & 0.00238 & 0.00223 & 0.00146 \end{pmatrix}$$

$$e(30) := \begin{pmatrix} 0.86326 & 1.11748 & 0.92138 & 1.172 & 0.88602 & 0.82757 & 0.54278 \\ 0.89831 & 1.16289 & 0.95891 & 1.21978 & 0.92228 & 0.86156 & 0.56519 \\ 0.9412 & 1.21831 & 1.00435 & 1.27745 & 0.96545 & 0.90148 & 0.59105 \\ 0.91537 & 1.18494 & 0.97701 & 1.24275 & 0.9395 & 0.87752 & 0.57554 \\ 0.65497 & 0.84828 & 0.70048 & 0.8915 & 0.67567 & 0.63271 & 0.41629 \\ 0.53398 & 0.69183 & 0.57194 & 0.72822 & 0.55297 & 0.5188 & 0.34214 \\ 0.44146 & 0.57203 & 0.47307 & 0.60242 & 0.45772 & 0.42969 & 0.28358 \end{pmatrix}$$

Figure 2.15 Transition probability (0,30).

The value for $P_{-3,-3}(0,30)$ means that the probability of staying in a 3-month prepaid state for 30 days is 0.0045, which could be explained as the probability that a loan will continue to be paid 3-month ahead is 0.0045. Also, $e_{-3,-3}(30) = 0.86326$ tells us that, during the interval $(0,30)$, staying in 3-month prepaid state is only 0.86326 unit of time. Furthermore, one can see that a small value for $P_{i,j}(0,t)$ is usually accompanied by a small value for $e_{i,j}(t)$, which is what one expects based on banking experience.

As can be seen, there is a large difference between the retail mortgage's book amount on the bank's financial statement and the estimated stochastic amount which take into consideration the prepayment, past due, and default after one month. The latter is often of most interest to the outside investors because this is the real assets amounts that could be used to buffer the liability due to the customer's deposit. In most cases, it could be used to evaluate the bank's operation efficiency as well as its bankruptcy potential.

Nevertheless, the discrete time and the continuous time Markov models are by no mean the only tools that could be deployed by bank management. In fact, the above models used only the occurrence frequencies of each state and did not consider the loan assets which, in a sense, are more important for risk management in the banking industry. In the next chapter, we provide a method to index a retail mortgage's health status and link it to the local macro-economic situation.

CHAPTER 3

ANALYSIS OF MORTGAGE LOANS STATUS INDEX

This chapter provides an indexing procedure for a mortgage loan by means of a finite Markov chain approach, which converts the loan health abstract idea into a workable number system. This method could be easily extended to other banking products as well. In the model section, a theoretical Stated-Space time series model is presented to analyze and to predict the loan health index's sensitivity to local macro-economic factors, such as GDP, inflation, unemployment, interest rate, and personal disposable income. A multivariate regression method is used to analyze the local macroeconomic factors' effects on the health index. The management of a bank could use these procedures to adjust its loan approval policies based on current characteristics and future prediction of the portfolio.

3.1 Model

A bank's portfolio pool, say 20 years of mortgage loans, is composed of distinct individuals, who behave independently. Some of the individual loans, having been prepaid, past due, or charged off at the beginning of the measuring period, will transfer to a different state or stay in their respective states. Once a loan being charged off, the balance is removed and it could never go back to the bank books. The model is based on the idea that it will measure the expected duration of stay in each state and the probability of the process going back to the normal payment state.

Table 3.1 Definitions of the different states of the Markov chain.

Past Due and Prepayment States $S_i; i=-3,-2,-1,0,1,2,3$		Default States R_k $R_k, k=1,2,3,4$	
S_{-3}	Prepaid More than 91 days	R_1	Sold by Bank
S_{-2}	Prepaid 61 days – 90 days	R_2	Foreclosure
S_{-1}	Prepaid 31 days – 60 days	R_3	Refuse to pay
S_0	No more than 30 days past due	R_4	All others reasons
S_1	31 days – 60 days past due		
S_2	61 days – 90 days past due		
S_3	More than 91 days past due		

Health states, $S_i, i=-3,-2,-1,0,1,2,3$ (Table 3.1), are defined as follows: S_{-3}, S_{-2}, S_{-1} are prepayment states, while S_1, S_2, S_3 are past due states. S_0, the only health state, refers to the normal payment. From the bank's point of view, although prepayment is not as adverse as past due, it is still undesirable. Behaviors of prepayment, in spite of the fact that they can insure early payback of the principle, reduce the total interests the bank could possibly earn on the outstanding loan balance at the beginning of the period. The different prepayment states are determined by the formula, $S_{-i} = \frac{X_i - Y_i}{Y_i}$, where X_i is the actual payment at month i, and Y_i is the expected payment at month i. It is seen that a S_{-i} state is defined as the extra payment. On the other hand, once a loan has been charged off, it would be eliminated from the bank's portfolio pool and transferred to a third collection company. As a result, the charge-off states are defined as $R_k, k = 1,2,3,4$, k referring to different causes of charge-offs.

3.1.1 Loan Health Index Model

Let H bet the index of a portfolio, which at time t has $S_j, j = -3,-2,-1,0,1,2,3$ health states and $R_k, k = 1,2,3,4$ charged-off or absorbing states. Here, H is given as

$$H = e_{-3}\theta_{-3,0} + e_{-2}\theta_{-2,0} + e_{-1}\theta_{-1,0} + e_0\theta_{0,0} + e_1\theta_{1,0} + e_2\theta_{2,0} + e_3\theta_{3,0} \tag{3.1}$$

where, e_j refers to the expected duration of stay in state $j, j = -3,-2,-1,0,1,2,3$, $\theta_{j,0}$ is an intensity function $j = -3,-2,-1,0,1,2,3$ measuring the transitions to the normal state, S_0. The expected duration of stay in a specific state is based on the Markov transition intensity matrix as shown in Figure 3.1:

$$V = \begin{array}{c} \\ S_{-3} \\ S_{-2} \\ S_{-1} \\ S_0 \\ S_1 \\ S_2 \\ S_3 \end{array} \begin{array}{c} \begin{array}{ccccccc} S_{-3} & S_{-2} & S_{-1} & S_0 & S_1 & S_2 & S_3 \end{array} \\ \begin{bmatrix} v_{-3,-3} & v_{-3,-2} & v_{-3,-1} & v_{-3,0} & v_{-3,1} & 0 & 0 \\ v_{-2,-3} & v_{-2,-2} & v_{-2,-1} & v_{-2,0} & v_{-2,1} & 0 & 0 \\ v_{-1,-3} & v_{-1,-2} & v_{-1,-1} & v_{-1,0} & v_{-1,1} & 0 & 0 \\ v_{0,-3} & v_{0,-2} & v_{0,-1} & v_{0,0} & v_{0,1} & 0 & 0 \\ v_{1,-3} & v_{1,-2} & v_{1,-1} & v_{1,0} & v_{1,1} & v_{1,2} & 0 \\ v_{2,-3} & v_{2,-2} & v_{2,-1} & v_{2,0} & v_{2,1} & v_{2,2} & v_{2,3} \\ v_{3,-3} & v_{3,-2} & v_{3,-1} & v_{3,0} & v_{3,1} & v_{3,2} & v_{3,3} \end{bmatrix} \end{array}$$

$$U = \begin{array}{c} \\ S_{-3} \\ S_{-2} \\ S_{-1} \\ S_0 \\ S_1 \\ S_2 \\ S_3 \end{array} \begin{array}{c} \begin{array}{cccc} R_1 & R_2 & R_3 & R_4 \end{array} \\ \begin{bmatrix} \mu_{-3,1} & \mu_{-3,2} & \mu_{-3,3} & \mu_{-3,4} \\ \mu_{-2,1} & \mu_{-2,2} & \mu_{-2,3} & \mu_{-2,4} \\ \mu_{-1,1} & \mu_{-1,2} & \mu_{-1,3} & \mu_{-1,4} \\ \mu_{0,1} & \mu_{0,2} & \mu_{0,3} & \mu_{0,4} \\ \mu_{1,1} & \mu_{1,2} & \mu_{1,3} & \mu_{1,4} \\ \mu_{2,1} & \mu_{2,2} & \mu_{2,3} & \mu_{2,4} \\ \mu_{3,1} & \mu_{3,2} & \mu_{3,3} & \mu_{3,4} \end{bmatrix} \end{array}$$

Figure 3.1 Transition intensities transient states and absorbing states.

The transitions within the S-states are defined as (Chiang, 1980):

$v_{ij}\Delta t$ = Pr {an individual in state S_i at time τ will be state S_j at time $\tau + \Delta t$ }, where $i \neq j; i, j = -3, -2, -1, 0, 1, 2, 3$,

$\mu_{ik}\Delta t$ = Pr {an individual in state S_i at time τ will be state R_k at time $\tau + \Delta t$ }, where, $k = -3, -2, -1, 0, 1, 2, 3$ and k refers to the default or absorbing states, $k = 1, 2, 3, 4$.

Furthermore, we assume that future transitions of an individual are independent of past transitions. In other words, the intensities v_{ij} and μ_{ik} are assumed to be independent of time τ. For $0 \le \tau \le t$. Thus, we are concerned here with a time homogenous Markov chain.

If an individual stays in its original state, its intensity is defined by

$v_{ii} = -(v_{ij} + \sum_{\delta=1}^{4} u_{jk}), i \neq j, i, j = -3, -2, -1, 0, 1, 2, 3, k = 1, 2, 3, 4$. By this definition, it is obviously that

$1 + v_{ii}\Delta t =$ Pr {an individual in state S_i at time τ will remain in state S_i at time $\tau + \Delta t$ }. Within any single time interval, { $\tau + \Delta t$ }, V is the prepayment and past due intensity matrix.

Thus, the expected duration of stay in state j is given by

$$e_j = \sum_{i=-3}^{3} \sum_{l=-3}^{3} \frac{\pi_i A_{ij}'(\rho_l)}{\prod_{\substack{j=1 \\ j \neq l}}^{s} (\rho_l - \rho_j) \rho_l} (e^{\rho_l t} - 1), i, j = -3, -2, -1, 0, 1, 2, 3 \qquad (3.2)$$

where, π_i's$, i = -3, -2, -1, 0, 1, 2, 3$ are the proportion of individuals (in the limit) in the portfolio pool who are initially in $S_i, i = -3, -2, -1, 0, 1, 2, 3$. Let c_i be the number of loans in state i at the initial starting date. Thus π_i, the steady state probability distribution of loans at time t, is estimated as

$$\pi_i = \frac{c_i}{\sum_{i=-3}^{3} c_i}, i = -3, -2, -1, 0, 1, 2, 3, t = 1, 2, \ldots, 16 \qquad (3.3)$$

Furthermore, A_{ij}' is the characteristic matrix of V', the transpose of the intensity matrix V, defined by

$$A_{ij}' = (\rho I - V'), \tag{3.4}$$

where ρ_l = Eigenvalue of the intensity matrix V.

On the other hand, it is obvious that $\theta_{i,0}$ measures an individual's ability to recover from the semi-health prepayment and past due state $S_j, j = -3, -2, -1, 1, 2, 3, j \neq 0$ to a pure health state, S_0. Thus, a Maximum Likelihood Estimate [Chiang (1975)] of $\theta_{i,0}$ is given as

$$\hat{\theta}_{i,0} = \frac{\sum_{r=1}^{N} n_{i,0,r}}{\sum_{r=1}^{N} t_{i,r}}, i = -3, -2, -1, 0, 1, 2, 3 \tag{3.5}$$

where, $n_{i,0,r}$ is the number of transitions from $S_i, i = -3, -2, -1, 0, 1, 2, 3$ to S_0 by the *rth* individual. As such, $\sum_{r=1}^{N} n_{i,0,r}$ is the total number of transitions made by all N individuals in the portfolio. By the same reasoning, $\sum_{r=1}^{N} t_{i,r}$ is the total length of time that all individuals in the portfolio stay in $S_i, i = -3, -2, -1, 0, 1, 2, 3$. Therefore, the portfolio health index is given as

$$H = \sum_{i=-3}^{3}\sum_{l=-3}^{3} \frac{\pi_i A_{ij}'(\rho_l)(e^{\rho_l t}-1)\sum_{r=1}^{N} n_{i,0,r}}{\prod_{\substack{j=-3\\ j\neq l}}^{3}(\rho_l-\rho_j)\rho_l \sum_{r=1}^{N} t_{i,r}}, i,j=-3,-2,-1,0,1,2,3, \quad (3.6)$$

3.1.2 State-Space Prediction Model

The macroeconomic environment is long believed to play a central role in the analysis of loan payment behaviors and the index alone cannot provide adequate information unless it has linkage to some benchmarks.

The health index discussed above will provide banking management a snap-shot of its portfolio quality. To predict the future health index under different economic conditions, we need a time series state-space mode to analyze the sensitivity of the health index to local macroeconomic factors.

The state space model represents a multivariate time series through auxiliary variables, some of which may not be directly observable (SAS Online Doc, 2005). These auxiliary variables constitute the state vector. The state vector summarizes all the information from the present and past values of the time series relevant to the prediction of future values of the series. The observed time series is expressed as a linear combination of the state variables. The state space model is also called a Markovian representation, or a canonical representation, of a multivariate time series process. The state space approach to modeling a multivariate stationary time series is summarized in (Wei, 1990).

The state space form encompasses a very rich class of models. Any Gaussian multivariate stationary time series can be written in a state space form, provided that the dimension of the predictor space is finite (Box and Jenkins, 1994). In particular, any autoregressive moving average (ARMA) process has a state space representation and, conversely, any state space process can be expressed in an ARMA form (Wei, 1990).

Let X_t be the r ×1 vector of observed variables, after differencing (if differencing is specified) and subtracting the sample mean. Let H_t be the state vector of dimension s, r, where the first r components of H_t consist of X_t. Let $X_{t+k|t}$ be the conditional expectation (or prediction) of X_{t+k} based on the information available at time t. Then, the last s - r elements of H_t consist of elements of $X_{t+k|t}$, where k > 0 is specified or determined automatically by the procedure (SAS Online Doc, 2005).

Various forms of the state space model are in use. The form of the state space model used by the STATESPACE procedure is based on Wei (1990). The model is defined by the following state transition equation:

$$H_{t+1} = F \times H_t + Ge_{t+1} \tag{3.7}$$

In the state transition equation, the s × s coefficient matrix F is called the transition matrix. It determines the dynamic properties of the model. The s ×r coefficient matrix G is called the input matrix. It determines the variance structure of the transition equation. For model identification, the first r rows and columns of G are set to an r × r identity matrix (SAS Online Doc, 2005). The input vector e_t is a sequence of independent normally distributed random vectors of dimension r with mean 0 and covariance matrix. The random error e_t is sometimes called the innovation vector or shock vector (SAS Online Doc, 2005).

3.1.3 Multivariate Regression Model

Although it is optimal to use the state space model to link the retail mortgages' health index to local macroeconomic factors, the stated space model produces accurate estimates of the parameters when more than 40 consecutive periods of data are available. Thus, a multivariate regression model would be an alternative because of its less strict requirement. We assume that the relation between retail mortgages' health and local macroeconomic factors is linear. This assumption seems realistic as most nonlinear models could be simply transferred to linear ones by taking the log transform. Thus, the model is given as

$$H = b_0 + b_1 Ir + b_2 Un + b_3 In + b_4 Dpi, \tag{3.8}$$

where, H, the health index, is the dependent variable. The 4 independent variables include Ir ,interest rate, Un ,unemployment, In , inflation, and Dpi ,disposable person income. The SAS software was used to fit the model to the data.

3.2 Application

The data of 18 periods of retail mortgage loans, provided by the Ohio local bank mentioned in chapter 2, were used in the regression model given in equation (3.6) to estimate the health index of the loans. Then, because there are no sufficient data to utilize efficiently the stated space procedure, or Markovian representation, we used the SAS multivariate regression procedure to analyze the relationship between the macroeconomic factors and the retail mortgages health index of this bank. The SAS output for the regression model results are presented in subsection 3.2.2.

3.2.1 Chiang's Health Index Model

For practical reasons, further modification must be made to deal with the data series structure. The numerator of equation (3.5) is actually the expected number of transitions from state i to state 0 for all individuals.

$$\hat{\theta}^{t}_{i,0} = \frac{p_{i,0}N_t}{30N_t\delta_i}, i = -3,-2,-1,0,1,2,3, t = 1,2,...,16, \tag{3.9}$$

where $\hat{\theta}^{t}_{i,0}$ is the intensity function at period t, N_t is the total number of retail mortgages at period t, $N_t = \sum_{S_i=-3}^{S_3} N_{S_i,t}$. Thus, N_t represents all individuals in the transient states. Also, $p_{i,0}N_t$ is the total number of transitions made by all individual loans, where, $p_{i,0}$ is the transition probability from S_i to S_0, and S_0 is the column vector of the transition matrix $\{P_{i,j}\}$ with

$$P_{i,j}(0,t) = \sum_{l=-3}^{3} \frac{A'_{ij}(\rho_l)}{\prod_{\substack{m=-3 \\ m\neq l}}^{3}(\rho_l - \rho_m)} e^{\rho_l t}, i,j = -3,-2,-1,0,1,2,3 \tag{3.10}$$

In equation (3.9), δ_i is defined as

$$\delta_i = \begin{cases} 1, \text{ if an individual is in state } S_i \\ \\ 0, \text{ otherwise} \end{cases}$$

We use $30N_t\delta_i$ to approximate the total length of time that all individuals in the portfolio stay in $S_i, i = -3, -2, -1, 0, 1, 2, 3$. As a result, $30N_t\delta_i$ gives the expected length of time for all individuals staying in state S_i during the two month period between check points. The following table gives N_t, the total number of retail mortgages in transient states at time t.

Table 3.2 Number of retail mortgages in transient state at time t.

time t	1	2	3	4	5	6	7	8
N_t	917	875	836	821	805	786	742	741
time t	9	10	11	12	13	14	15	16
N_t	680	668	641	634	598	563	521	517

Also, the transition probabilities and expected duration with $t = 1$, calculated by equation (3.10) and (3.2), respectively, are given in Figure 3.2:

$$P_{ij}(0,1) := \begin{pmatrix} 0.08248 & 0.10673 & 0.08789 & 0.11175 & 0.08431 & 0.07858 & 0.05141 \\ 0.0858 & 0.11103 & 0.09146 & 0.1163 & 0.08779 & 0.08186 & 0.05359 \\ 0.09 & 0.11643 & 0.09583 & 0.12181 & 0.09181 & 0.08549 & 0.05586 \\ 0.08746 & 0.11318 & 0.0932 & 0.1185 & 0.0894 & 0.08332 & 0.05451 \\ 0.06219 & 0.0806 & 0.0667 & 0.08496 & 0.06462 & 0.06073 & 0.04014 \\ 0.05046 & 0.06547 & 0.05438 & 0.06937 & 0.05309 & 0.0502 & 0.03342 \\ 0.04165 & 0.05407 & 0.04496 & 0.05738 & 0.044 & 0.04168 & 0.02781 \end{pmatrix}$$

$$e(1) := \begin{pmatrix} 0.65951 & 0.8416 & 0.72246 & 0.88976 & 0.70713 & 0.64828 & 0.42881 \\ 0.68767 & 0.87756 & 0.75341 & 0.9279 & 0.73757 & 0.6763 & 0.44743 \\ 0.72122 & 0.92029 & 0.78987 & 0.97272 & 0.77282 & 0.7083 & 0.46834 \\ 0.7006 & 0.89404 & 0.76747 & 0.9452 & 0.75119 & 0.68867 & 0.45552 \\ 0.49428 & 0.6311 & 0.5426 & 0.6686 & 0.53282 & 0.48972 & 0.32494 \\ 0.40626 & 0.5189 & 0.44659 & 0.55049 & 0.43949 & 0.40463 & 0.26903 \\ 0.3362 & 0.42946 & 0.36974 & 0.45581 & 0.36412 & 0.33541 & 0.22316 \end{pmatrix}$$

Figure 3.2 Transition probability matrix and expected duration of stay.

As an example, the intensity function, $\theta_{-3,0}$, in equation (3.1) for the health index at time $t = 1$, is estimated as

$$\begin{aligned} \hat{\theta}^{1}_{-3,0} &= \frac{p_{-3,0}N_1}{30N_1\delta_{-3}} \\ &= \frac{0.11175 \times 917}{30 \times 20} \\ &= 0.1708 \end{aligned} \qquad (3.11)$$

Table (3.3) presents estimates of the intensity functions, $\hat{\theta}^{1}_{S_i,0}$ (i= -3, -2, -1, 0, 1, 2, 3) and expected duration of stay in a transient state, e_i, for $t = 1$, based on the calculation from Excel's function:

Table 3.3 Estimates of the intensity functions.

State	S_{-3}	S_{-2}	S_{-1}	S_0	S_1	S_2	S_3
$p_{i,0}$	0.11175	0.1163	0.1218	0.1185	0.0849	0.0693	0.0573
$N_1\delta_i$	20	19	75	778	19	2	4
$\hat{\theta}^1_{-3,0}$	0.1708	0.1871	0.0496	0.0047	0.1366	1.0591	0.4379
e_i	4.89753	5.10785	5.35356	5.20269	3.68406	3.0354	2.5139

Here, $e_i = \sum_{j=-3}^{3} e_{ij}$, from e(1) in Figure 3.2 and $p_{i,0}$ is the element of transition probability matrix given in Figure 3.2 at row S_{-3} and column S_0. From the data in Table (3.3), the health index in equation (3.1) is estimated to be 6.9009. By the same method, we calculated the health indexes from period 1 to period 16 which are given in Table (3.4).

Table 3.4 Health indices from period 1 to period 16.

time t	1	2	3	4	5	6	7	8
N_t	6.9009	6.5848	7.6233	7.2478	7.0647	6.4571	6.6478	6.2145
time t	9	10	11	12	13	14	15	16
N_t	6.1784	6.0658	5.4783	5.9847	5.8473	5.8741	5.6478	5.3421

3.2.2 Multivariate regression model

The purpose of the multivariate regression model is to find the relationship between the retail mortgage payment behaviors indicated by the loans' health indexes and the local macroeconomic factors. Although regression might not be the optimal method, it is perhaps best to use under the circumstance where the data set is too small for the state space analysis. The local macroeconomic data extracted by econmagic.com, the commercial economic database is given in Table (3.5).

Table 3.5 Local macroeconomic variables.

	Un_i	Ir_i	In_i	Dp_i
2005 04	5.90	5.86	5.74	4.93
2005 05	5.60	5.72	4.62	4.78
2005 06	6.10	5.58	5.62	4.63
2005 07	5.80	5.70	7.69	4.48
2005 08	5.50	5.82	6.98	4.30
2005 09	5.60	5.77	5.71	4.20
2005 10	5.30	6.07	3.06	3.90
2005 11	5.60	6.33	7.56	3.80
2005 12	5.50	6.27	6.05	4.30
2006 01	6.10	6.15	8.18	3.30
2006 03	5.30	6.32	4.30	3.20
2006 04	5.40	6.51	7.45	2.70
2006 05	4.90	6.60	5.51	2.50
2006 06	5.20	6.68	2.40	2.50
2006 07	5.80	6.76	5.47	2.50

Thus, using the SAS regression procedure, we have the following multivariate regression model, representing the relation between a retail mortgage loan health index and local macroeconomic factors. Figure 3.3 gives the SAS output for the multivariate regression model

$$H = 17.49997 - 0.41195 \times Un - 1.51859 \times Ir + 0.02949 \times In + 0.08064 \times Dpi \quad (3.12)$$

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The REG Procedure

Model: MODEL1

Dependent Variable: h

Number of Observations Read	16
Number of Observations Used	16

Analysis of Variance

Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
Model	4	5.22384	1.30596	12.27	0.0005
Error	11	1.17096	0.10645		
Corrected Total	15	6.39479			

Root MSE	0.32627	R-Square	0.8169
Dependent Mean	6.32246	Adj R-Sq	0.7503
Coeff Var	5.16046		

Figure 3.3 SAS output for the multivariate regression model.

Parameter Estimates

Variable	DF	Parameter Estimate	Standard Error	t Value	Pr > |t|
Intercept	1	17.49997	4.61002	3.80	0.0030
un	1	-0.41195	0.30259	-1.36	0.2006
ir	1	-1.51859	0.54755	-2.77	0.0181
in	1	0.02949	0.05867	0.50	0.6251
dp	1	0.08064	0.23900	0.34	0.7422

Figure 3.3 Continued.

The most important indications of model performance as a whole are the P-value of the F test for the model and R^2_{Adj}, or adjusted R-square, which are 0.0005 and 75.03%, respectively. Because the critical value is $\alpha = 0.05$ and the P-value for the F test is far smaller than 0.05, the model is highly significant as a whole. Also, $R^2_{Adj} = 75.03\%$ means that 75.03% of the total variation of the dependent variable, which is the loan health index, could be explained by the model. The parameter estimates, however, are not significant except for the independent variable ir (p = 0.0181).

3.3 Conclusion

The models presented in this chapter include a health index model and a multivariate regression model. The former provide a stochastic measurement for the loan payment behavior and the latter could be used to analyze and predict the behavior under different macroeconomic environments. Also, the sign of the parameters given in equation (3.11) confirm the empirical evidence of the macroeconomic effects on the loans payment behavior, which is summarized in the following table:

Table 3.6 Macroeconomic effects on the loans payment behavior.

	Unemployment	Interest Rate	Inflation	Disposable Income
Health Index	Negative	Negative	Positive	Positive

Although effects of unemployment and Disposable income require little explanation, the rising of interest rate requires more mortgage payment because of the increasing financial charges if the mortgage rate is not fixed (use market rate as a reference). The direct effect of inflation is to decrease the money value and increase the real estate value. Thus, using less worthy money to pay off more worthy property might be a good idea under the circumstance.

CHAPTER 4

A MARKOV CHAIN DECISION MODEL WITH REGARD TO LOAN OFFICER COMPENSATION AND LOAN COLLECTION

In this chapter, two Markov decision models would be used to estimate the appropriate compensation for loan officers and optimal credit collection policies. The appropriate compensation for an individual loan officer should be based on a sophisticated balance between benefits and costs for the bank that he or she is representing. Benefits refer to the investment returns from the different Markov states in the portfolio, such as past due, prepayment, and default.. Costs include the collection costs for the portfolio, associating with each Markov state, and compensations for the loan officer.

The effect of credit risk asset management calls for the use of dynamic stochastic techniques for optimizing decision making. In this chapter, a stochastic transition model is presented to analyze the balance between the recovered credit assets from a variety of collection policies and the collection cost associated with each policy. Thus, a Markov decision model is used to identify the optimal policy package to maximize benefits for the bank. A policy package is defined as a series of actions to be taken corresponding to each of the past due and default states. Without loss of generality, it is assumed there is no delay of customers' reaction on reception of a collection notice from the bank and the prepayment is generally in the best interest of the bank inspite of interest losses because of prepayment.

One purpose of this study is to analyze the duration of the loan system in an "up" or "down" state where "up" or "down" refers to the bank's stochastic portfolio value being larger or smaller than the bank's liability balance, which is the customers deposit in the bank. For predicting the rate of the loan system breakdown, other stochastic models are used to estimate the portfolio value and the liability balance, respectively. These models are useful to approximate the bank's ability to take risk and to avoid bankruptcy due to over-issuing of loans.

4.1 The Models

In this section, we present two Markov decision models to analyze the optimal loan officer compensation policies and optimal credit collection policies. In the former model, we need to maintain a sensitive balance between the benefits of stochastic credit asset contributed by each of the loan officers and the bonus plan or compensation policies to motivate the loan officers. Generally speaking, with more aggressive compensation policies for loan officers, one expects more credit assets that could earn more interest for the bank. However, the purpose for the optimal credit collection model is to choose a feasible policy package such that the positive difference between the benefits from the collected credit assets and the cost of the collection policy is maximal. In subsection 4.1.1 we present the model for optimal loan officer compensation while the optimal loan collection policy is introduced in subsection 4.1.2. Finally, the model to analyze the duration of the loan system ups and downs is given in subsection 4.1.3

4.1.1 A Dynamic Model for Loan Officer Compensation

We use continuous time Markov chain theory to build a stochastic model in order to estimate the portfolio values belonging to each of the states, prepayment, past due, and default. Combined with the cost estimate model, including the collection cost and compensation cost, the dynamic model is used to find the optimal policy so that maximum profits could be achieved. The validity of the model depends on the following assumptions:

1. The investment return rate is independent from state to state.

2. The relationship between the performance of a loan officer and his compensation could be represented by a linear regression function, namely $R(r) = a + br$. This assumption was confirmed by Magnan and St-Onge (1997)

3. The portfolio asset under analysis is associated with only one loan officer. Thus, the flexibility of the model presented in this paper would let the bank's management to specify the optimal compensation policy for each officer. As a result, the compensation policy could be optimized and benefits could be achieved for the bank as a whole.

4. The collection cost is associated only with assets in past due states. Due to the fact that the defaulted asset would be transferred to a third independent party for collection, the costs for collecting defaulted assets would not therefore be encountered by the bank which issues the portfolio.

4.1.1.1 Benefits Associated with Each State

Following the state definition given in Table 2.2, the expected portfolio size in state S_j at time t, and is given as

$$q_j(t) = \sum_{i=-3}^{3} \int_0^t \lambda_i \cdot P_{ij}(\tau, t) d\tau \tag{4.1}$$

Here, λ_i is the immigration rate to state S_i and $P_{ij}(\tau,t)$ is the probability of being in state S_j at time t given that the process was in state S_i at time τ. The solution of $P_{ij}(\tau,t)$ depends on A_{ij}', the *ijth* element of the characteristic matrix of V', the transpose of the intensity matrix V, defined as

$$A_{ij}' = (\rho I - V'), \tag{4.2}$$

where $\rho_l =$ eigenvalue of the intensity matrix V, which is given as

$$V = \begin{array}{c} \\ S_{-3} \\ S_{-2} \\ S_{-1} \\ S_0 \\ S_1 \\ S_2 \\ S_3 \end{array} \begin{array}{c} \begin{array}{ccccccc} S_{-3} & S_{-2} & S_{-1} & S_0 & S_1 & S_2 & S_3 \end{array} \\ \begin{bmatrix} v_{-3,-3} & v_{-3,-2} & v_{-3,-1} & v_{-3,0} & v_{-3,1} & 0 & 0 \\ v_{-2,-3} & v_{-2,-2} & v_{-2,-1} & v_{-2,0} & v_{-2,1} & 0 & 0 \\ v_{-1,-3} & v_{-1,-2} & v_{-1,-1} & v_{-1,0} & v_{-1,1} & 0 & 0 \\ v_{0,-3} & v_{0,-2} & v_{0,-1} & v_{0,0} & v_{0,1} & 0 & 0 \\ v_{1,-3} & v_{1,-2} & v_{1,-1} & v_{1,0} & v_{1,1} & v_{1,2} & 0 \\ v_{2,-3} & v_{2,-2} & v_{2,-1} & v_{2,0} & v_{2,1} & v_{2,2} & v_{2,3} \\ v_{3,-3} & v_{3,-2} & v_{3,-1} & v_{3,0} & v_{3,1} & v_{3,2} & v_{3,3} \end{bmatrix} \end{array}$$

The intensity v_{ii} that an individual stays in its original state in the time interval $(\tau, \tau + \Delta t)$, is defined as: $v_{ii} = -(v_{ij} + \sum_{\delta=1}^{4} u_{jk}), i \neq j, i, j = -3,-2,-1,0,1,2,3, k = 1,2,3,4$. By this definition, it is obvious that $1+v_{ii}\Delta t=$ Pr {an individual in state S_i at time τ will be state S_i at time $\tau+\Delta t$ }. The transition probability in the time interval $(0,t)$, is given as

$$P_{ij}(0,t)=\sum_{l=-3}^{3}\frac{A'_{ij}(\rho_l)}{\prod_{\substack{m=-3\\m\neq l}}^{3}(\rho_l-\rho_m)}e^{\rho_l t}, i,j=-3,-2,-1,0,1,2,3 \tag{4.3}$$

Thus, equation (4.1) becomes

$$\begin{aligned} q_j(t)&=\sum_{i=-3}^{3}\int_0^t \lambda_i\cdot P_{ij}(\tau,t)d\tau \\ &=\sum_{i=-3}^{3}\int_0^t \lambda_i\cdot\sum_{l=-3}^{3}\frac{A'_{ij}(\rho_l)}{\prod_{\substack{m=-3\\m\neq l}}^{3}(\rho_l-\rho_m)}e^{\rho_l(t-\tau)}d\tau, \\ &=\sum_{i=-3}^{3}\sum_{l=-3}^{3}\lambda_i\frac{A'_{ij}(\rho_l)}{\prod_{\substack{m=-3\\m\neq l}}^{3}(\rho_l-\rho_m)}(e^{\rho_l t}-1), i,j=-3,-2,-1,0,1,2,3 \end{aligned} \tag{4.4}$$

Let r_j be the investment return rate from the portfolio asset belonging to state S_j. Hence, the total expected investment benefits can be expressed as

$$\begin{aligned} R&=\sum_{j=0}^{6}E(X_j r_j)=\sum_{j=0}^{6}E(X_j)E(r_j) \\ &=\sum_{j=0}^{6}r_j[\sum_{i=0}^{6}b_i p_{ij}(0,t)+q_j(t)], i,j=0,1,2,3,4,5,6 \end{aligned} \tag{4.5}$$

where X_j, the portfolio asset , is given in equation (2.13), and b_i is the loan balance in state i.

4.1.1.2 Cost Associated with Each State

The expected portfolio value for each state S_j is $E[X_j(t)] = \sum_{i=0}^{6} i_i p_{ij}(0,t) + q_j(t)$, $i, j = 0,1,2,3,4,5,6$. Suppose that the bank has a collection policy for each of the states, $C(c_j)$. Thus, the total expected collection cost will be

$$\begin{aligned} E[C[X(t)]] &= E[X_j(t)C(c_j)] \\ &= c_j[\sum_{i=0}^{6} b_i p_{ij}(0,t) + q_j(t)], i, j = 0,1,2,3,4,5,6 \end{aligned} \tag{4.6}$$

4.1.1.3 Optimal Compensation Policy

Let $x_j, j = 0,1,2,3,4,5,6$ be the additional portfolio asset because of the implementation of the loan officer compensation policy, k, presented as a percentage of the extra portfolio asset the loan officer brings to the bank. If e^{-it} is the discount factor, where i is the discount rate, then the estimated benefits from implementing policy k, is given as

$$\begin{aligned} e^{-it}E(\sum_{j=0}^{6} R'_j) &= \sum_{j=0}^{6} E(R'_j) = \sum_{j=0}^{6} E[(X_j + x_j)r_j] = \sum_{j=0}^{6} E((X_j + x_j))E(r_j) \\ &= e^{-it}\sum_{j=0}^{6} r_j x_j [\sum_{i=0}^{6} i_i p_{ij}(0,t) + q_j(t)] \end{aligned} \tag{4.7}$$

By including the compensation policy as cost in the total cost in equation (4.6), we have the discount total cost equation, including compensation and collection costs, which can be expresses as

$$e^{-it}E[C[X(t)]] = e^{-it}E[X_j(t)C(c_j)]$$
$$= e^{-it}\{c_j[\sum_{i=0}^{6} i_i p_{ij}(0,t) + q_j(t)] + x_j k\}, i, j = 0,1,2,3,4,5,6 \tag{4.8}$$

Here, $x_j k$ is the compensation cost to the bank. As a result, the dynamic equations for obtaining an optimal compensation policy is given as

Maximize:

$$f(k) = e^{-it}\{\sum_{j=0}^{6} r_j x_j [\sum_{i=0}^{6} i_i p_{ij}(0,t) + q_j(t)] - c_j [\sum_{i=0}^{6} i_i p_{ij}(0,t) + q_j(t)] + x_j k\} \tag{4.9}$$

Subject to: $p_{ij}(0,t) \geq 0$;

$$\sum_i \sum_j p_{ij}(0,t) = 0$$

4.1.2 Loan Collection Policy

The purpose for building this model is to find the optimal loan collection policy package. The package includes letters, emails, phone calls, corresponding to each of the state. In practice, there is no collection method for the prepayment state. Also, for completeness, the collection method for the normal state is included. In practice, this could be defined as sending a statement letter, which is normal operation for banks. The model is given as:

$$B_{t=0} = \int_0^{\infty} e^{-it}[R(t) - e^{-it}C(t)]dt$$
$$\Rightarrow E(B_{t=0}) = E\{\int_0^{\infty} e^{-it}[R(t) - e^{-it}C(t)]dt\} \tag{4.10}$$
$$\Rightarrow E(B_{t=0}) = \int_0^{\infty} e^{-it}E[R(t)]dt - \int_0^{\infty} e^{-it}E[C(t)]dt$$

where $B_{t=0}(k)$ is the benefits function of policy package k at time $t=0$, $R(k)$ is the recovered credit asset, $C(k)$ is the cost function for each policy package, and e^{-ix} is the continuous discount factor.

4.1.2.1 Effective Recovered Economic Assets

The estimation of portfolio value is given again by equation (4.4) as

$$q_j(t) = \sum_{i=-3}^{3}\sum_{l=-3}^{3} \lambda_i \frac{A'_{ij}(\rho_l)}{\prod_{\substack{m=-3 \\ m\neq l}}^{3} (\rho_l - \rho_m)} (e^{\rho_l t} - 1)i, j = -3,-2,-1,0,1,2,3 \qquad (4.11)$$

For simplicity, we assume that a transition to a lower state is due to the implementation of a collection policy. Thus, an auxiliary function $I(x)$ is defined as

$$I(t) = \begin{cases} 1, \text{ if } i < j, \text{ which means the collection policy is effective} \\ 0, \text{ if } i > j, \text{ which means the collection policy is not effective} \end{cases} \qquad (4.12)$$

Thus, we have the following derivation:

$$\begin{aligned}
&\int_0^T X(t)dt = \int_0^\infty X(t)I(t)dt \\
&\Rightarrow E[\int_0^T X(t)dt] = E[\int_0^\infty X(t)I(t)dt] \\
&\Rightarrow E[\int_0^T X(t)dt] = \int_0^\infty E[X(t)]E[I(t)]dt \\
&\Rightarrow E[\int_0^T X(t)dt] = \int_0^\infty E[X(t)]\sum_i \chi_{(i<j)}dt
\end{aligned}$$

$$\Rightarrow E[\int_o^T X(t)dt] = \int_o^\infty E[X(t)]dt \sum_i \chi_{(i<j)}$$

$$\Rightarrow \int_o^\infty E[X(t)]dt = \frac{E[\int_o^T X(t)dt]}{\sum_i \chi_{(i<j)}} as T \to \infty \tag{4.13}$$

$$\Rightarrow \int_o^\infty E[X(t)]dt = \frac{\int_o^T E[X(t)]dt}{\sum_i \chi_{(i<j)}} as T \to \infty$$

Discounting the above equation by the compound discount factor, e^{-it}, where i is the discreet interest rate available from the market, we have the present value of the portfolio from the effective collection policy:

$$E(R) = \frac{e^{-it}\int_o^T E[R(t)]dt}{\sum_i \chi_{(i<j)}} as T \to \infty \tag{4.14}$$

where $\sum_i \chi_{(i<j)}$ could be reached by a limiting method provided by Ross (2002).

Ross (2002) provided a very useful model to predict the long run probabilities in each of the states. Let V' be the reduced-form transition probability matrix within a single time interval $(\tau, \tau + \Delta t)$, and V'^n be the transition probability after n time periods. Furthermore, assume $\chi_j = \lim_{n\to\infty} v_{ij}'^n$, where v_{ij}' is the ith row and jth column component of matrix V' defined in Figure 4.1,

$$
V' = \begin{array}{c|ccccccc|}
 & S_0 & S_1 & S_2 & S_3 & S_4 & S_5 & S_6 \\
\hline
S_0 & v'_{0,0} & 0 & 0 & 0 & 0 & 0 & 0 \\
S_1 & v'_{1,0} & v'_{1,1} & 0 & 0 & 0 & 0 & 0 \\
S_2 & v'_{2,0} & v'_{2,1} & v'_{2,2} & 0 & 0 & 0 & 0 \\
S_3 & v'_{3,0} & v'_{3,1} & v'_{3,2} & v'_{3,3} & 0 & 0 & 0 \\
S_4 & v'_{4,0} & v'_{4,1} & v'_{4,2} & v'_{4,3} & v'_{4,4} & 0 & 0 \\
S_5 & v'_{5,0} & v'_{5,1} & v'_{5,2} & v'_{5,3} & v'_{5,4} & v'_{5,5} & 0 \\
S_6 & v'_{6,0} & v'_{6,1} & v'_{6,2} & v'_{6,3} & v'_{6,4} & v'_{6,5} & v'_{6,6}
\end{array}
$$

Figure 4.1 Reduced-Form Transition matrix.

In which,

$$
v'_{i,j} = \begin{cases} v_{i,j}k_i,\ i > j \text{ which means the collection policy } k_i \text{ is effective} \\ \\ 0,\ i < j \text{ which means the collection policy } k_i \text{ is not effective} \end{cases} \tag{4.15}
$$

Then,

$$
\begin{aligned}
&\chi_j = \sum_{i=-3}^{2} \chi_i v_{ij}, j = -3,-2,-1,0,1,2 \\
&\sum_j \chi_j = 1,, j = -3,-2,-1,0,1,2,3
\end{aligned} \tag{4.16}
$$

is the percentage of the individuals in state j, where

$$
\begin{cases}
\chi_{-3} = \chi_{-3}v_{-3,-3} + \chi_{-2}v_{-2,-3} + \chi_{-1}v_{-1,-3} + \chi_0 v_{0,-3} + \chi_1 v_{1,-3} + \chi_2 v_{2,-3} + \chi_3 v_{3,-3} \\
\chi_{-2} = \chi_{-3}v_{-3,-2} + \chi_{-2}v_{-2,-2} + \chi_{-1}v_{-1,-2} + \chi_0 v_{0,-2} + \chi_1 v_{1,-2} + \chi_2 v_{2,-2} + \chi_3 v_{3,-2} \\
\ldots\ldots \\
\chi_2 = \chi_{-3}v_{-3,2} + \chi_{-2}v_{-2,2} + \chi_{-1}v_{-1,2} + \chi_0 v_{0,2} + \chi_1 v_{1,2} + \chi_2 v_{2,2} + \chi_3 v_{3,2} \\
\chi_{-3} + \chi_{-2} + \chi_{-1} + \chi_0 + \chi_1 + \chi_2 + \chi_3 = 1
\end{cases} \tag{4.17}
$$

4.1.2.2 Total Cost

By the same reasoning, we have the expected economic portfolio value for each state S_j, $E[X_j(t)] = \sum_{i=0}^{6} i_i p_{ij}(0,t) + q_j(t), i, j = 0,1,2,3,4,5,6$, and suppose that the bank has a collection policy for each of the state, $C(c_j)$. Thus, the total expected collection cost is given by the following equation:

$$\begin{aligned} E[C[X(t)]] &= E[X_j(t)C(c_j)] \\ &= c_j k_j [\sum_{i=0}^{6} i_i p_{ij}(0,t) + q_j(t)], i, j = 0,1,2,3,4,5,6 \end{aligned} \tag{4.18}$$

where $C(c_j) = c_j k_j$ is the cost increments factor from the implementation of collection policy k_j.

4.1.2.3 Dynamic Decision Making

Let $x_j, j = 0,1,2,3,4,5,6$ be the portfolio asset due to the implementation of the loan officer compensation policy, k, presented as a percentage of the extra portfolio asset the loan officer brings to the bank. If e^{-it} is the discount factor, where i is the discrete discount rate, then the estimated cost from the implementation of policy, k, is given as

$$\begin{aligned} e^{-it} E[C[X(t)]] &= e^{-it} E[X_j(t)C(c_j)] \\ &= e^{-it} \{c_j k_j [\sum_{i=0}^{6} i_i p_{ij}(0,t) + q_j(t)]\}, i, j = 0,1,2,3,4,5,6 \end{aligned} \tag{4.19}$$

As a result, the dynamic approach for obtaining an optimal compensation policy is given as

$$\text{Maximize: } B_{t=0} = e^{-it}\{\frac{\int_o^T E[R(t)]dt}{\sum_i \chi_{(i<j)}} - c_j k_j [\sum_{i=0}^{6} i_i p_{ij}(0,t) + q_j(t)]\} \tag{4.20}$$

Subject to: $p_{ij}(0,t) \geq 0$, and $\sum_i \sum_j p_{ij}(0,t) = 0$

4.1.2.4 Proceedings from the Past Due State

The approach by Ross (2002) could also be used to estimate the proceedings for past due. Let V be the transition probability matrix within a single time interval $(\tau, \tau + \Delta t)$, and V^n be the transition probability after n time periods. Furthermore, we assume that $\chi_j = \lim_{n \to \infty} v^n{}_{ij}$, where v_{ij} is the ith row and jth column component of the full matrix V. Then,

$$\begin{aligned} \chi_j &= \sum_{i=-3}^{2} \chi_i v_{ij}, j = 1,2,3 \\ \sum_j \chi_j &= 1,, j = 1,2,3 \end{aligned} \tag{4.21}$$

is the percentage of individuals in state S_j, where

$$\begin{cases} \chi_1 = \chi_1 v_{1,2} + \chi_2 v_{2,2} + \chi_3 v_{3,2} \\ \chi_2 = \chi_1 v_{1,2} + \chi_2 v_{2,2} + \chi_3 v_{3,2} \\ \chi_{-3} + \chi_{-2} + \chi_{-1} + \chi_0 + \chi_1 + \chi_2 + \chi_3 = 1 \end{cases} \tag{4.22}$$

If we let f_i be the penalty amount for being in past due state i, then the total expected proceedings from customers being past due during the period $1, 2, \ldots., N$ is given by

$$\sum_{i,N} f = \sum_{i=-3}^{3} f_i \chi_i \qquad (4.23)$$

4.1.3 Analysis of Loan System Status

The idea here is based on the fact that the bank's economic assets provide a warranty for its liability, which means, in the long run, the assets have to be more than liability to guarantee its insolvency. Thus, if we use the random variables $X(t)$ and $D(t)$ to describe the bank's economic assets and liability, respectively, the expected system status, $S(t) = X(t) - D(t)$, should be positive over a long period, although it might be negative from time to time. In reality, the liability of a bank is generally represented by the deposits from customers, and the economic assets could be considered as its loan portfolio economic value. By economic, one means the actual value to the bank, considering its potential risk, other than the numbers on the books.

We will use a continuous time Markov chain model to estimate the economic portfolio value, $X(t)$, where $t = 0,1,2,..,T$. As such, the deposit process is represented by a compound Poisson process, $D(t), t = 0,1,2,..,T$, in which we assume that the arrival of a customer follows the Poisson distribution. Also, the deposit or withdrawal amount of each customer follows an exponential distribution. Finally, a Markov model is used to estimate the expected rate of the whole loan system as well as the expected duration that the system stays in an "up" state.

4.1.3.1 Portfolio Value

We will use the estimation of economic portfolio is given by equation (4.4) with exactly the definition:

$$q_j(t) = \sum_{i=-3}^{3}\sum_{l=-3}^{3} \lambda_i \frac{A'_{ij}(\rho_l)}{\prod_{\substack{m=-3 \\ m\neq l}}^{3} (\rho_l - \rho_m)} (e^{\rho_l t} - 1) j, j = -3,-2,-1,0,1,2,3 \quad (4.24)$$

4.1.3.2 Compound Poisson Model

Ross (2002) and Cummins (1991) provided a method to compute the expectations by conditioning. We can use this technique to find the expected daily net changes of deposit balance. For simplicity of the model, we consider all transactions are either deposits or withdrawals, no matter how the transaction is fulfilled, whether by wire, direct transfer, or branch operation. Thus, the expected month-end deposit pool balance is given by:

$$E[D(x)] = E[\sum_{i=1}^{n} y_i] - E[\sum_{j=1}^{\bar{n}} \bar{y}_j], i = 1,2,3,...,n, j = 1,2,3,...,\bar{n}. \quad (4.25)$$

where y_i refers to a deposit transaction, $\bar{y}_i$ is a withdraw transaction, and n is the number of daily transactions. Also, by assumption, n follows a Poisson distribution and y_i or $\bar{y}_i$ an exponential distribution. Using the conditional approach, we have

$$\begin{aligned}
E[D(x)] &= E[E[D(x)] \mid N(t) = n] \\
&= E[E[\sum_{i=1}^{n} y_i] \mid N(t) = n]] - E[E[\sum_{j=1}^{\bar{n}} \bar{y}_j \mid \bar{N}(t) = \bar{n}]] \\
&= E[E[\sum_{i=1}^{n} y_i]] - E[E[\sum_{j=1}^{\bar{n}} \bar{y}_j]] \\
&= E[nE[y_i]] - E[\bar{n}E[\bar{y}_j]] \\
&= \lambda t E[y_i] - \bar{\lambda} t E[\bar{y}_i]
\end{aligned}$$

(4.26)

The result follows from the fact that $\bar{y}_i$, and y_i are independent of $N(t) = n$ and $\bar{N}(t) = \bar{n}$.

4.1.3.3 Loan System Status

After $S(t) = X(t) - D(t)$ is determined for each of the bank's products, namely Credit Card, Mortgages, Line of Credit, etc., one can define the Markov states as follows in Table 4.1:

Table 4.1 Different states of the Markov chain.

$S_{i,j}, i,j = -3,-2,-1,0,1,2,3$		
A Accepted states	S_{-3}	$E[x(i)] - E[d(i)] > 3\delta$
	S_{-2}	$2\delta < E[x(i)] - E[d(i)] < 3\delta$
	S_{-1}	$\delta < E[x(i)] - E[d(i)] < 2\delta$
	S_0	$-\delta < E[x(i)] - E[d(i)] < \delta$
A^c Unaccepted states	S_1	$-\delta < E[x(i)] - E[d(i)] < -2\delta$
	S_2	$-2\delta < E[x(i)] - E[d(i)] < -3\delta$
	S_3	$E[x(i)] - E[d(i)] < -3\delta$

Let the process be in state S_i, if $S(t), t = 0,1,2,..,T$ is within the interval $[i\delta, (i+1)\delta]$, where $\delta = \frac{|x_i - d_i|}{(x_i + d)/2}, i = 0,1,2,...,T$. By assumption, $X(t)$ and $D(t)$ are Markovian. Thus, each state of $S(t), t = 0,1,2,..,T$ could be represented by a continuous time Markov chain.

From Table (4.1), it is obvious that the S_{-3}, S_{-2}, S_{-1}, S_0 states, in which the economic portfolio provides liability insurance with regard to bank deposits, they may be defined as the accepted states. On the other hand, S_1, S_2, S_3 refer to the unaccepted states because they do not provide such insurance or protection.

The matrix of transition intensities, given by the V matrix in Fig 4.2, is regular or ergodic since the system is a closed set of communicative states.

$$
V = \begin{array}{c} \\ S_{-3} \\ S_{-2} \\ S_{-1} \\ S_0 \\ S_1 \\ S_2 \\ S_3 \end{array}
\begin{array}{c}
\begin{array}{ccccccc} S_{-3} & S_{-2} & S_{-1} & S_0 & S_1 & S_2 & S_3 \end{array} \\
\begin{bmatrix}
v_{-3,-3} & v_{-3,-2} & v_{-3,-1} & v_{-3,0} & v_{-3,1} & v_{-3,2} & v_{-3,3} \\
v_{-2,-3} & v_{-2,-2} & v_{-2,-1} & v_{-2,0} & v_{-2,1} & v_{-2,2} & v_{-2,3} \\
v_{-1,-3} & v_{-1,-2} & v_{-1,-1} & v_{-1,0} & v_{-1,1} & v_{-1,2} & v_{-1,3} \\
v_{0,-3} & v_{0,-2} & v_{0,-1} & v_{0,0} & v_{0,1} & v_{0,2} & v_{0,3} \\
v_{1,-3} & v_{1,-2} & v_{1,-1} & v_{1,0} & v_{1,1} & v_{1,2} & v_{1,3} \\
v_{2,-3} & v_{2,-2} & v_{2,-1} & v_{2,0} & v_{2,1} & v_{2,2} & v_{2,3} \\
v_{3,-3} & v_{3,-2} & v_{3,-1} & v_{3,0} & v_{3,1} & v_{3,2} & v_{3,3}
\end{bmatrix}
\end{array}
$$

Figure 4.2 Transition intensities within the S-states (V matrix).

For the time interval $(0,t), t < \infty$, the transitions intensities among the states are defined as follows:

$v_{ij}\Delta t$ = Pr {an individual in state S_i at time τ will be state S_j at time $\tau + \Delta t$ }, where $i \neq j; i, j = -3, -2, -1, 0, 1, 2, 3,$

$1 + v_{ii}\Delta t$ = Pr {an individual in state S_i at time τ will be state S_i at time $\tau + \Delta t$ }.

Furthermore, we assume that future transitions of an individual are independent of past transitions. In other word, the intensities v_{ij} are assumed to be independent of time. Thus, we are concerned here with a time homogenous Markov chain.

4.1.3.4 Transition Probabilities

Let $P_{ij}(\tau, t)$ = Pr {an individual in state S_i at time τ will be state S_j at time t }, $i, j = -3, -2, -1, 0, 1, 2, 3$ .Considering three points, $\tau < t < t + \Delta t$, by definition we have

$$\begin{aligned} P_{ij}(t, t + \Delta t) &= v_{\gamma j}(t)\Delta t \\ P_{jj}(t, t + \Delta t) &= 1 + v_{jj}(t)\Delta t \end{aligned} \tag{4.27}$$

$$P_{ij}(\tau, t + \Delta t) = P_{ij}(\tau, t)P_{jj}(t, t + \Delta t) + \sum_{\gamma \neq j} P_{i\gamma}(\tau, t)P_{\gamma j}(t, t + \Delta t) \tag{4.28}$$

By substituting Eq. (3) into Eq. (4) and rearranging the equation we have:

$$\begin{aligned} &\frac{P_{ij}(\tau, t + \Delta t) - P_{ij}(\tau, t)}{\Delta t} = P_{ij}(\tau, t)v_{jj}(t)\Delta t + \sum_{\gamma \neq j} P_{i\gamma}(\tau, t)v_{\gamma j}(t) \\ &\Rightarrow \lim_{\Delta \to 0} \frac{P_{ij}(\tau, t + \Delta t) - P_{ij}(\tau, t)}{\Delta t} = \sum_{\gamma \neq j} P_{i\gamma}(\tau, t)v_{\gamma j}(t) \\ &\Rightarrow \frac{\delta}{\delta t_{ij}} P_{ij}(\tau, t) = \sum_{\gamma \neq j} P_{i\gamma}(\tau, t)v_{\gamma j}(t); i, j = -3, -2, -1, 0, 1, 2, 3 \end{aligned} \tag{4.29}$$

This is called the Kolmogorov Forward Differential Equation, and its solution is given by (Chiang, 1980):

$$P_{ij}(0,t) = \sum_{l=-3}^{3} \frac{A'_{ij}(\rho_l)}{\prod_{\substack{m=1 \\ m\neq l}}^{s} (\rho_l - \rho_m)} e^{\rho_l t}, i, j = -3,-2,-1,0,1,2,3 \quad (4.30)$$

Here, A_{ij}' is the characteristic matrix of V', the transpose of the intensity matrix V, defined by

$$A_{ij}' = (\rho I - V'), \quad (4.31)$$

where ρ_l = Eigenvalue of the intensity matrix V.

<u>4.1.3.5 Expected duration of Stay in a State</u>

For an individual in state S_i at time 0, let $e_{ij}(t)$ = the expected duration of stay in S_j during the interval $(0,t), j = -3,-2,-1,0,1,2,3$. In terms of our process, $e_{ij}(t)$ evaluates the expected duration of the loan before default occurs. This expected duration, $e_{ij}(t)$, is given by (Chiang, 1980).

$$e_{ij}(t) = \int_0^t P_{ij}(0,\pi)d\pi \quad (4.32)$$

Or

$$e_{ij}(0,t) = \sum_{l=-3}^{3} \frac{A_{ij}'(\rho_l)}{\prod_{\substack{j=-3 \\ j\neq l}}^{3} (\rho_l - \rho_j)\rho_l} (e^{\rho_l t} - 1), i, j = -3,-2,-1,0,1,2,3 \quad (4.33)$$

4.1.3.6 Probabilities in the Limit

Ross (2002) provided a very useful set of equations to predict the long run probabilities in each of the states. Let V be the transition probability matrix within a single time interval $(\tau, \tau+\Delta t)$, and V^n be the transition probability after n time. periods. Furthermore, assume that $\chi_j = \lim_{n\to\infty} v^n{}_{ij}$, where v_{ij} is the ith row and jth column component of matrix V. Then,

$$\begin{aligned} &\chi_j = \sum_{i=-3}^{2} \chi_i v_{ij}, j = -3,-2,-1,0,1,2,3 \\ &\sum_j \chi_j = 1,, j = -3,-2,-1,0,1,2,3 \end{aligned} \tag{4.34}$$

is the percentage of the individuals in state j, where

$$\begin{cases} \chi_{-3} = \chi_{-3}v_{-3,-3} + \chi_{-2}v_{-2,-3} + \chi_{-1}v_{-1,-3} + \chi_0 v_{0,-3} + \chi_1 v_{1,-3} + \chi_2 v_{2,-3} + \chi_3 v_{3,-3} \\ \chi_{-2} = \chi_{-3}v_{-3,-2} + \chi_{-2}v_{-2,-2} + \chi_{-1}v_{-1,-2} + \chi_0 v_{0,-2} + \chi_1 v_{1,-2} + \chi_2 v_{2,-2} + \chi_3 v_{3,-2} \\ \ldots\ldots \\ \chi_2 = \chi_{-3}v_{-3,2} + \chi_{-2}v_{-2,2} + \chi_{-1}v_{-1,2} + \chi_0 v_{0,2} + \chi_1 v_{1,2} + \chi_2 v_{2,2} + \chi_3 v_{3,2} \\ \chi_{-3} + \chi_{-2} + \chi_{-1} + \chi_0 + \chi_1 + \chi_2 + \chi_3 = 1 \end{cases} \tag{4.35}$$

Thus,

$$\text{Breakdown Rate} = \frac{1}{\dfrac{\sum_{i\in A^c} \chi_i}{\sum_{i\in A^c}\sum_{i\in A} \chi_i p_{ij}} + \dfrac{\sum_{i\in A} \chi_i}{\sum_{i\in A^c}\sum_{i\in A} \chi_i p_{ij}}} \tag{4.36}$$

and

$$\text{Proportion of Up Time} = \frac{\dfrac{\sum_{i\in A}\chi_i}{\sum_{i\in A^c}\sum_{i\in A}\chi_i p_{ij}}}{\dfrac{\sum_{i\in A^c}\chi_i}{\sum_{i\in A^c}\sum_{i\in A}\chi_i p_{ij}} + \dfrac{\sum_{i\in A}\chi_i}{\sum_{i\in A^c}\sum_{i\in A}\chi_i p_{ij}}} \tag{4.37}$$

4.2 Conclusion

The above models are useful in that they provide the management in a bank practical tools for analyzing a loan status for each single portfolio or financial service. Also, these models help considerably in decision making and can be easily integrated into management software packages for the banking industry. Future availability of data will help in demonstrating the applicability of these models.

CHAPTER 5

A HIGHER-ORDER MULTIVARIATE MARKOV CHAIN MODEL FOR RETAIL MORTGAGES AND CREDIT CARDS

This chapter presents a high-order multivariate Markov chain model to analyze the correlation between retail mortgage loans and consumer credit cards (other than commercial cards). This model provides a quantitatively theoretical evidence for the empirical phenomenon concerning the historically high correlation between those two retail financial products. Also, conclusions about the correlation will be presented after the model is tested by real data provided by the Ohio local bank.

5.1 The Model

Multivariate Markov chain models have been successfully used in representing the behavior of multiple data sequences generated by the same source. Years of operation experience convinced the bank management of the importance of the correlation between retailed mortgage loans and personal credit cards, both of them are usually offered by a local bank to the same group of consumers in the area. In most cases, credit cards are used to purchase daily supplies, such as food and consumer goods. Thus, with the fluctuation of the macro-economic and employment situations, the question becomes: what is more important, house or food?

To answer this question, we need to have information not only about the direction of the correlation, but also about its magnitude. The high-order multivariate Markov chain model introduced by Ching and NG (2006) could be a good candidate to analyze and quantify the correlation that has been long observed by the credit risk management personals in banking.

5.1.1 Multivariate Markov Chain

Multivariate Markov chain models have many applications in multi-product demand estimation, credit rating, DNA sequence, and genetic networks. In this chapter, we will use the model proposed by Ching and Ng (2006).

$$\begin{cases} F_{n+1} = \sum_{\alpha=1}^{2} \lambda_{\alpha\beta} V^{\alpha\beta} F_n, \beta = 1,2 \\ \lambda_{\alpha\beta} \geq 1, 1 \leq \alpha, \beta \leq 2, \\ \sum_{\beta=1}^{2} \lambda_{\alpha\beta} = 1 \end{cases} \tag{5.1}$$

In this model, the parameter $\lambda_{\alpha\beta}$ that gives the direction and magnitude of the correlation is the model outcome. We define $\alpha, \beta = 1,2$ as the data sets for retail mortgage loans and personal credit cards, respectively. $F_{n+1} = (F^{\alpha}_{n+1}, F^{\beta}_{n+1})^T$ refers to the probability distribution vector in each of the states. We follow the definition of state in chapter two. That is, $S \in (S_i, R_k), i = -3,-2,-1,0,1,2,3, k = 1,2,3,4$ (please refer to Table 2.1 for detailed definition of the states). So, $F^{\alpha}_{n+1} = (F^{\alpha}_{n+1,S_i}, F^{\alpha}_{n+1,R_k})^T$ $, i = -3,-2,-1,0,1,2,3$;

$k = 1,2,3,4$ at time $t = n+1$ is the probability distribution vector for the retail mortgage, while F^{β}_{n+1} is that for personal credit cards at time $t = n+1$. $V^{\alpha\beta}$ is defined as the intensity of transition between states of retail mortgage and personal credit cards. The matrix form of model (5.1) is given as

$$F_{n+1} = \begin{pmatrix} F^{\alpha}_{n+1} \\ F^{\beta}_{n+1} \end{pmatrix} = \begin{pmatrix} \lambda_{\alpha\alpha} V^{\alpha\alpha} & \lambda_{\alpha\beta} V^{\alpha\beta} \\ \lambda_{\beta\alpha} V^{\beta\alpha} & \lambda_{\beta\beta} V^{\beta\beta} \end{pmatrix} \begin{pmatrix} F^{\alpha}_{n} \\ F^{\beta}_{n} \end{pmatrix}, V^{ij} = \begin{bmatrix} I^{ij} & O^{ij} \\ R^{ij} & Q^{ij} \end{bmatrix}, i,j = \alpha,\beta \qquad (5.2)$$

$$\text{or, } F_{n+1} = WF_n, W = \begin{pmatrix} \lambda_{\alpha\alpha} V^{\alpha\alpha} & \lambda_{\alpha\beta} V^{\alpha\beta} \\ \lambda_{\beta\alpha} V^{\beta\alpha} & \lambda_{\beta\beta} V^{\beta\beta} \end{pmatrix}$$

where, $F^{j}_{n+1} = (F_{n+1,S_i}, F_{n+1,R_k})^T, i = -3,-2,-1,0,1,2,3, k = 1,2,3,4, j = \alpha,\beta$, while I^{ij}, O^{ij}, R^{ij}, Q^{ij} are given in Figure 5.1:

$$I^{ij}_{4\times4} = \begin{array}{c} \\ R_1 \\ R_2 \\ R_3 \\ R_4 \end{array}\!\!\begin{array}{c} \begin{array}{cccc} R_1 & R_2 & R_3 & R_4 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}, \quad O^{ij}_{4\times7} = \begin{array}{c} \\ R_1 \\ R_2 \\ R_3 \\ R_4 \end{array}\!\!\begin{array}{c} \begin{array}{ccccccc} S_{-3} & S_{-2} & S_{-1} & S_0 & S_1 & S_2 & S_3 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$R^{ij}_{7\times4} = \begin{array}{c} \\ S_{-3} \\ S_{-2} \\ S_{-1} \\ S_0 \\ S_1 \\ S_2 \\ S_3 \end{array}\!\!\begin{array}{c} \begin{array}{cccc} R_1 & R_2 & R_3 & R_4 \end{array} \\ \begin{bmatrix} \mu_{-3,1} & \mu_{-3,2} & \mu_{-3,3} & \mu_{-3,4} \\ \mu_{-2,1} & \mu_{-2,2} & \mu_{-2,3} & \mu_{-2,4} \\ \mu_{-1,1} & \mu_{-1,2} & \mu_{-1,3} & \mu_{-1,4} \\ \mu_{0,1} & \mu_{0,2} & \mu_{0,3} & \mu_{0,4} \\ \mu_{1,1} & \mu_{1,2} & \mu_{1,3} & \mu_{1,4} \\ \mu_{2,1} & \mu_{2,2} & \mu_{2,3} & \mu_{2,4} \\ \mu_{3,1} & \mu_{3,2} & \mu_{3,3} & \mu_{3,4} \end{bmatrix} \end{array}, \quad Q^{ij}_{7\times7} = \begin{array}{c} \\ S_{-3} \\ S_{-2} \\ S_{-1} \\ S_0 \\ S_1 \\ S_2 \\ S_3 \end{array}\!\!\begin{array}{c} \begin{array}{ccccccc} S_{-3} & S_{-2} & S_{-1} & S_0 & S_1 & S_2 & S_3 \end{array} \\ \begin{bmatrix} v_{-3,-3} & v_{-3,-2} & v_{-3,-1} & v_{-3,0} & v_{-3,1} & 0 & 0 \\ v_{-2,-3} & v_{-2,-2} & v_{-2,-1} & v_{-2,0} & v_{-2,1} & 0 & 0 \\ v_{-1,-3} & v_{-1,-2} & v_{-1,-1} & v_{-1,0} & v_{-1,1} & 0 & 0 \\ v_{0,-3} & v_{0,-2} & v_{0,-1} & v_{0,0} & v_{0,1} & 0 & 0 \\ v_{1,-3} & v_{1,-2} & v_{1,-1} & v_{1,0} & v_{1,1} & v_{1,2} & 0 \\ v_{2,-3} & v_{2,-2} & v_{2,-1} & v_{2,0} & v_{2,1} & v_{2,2} & v_{2,3} \\ v_{3,-3} & v_{3,-2} & v_{3,-1} & v_{3,0} & v_{3,1} & v_{3,2} & v_{3,3} \end{bmatrix} \end{array}$$

$$i,j = \alpha,\beta$$

Figure 5.1 Transition intensity matrices between retail mortgages and credit cards.

$I^{ij}{}_{4\times4}, O^{ij}{}_{4\times7}$ are the transitions within transient states and transitions from transient to absorbing states, respectively. By the definition of an absorbing state, $I^{ij}{}_{4\times4}, O^{ij}{}_{4\times7}$ are a 4×4 identity matrix and a 4×7 zero matrix, respectively.

Furthermore, letting $c_{lk}{}^{ij}, i, j = \alpha, \beta$ be the transition probability between state l in data set i and state k in dataset j, the elements of $R^{ij}{}_{7\times4}$ and $Q^{ij}{}_{7\times7}$ are calculated from the following equations:

$$C_{lk}{}^{ij} = \begin{pmatrix} c_{-3,-3}{}^{ij} & \cdots & c_{-3,4}{}^{ij} \\ \vdots & \ddots & \vdots \\ c_{4,-3}{}^{ij} & \cdots & c_{4,-3}{}^{ij} \end{pmatrix} V_{lk}{}^{ij} = \begin{pmatrix} v_{-3,-3}{}^{ij} & \cdots & v_{-3,4}{}^{ij} \\ \vdots & \ddots & \vdots \\ v_{4,-3}{}^{ij} & \cdots & v_{4,-3}{}^{ij} \end{pmatrix} \quad (5.3)$$

$$v_{m,n}{}^{ij} = \begin{cases} \dfrac{c_{-3,-3}{}^{ij}}{\sum_{n=S_{-3}}^{R_4} c_{m,n}}, & \text{if } \sum_{n=S_{-3}}^{R_4} c_{m,n} \neq 0 \\ 0, & \text{Otherwise} \end{cases}$$

Based on the assumptions that $V^{ij} = \begin{bmatrix} I^{ij} & O^{ij} \\ R^{ij} & Q^{ij} \end{bmatrix}$ is irreducible and $\lambda_{\alpha\beta} > 1$, Ching and Ng (2006) proved that there is a unique stationary vector $F = \begin{pmatrix} F^{\alpha} \\ F^{\beta} \end{pmatrix}$, such that $F = WF$, and $\sum_{i=S_{-3}}^{R_k} | F^j |_i, j = \alpha, \beta$. Thus, (5.4) could also be written as:

$$\left\{\begin{array}{ll} & F=\begin{pmatrix}\lambda_{\alpha\alpha}V^{\alpha\alpha} & \lambda_{\alpha\beta}V^{\alpha\beta}\\ \lambda_{\beta\alpha}V^{\beta\alpha} & \lambda_{\beta\beta}V^{\beta\beta}\end{pmatrix}F, \\ \text{or} & \sum_{\alpha=1}^{\beta}\lambda_{\alpha\beta}V^{\alpha\beta}F^{\beta}=F^{\alpha}\Rightarrow\sum_{\alpha=1}^{\beta}\lambda_{\alpha\beta}V^{\alpha\beta}F^{\beta}-F^{\alpha}=0 \\ \text{where,} & V^{\alpha\alpha}=\begin{pmatrix}v_{-3,-3}{}^{\alpha\alpha} & \cdots & v_{-3,4}{}^{\alpha\alpha}\\ \vdots & \ddots & \vdots\\ v_{4,-3}{}^{\alpha\alpha} & \cdots & v_{4,-3}{}^{\alpha\alpha}\end{pmatrix} \\ & V^{\alpha\beta}=\begin{pmatrix}v_{-3,-3}{}^{\alpha\alpha} & \cdots & v_{-3,4}{}^{\alpha\alpha}\\ \vdots & \ddots & \vdots\\ v_{4,-3}{}^{\alpha\alpha} & \cdots & v_{4,-3}{}^{\alpha\alpha}\end{pmatrix} \\ & V^{\beta\alpha}=\begin{pmatrix}v_{-3,-3}{}^{\alpha\alpha} & \cdots & v_{-3,4}{}^{\alpha\alpha}\\ \vdots & \ddots & \vdots\\ v_{4,-3}{}^{\alpha\alpha} & \cdots & v_{4,-3}{}^{\alpha\alpha}\end{pmatrix} \\ & V^{\beta\beta}=\begin{pmatrix}v_{-3,-3}{}^{\alpha\alpha} & \cdots & v_{-3,4}{}^{\alpha\alpha}\\ \vdots & \ddots & \vdots\\ v_{4,-3}{}^{\alpha\alpha} & \cdots & v_{4,-3}{}^{\alpha\alpha}\end{pmatrix} \end{array}\right. \tag{5.4}$$

According to Ching and Ng (2006), by letting $\psi=\left\|\sum_{\alpha=1}^{\beta}\lambda_{\alpha\beta}V^{\alpha\beta}F^{\beta}-F^{\alpha}\right\|_{\infty}$ be the vector norm for measuring the difference in (5.4), where ψ is defined as $\max\{\psi_{\alpha},\psi_{\beta}\}$ by Burden and Faires (2001), the parameters of the above model could be solved by linear programming:

$$
\begin{cases}
Min_{\lambda}\{Max\{\sum_{\alpha=1}^{\beta}\lambda_{\alpha\beta}V^{\alpha\beta}F^{\beta}-F^{\alpha}\}\} \\
\text{subject to } \sum_{\alpha=1}^{\beta}\lambda_{\alpha\beta}=1 \text{ and } \lambda_{\alpha\beta}\geq 0,\alpha,\beta=1,2
\end{cases}
\tag{5.5}
$$

In the next subsection, we will introduce a high-order Markov chain, which, under a normal macroeconomic environment, could produce more accurate results for analyzing loans payment behavior.

5.1.2 High-Order Markov Chain

In analysis of real-world problems like retail mortgage loans and credit cards payments, the behaviors of the payments are supposed to be affected by the prevailed macro-economic factors like local interest rates and employment. On the other hand, past payment pattern could also play a role in the current and future payment. When these are indeed the case, a high-order Markov chain model might give a more accurate description of the real payment behavior and offer better prediction. Ching and Ng (2004) proved that a second-order Markov chain model predicted a product's sale demand with 83% accuracy while a first-order version provided only 74% accuracy with the same data set.

Unfortunately, an kth order Markov chain with m states will have $(m-1)m^k$ model parameters, and the number of parameters (the transition probabilities) will increase exponentially with the increase order of the model. Raftery (1985) introduced a higher-order Markov chain model with only one additional parameter for each extra lag. By assuming $Q=[q_{ij}]$ is a stationary transition matrix which means it doesn't change with different lags, his model could be written as:

$$P(X^n = j_0 \mid X^{(n-1)} = j_1, \ldots, X^{(n-k)} = j_k) = \sum_{i=1}^{k} \lambda_i v_{j_0 j_i} \tag{5.6}$$

where, $\sum_{i=1}^{k} \lambda_i = 1, 0 \le \sum_{i=1}^{k} \lambda_i v_{j_0 j_i} \le 1$. It could be also presented in matrix form as

$$P^{(n+k+1)} = \sum_{i=1}^{k} \lambda_i V P^{(n+k+1-i)}, \sum_{i=1}^{k} \lambda_i = 1 \tag{5.7}$$

where, $P^{(n+k+1)} = (P_{S_i}^{(n+k+1)})^T, i = 1,2,..,m$ is the probability distribution of states at time $n+k+1$, $S_i = \{i \in 1,2,..,m\}$. Ching and Ng (2006) generalized Raftery's model in (5.7) by allowing the transition matrix $V = [v_{ij}]$ to vary in different lags, that is, $V_i \ne V_j, i \ne j$. Thus, the model reduces to

$$P^{(n+k+1)} = \sum_{i=1}^{k} \lambda_i V_i P^{(n+k+1-i)} \quad . \tag{5.8}$$

It is seen that if $V_1 = V_2 = \ldots = V_k$, Ching and Ng's model in (5.8) is reduced to Raftery's model in (5.7). Also, Ching and Ng (2002) proved that if V_k is irreducible and $\lambda_k > 0$ such that $0 \le \lambda_k \le 1$ and $\sum_{i=1}^{k} \lambda_i = 1$, then $P^{(n+k+1)} = (P_{S_i}^{(n+k+1)})^T, i = 1,2,..,m$ is a stationary distribution, that is

$$\begin{aligned} \lim_{n\to\infty} P^{(n+k+1)} &= \lim_{n\to\infty} \sum_{i=1}^{k} \lambda_i V_i P^{(n+k+1-i)} \\ \Rightarrow P &= \sum_{i=1}^{k} \lambda_i V_i P \\ \Rightarrow (I - \sum_{i=1}^{k} \lambda_i V_i) P &= 0 \end{aligned} \tag{5.9}$$

where I is a $m \times m$ identity matrix, and m is the number of transition states. One can also show that $1^T P = 1, 1^T = (1 \;\; \dots \;\; 1)_{1\times m}$. Given the probability distribution matrix P and the transition intensity matrix V which could be observed from the data sequence and calculated by the scheme in (5.3), respectively, we can solve $\lambda_i, i = 1,2,..,m$ by this linear system. However, a better way to solve this linear system is use the algorithmic proposed by Ching and Ng (2006). They used a linear programming technique, similar to the one in (5.6), defined as

$$\begin{cases} Min_\lambda \{ \left\| \sum_{i=1}^{k} \lambda_i V_i P - P \right\|_l \}, \\ \\ \text{subject to} \sum_{i=1}^{k} \lambda_i = 1, \lambda_i \geq 0 \end{cases} \tag{5.10}$$

where $\|\square\|_l$ is a vector norm, and $l \in \{1,2,\infty\}$. For simplicity, we choose $l = 1$. Thus, an equivalent linear programming technique proposed by Ching and Ng (2006) is as follows:

$$Min_\lambda \sum_{l=1}^{m} w_l \text{, subject to} \tag{5.11}$$

$$\begin{pmatrix} w_1 \\ w_2 \\ \dots \\ w_m \end{pmatrix} \geq X - [V_1 X \mid V_2 X \dots \mid V_k X] \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \dots \\ \lambda_k \end{pmatrix}$$

$$\begin{pmatrix} w_1 \\ w_2 \\ \dots \\ w_m \end{pmatrix} \geq X + [V_1 X \mid V_2 X \dots \mid V_k X] \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \dots \\ \lambda_k \end{pmatrix}$$

In the application section, due to the seasonal fluctuation, we will use a fourth order Markov chain in the hope that it will result in a better representation of the loan behavior.

5.1.3 High-Order Multivariate Markov Chain

By assuming that the state probability distribution of the $j \cdot th$ sequence at time $t = r+1$ depends on the state probability distribution of all sequences at times $t = r, r-1, \ldots, r-n+1$, Ching and Ng (2006) proposed a higher-order multivariate Markov chain model:

$$\begin{aligned} &F_{r+1}^{j} = \sum_{k=1}^{s}\sum_{h=1}^{n} \lambda_{jk}^{h} V_h^{jk} F_{r-h+1}^{k}, j = 1,2,\ldots,s \\ &\lambda_{jk}^{h} \geq 0, 1 \leq j,k \leq s, 1 \leq h \leq n \\ &\sum_{k=1}^{s}\sum_{h=1}^{n} \lambda_{jk}^{h} = 1, j = 1,2,\ldots,s \end{aligned} \tag{5.12}$$

where V_h^{jk} is the $h \cdot th$ intensity transition matrix indicating the $h \cdot th$ intensity transition from states in the $j \cdot th$ sequence at time $t = r-h+1$ to states in the $k \cdot th$ sequence at time $t = r+1$. In fact, each V_h^{jk} is a $m \times m$ matrix represented by

$$V_h^{jk} = \begin{pmatrix} v_{-3,-3} & \cdots & v_{-3,4} \\ \vdots & \ddots & \vdots \\ v_{4,-3} & \cdots & v_{4,4} \end{pmatrix}_h^{jk} \tag{5.13}$$

Equation (5.11) could also be written in matrix form:

$$\begin{cases} F_{r+1}^{\alpha} = B^{\alpha\alpha} F_r^{\alpha} + B^{\alpha\beta} F_r^{\beta} \\ F_{r+1}^{\beta} = B^{\beta\alpha} F_r^{\alpha} + B^{\beta\beta} F_r^{\beta} \end{cases} \tag{5.14}$$

$$B^{\alpha\beta} = \begin{pmatrix} \lambda_{\alpha\beta}^{4} V_4^{\alpha\beta} & \lambda_{\alpha\beta}^{3} V_3^{\alpha\beta} & \lambda_{\alpha\alpha,\beta\beta}^{2} V_2^{\alpha\beta} & \lambda_{\alpha\beta}^{1} V_1^{\alpha\beta} \\ I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \end{pmatrix}_{7\square 4\times 7\square 4} \qquad \alpha \neq \beta$$

$$B^{\alpha\alpha,\beta\beta} = \begin{pmatrix} \lambda_{\alpha\alpha,\beta\beta}^{4} V_4^{\alpha\alpha,\beta\beta} & \lambda_{\alpha\alpha,\beta\beta}^{3} V_3^{\alpha\alpha,\beta\beta} & \lambda_{\alpha\alpha,\beta\beta}^{2} V_2^{\alpha\alpha,\beta\beta} & \lambda_{\alpha\alpha,\beta\beta}^{1} V_1^{\alpha\alpha,\beta\beta} \\ I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \end{pmatrix}_{7\square 4\times 7\square 4}$$

where $F_r^{(j)} = (F_r^{(j)}, F_{r-1}^{(j)}, \ldots, F_{r-n+1}^{(j)})^T, j = 1,2,\ldots,s$, and V_n^{ii} , V_n^{ij} is specified by equation (5.12). In our case, $s \in \{\alpha, \beta\}$. The model introduced in equation (5.14) is too complicated to be solved by linear programming. We will use the direct algorithm in MathCAD® to solve this model in matrix form.

5.2 Application

In this section, we will use the multivariate higher-order model introduced in the previous section to analyze the correlation between retail mortgage loans and credit cards. Also, performance comparison of the three models to predict the probability distribution in the next period, namely, a multivariate model by equation (5.5), a higher-order model by equation (5.11), and a multivariate higher-order model by equation (5.14) would be provided for the conclusion in the next section.

An Ohio local bank provided us with 18 months of consecutive data sequences on retail mortgage loans and credit cards, from April 2005 to September 2006. Based on the results, a report will be issued to the bank management. We will deal with each of the models separately followed by a summary of a model's performance according to its prediction ability.

5.2.1 Multivariate Model for Correlation and Prediction

One of the results that the bank wishes to know is the direction and the magnitude of the correlation between retail mortgage and credit card because, normally, these services are taken by the same group of people in a local area. Macroeconomic factors could be common drivers that have effects on payment patterns and behaviors of both retail mortgage and credit card. The following are the notations we will be using throughout this chapter for all three models. We will use α, β for retail mortgage and credit cards dataset, respectively. Following the definition in Table 2.1 of chapter 2, for each of the dataset, there are 7 + 4 states, 7 for transient states and 4 for absorbing states which are represented by $i,k \in \{S_i, R_k\} i = -3, \ -2,-1,0,1,2,3; k = 1,2,3,4$. Different lags or orders will be referred to as $j \in \{1,2,3,4\}$. For example, when j = 4, the transition intensity matrix: $V_4^{\alpha\beta} = \begin{pmatrix} v_{-3,-3} & \mathrm{K} & v_{-3,4} \\ \mathrm{M} & \mathrm{O} & \mathrm{M} \\ v_{4,-3} & \mathrm{L} & v_{4,-3} \end{pmatrix}_4^{\alpha\beta}$ is referring to 4-month lagged transition from retail mortgage states to credit cards states. Thus, the multivariate Markov model proposed by Ching and Ng (2006) is given as by equation (5.4), or $F = \begin{pmatrix} \lambda_{\alpha\alpha} V^{\alpha\alpha} & \lambda_{\alpha\beta} V^{\alpha\beta} \\ \lambda_{\beta\alpha} V^{\beta\alpha} & \lambda_{\beta\beta} V^{\beta\beta} \end{pmatrix} F$, which could also be written as

$$\sum_{\alpha=1}^{\beta} \lambda_{\alpha\beta} V^{\alpha\beta} F^{\beta} = F^{\alpha} \Rightarrow \sum_{\alpha=1}^{\beta} \lambda_{\alpha\beta} V^{\alpha\beta} F^{\beta} - F^{\alpha} = 0$$

The sign and values of the parameters $\lambda = \{\lambda^{\alpha\alpha}, \lambda^{\alpha\beta}, \lambda^{\beta\alpha}, \lambda^{\beta\beta}\}$ provide the direction and the magnitude of the correlation. Also, given the probability distribution at time t, the model can predict the distribution at time $t+1$. For simplicity, we choose $l = 1$ in the vector norm $\|\cdot\|_l$. Thus, the linear programming in equation (5.5) becomes

$$\begin{cases} Min_\lambda \{\sum_{\alpha=1}^{\beta} \lambda_{\alpha\beta} V^{\alpha\beta} F^\beta - F^\alpha\} \\ \\ \text{subject to } \sum_{\alpha=1}^{\beta} \lambda_{\alpha\beta} = 1 \text{ and } \lambda_{\alpha\beta} \geq 0, \alpha, \beta = 1,2 \end{cases} \tag{5.15}$$

We will provide methods, direct solution of (5.4) and linear programming solution of equation (5.15), for solving the parameter $\lambda = \{\lambda^{\alpha\alpha}, \lambda^{\alpha\beta}, \lambda^{\beta\alpha}, \lambda^{\beta\beta}\}$ by the ***Minerr()*** method of MathCAD and the ***Solver()*** function of Microsoft Excel, respectively. The models are built based on the datasets 1 – 15 periods, and data of the last period or period 16 is used to check and compare model performances.

The followings are examples of transition intensities calculated by equation (5.3)

$V^{\beta\beta} =$

	R_1	R_2	R_3	R_4	S_{-3}	S_{-2}	S_{-1}	S_0	S_1	S_2	S_3
R_1	1	0	0	0	0	0	0	0	0	0	0
R_2	0	1	0	0	0	0	0	0	0	0	0
R_3	0	0	1	0	0	0	0	0	0	0	0
R_4	0	0	0	1	0	0	0	0	0	0	0
S_{-3}	0	0	0	0	0	0	0	0	0	0	0
S_{-2}	0	0	0	0	0	0	0	0	0	0	0
S_{-1}	0	0	0	0	0	0	0	0.0014	0.0005	0	0.0012
S_0	0.0214	0	0.0012	0	0	0	0.0014	0.9514	0.0345	0	0.0011
S_1	0.0104	0.0014	0.0021	0.0045	0	0.0005	0.0021	0.0285	0.0014	0.9124	0.0024
S_2	0.0124	0.0072	0.0017	0.1040	0	0	0	0.0012	0.0041	0.0741	0.0076
S_3	0.0001	0.0017	0.0065	0.0032	0	0.0012	0	0.0034	0	0.0018	0.0015

Figure 5.2(A) Transition intensity matrix within credit cards.

$$
V^{\alpha\beta} = \begin{array}{c} \\ R^{\alpha}{}_{1} \\ R^{\alpha}{}_{2} \\ R^{\alpha}{}_{3} \\ R^{\alpha}{}_{4} \\ S^{\alpha}{}_{-3} \\ S^{\alpha}{}_{-2} \\ S^{\alpha}{}_{-1} \\ S^{\alpha}{}_{0} \\ S^{\alpha}{}_{1} \\ S^{\alpha}{}_{2} \\ S^{\alpha}{}_{3} \end{array}
\begin{array}{c} \begin{array}{ccccccccccc} R^{\beta}{}_{1} & R^{\beta}{}_{2} & R^{\beta}{}_{3} & R^{\beta}{}_{4} & S^{\beta}{}_{-3} & S^{\beta}{}_{-2} & S^{\beta}{}_{-1} & S^{\beta}{}_{0} & S^{\beta}{}_{1} & S^{\beta}{}_{2} & S^{\beta}{}_{3} \end{array} \\
\begin{bmatrix}
0.5901 & 0 & 0.1478 & 0.0471 & 0.1214 & 0.0124 & 0.0110 & 0 & 0.0014 & 0.0011 & 0.0001 \\
0.0012 & 0.4748 & 0.1478 & 0.0014 & 0.0142 & 0.0301 & 0.0018 & 0.0031 & 0.0784 & 0.0145 & 0.0984 \\
0.0651 & 0.1245 & 0.5684 & 0.0145 & 0.0321 & 0.0245 & 0.0781 & 0.0214 & 0.0321 & 0.0141 & 0.0148 \\
0.0914 & 0 & 0.1024 & 0.4512 & 0.1224 & 0.0001 & 0 & 0 & 0 & 0.2147 & 0.1473 \\
0 & 0.2541 & 0 & 0.0142 & 0.3871 & 0.1748 & 0.1201 & 0.2415 & 0.0012 & 0.1457 & 0.0007 \\
0.0012 & 0 & 0.0014 & 0.0661 & 0.0547 & 0.4517 & 0 & 0.1454 & 0.0047 & 0.0018 & 0.0009 \\
0.0019 & 0 & 0 & 0.0008 & 0.1233 & 0.4154 & 0.2315 & 0.0594 & 0.0124 & 0.0142 & 0.0005 \\
0.0001 & 0 & 0.0014 & 0 & 0.0025 & 0.2345 & 0.7841 & 0.1484 & 0.0978 & 0.0014 & 0.0078 \\
0.1025 & 0 & 0 & 0.0874 & 0.0019 & 0.0009 & 0.1269 & 0.1487 & 0.3412 & 0.0002 & 0.4816 \\
0.0021 & 0.1721 & 0.2365 & 0 & 0.0065 & 0.1471 & 0 & 0.1475 & 0.3874 & 0.2314 & 0.1673 \\
0 & 0.6748 & 0.0002 & 0 & 0.1781 & 0.0014 & 0.0014 & 0.0987 & 0.1114 & 0.1387 & 0.1991
\end{bmatrix} \end{array}
$$

Figure 5.2(B) Transition intensity matrix between retail mortgages and credit cards.

Figure 5.2(A) and Figure 5.2(A) are examples of transition intensities matrice given by equation (5.3).

Please note that the transition intensities in $V^{\alpha\beta}$ between states $R_k, k = 1,2,3,4$ and between $S_i, i = -3,-2,-1,0,1,2,3$ are no longer necessarily 1 and 0 because the charge-off in a retail mortgage loan does not always transit to the charge-off of credit cards and vice verse. Also, we can find that the prepayment in credit card is not as significant as the retail mortgage which has also been confirmed by many empirical analyses. The calculation of its elements is given as

$$
v_{ij}^{*} = \frac{\sum_{t=1}^{15} c_{ijt}}{\left.\sum_{j=-3}^{k=4} \sum_{t=1}^{15} c_{ijt}\right|_{i}}, i = -3,-2,-1,-,1,2,3, i \neq j, k = 1,2,3,4, * \in \{\alpha, \beta\} \quad (5.15)
$$

where c_{ijt} is the occurrence frequency counts of the transition between states at time t. Furthermore, the probability distribution vectors in each of the states are given in Figure 5.3:

$$F = \begin{pmatrix} F^{\alpha} \\ F^{\beta} \end{pmatrix}$$

$$F_{\alpha} = (0.0131 \quad 0.0286 \quad 0.1025 \quad 0.7523 \quad 0.0246 \quad 0.0321 \quad 0.0125 \quad 0.0098 \quad 0.0078 \quad 0.0115 \quad 0.0051)^{T}$$
$$F_{\beta} = (0.0000 \quad 0.0000 \quad 0.0001 \quad 0.7958 \quad 0.0212 \quad 0.0565 \quad 0.0814 \quad 0.0165 \quad 0.0158 \quad 0.0107 \quad 0.0014)^{T}$$

Figure 5.3 Probability distribution vectors.

Thus, the model in (5.4) solved by the *Minerr()* method of MathCAD is given as

$$\begin{cases} F_{n+1}^{\alpha} = 0.2955V^{\alpha\alpha}F_{n}^{\alpha} + 0.7045V^{\alpha\beta}F_{n}^{\beta} \\ F_{n+1}^{\beta} = 0.6077V^{\beta\alpha}F_{n}^{\alpha} + 0.3923V^{\beta\beta}F_{n}^{\beta} \end{cases} \quad (5.16)$$

where, $\lambda = \{\lambda^{\alpha\alpha}, \lambda^{\alpha\beta}, \lambda^{\beta\alpha}, \lambda^{\beta\beta}\} = \{0.2955, 0.7045, 0.6077, 0.3923\} and \sum_{\beta} \lambda^{\alpha\beta} = 1$

From the elements of the vector λ, it is seen that there is a relatively strong positive correlation between retail mortgage and credit cards payment. Also, the correlation is not symmetric ($\lambda_{\alpha\beta} = 0.7045 \neq 0.6077 = \lambda_{\alpha\beta}$). This result could be explained by the payment sequence for each month's bills, or the inelasticity of the mortgage payments. On the other hand, the function of credit cards could be easily replaced by cash or other payment method. As a result, credit cards payment seems more contingent on the payment of the mortgages.

5.2.2 Higher-Order Model for Prediction

In this subsection, we will apply a t^{th}-order Markov chains model, $t = 4$, to predict the probability distribution between states defined in Table 2.1. Data for this model are provided by the same Ohio local bank mentioned at the beginning of this application section. The parameters in model (5.8) provide information about the correlation between states of different lags. This correlation will reveal which lag has most influence over current states. That is, by taking past several transitions into consideration, we hope the model will offer better predictions.

According to model (5.8), or $P^{(n+k+1)} = \sum_{i=1}^{k} \lambda_i V P^{(n+k+1-i)}, \sum_{i=1}^{k} \lambda_i = 1$, $V = (V^t)$, $t = 1,2,3,4$ are the transition matrices from time $n-t$ to n where n is referring to the current time. Please note that when $t = 1$, the model is just a regular first-order Markov chain.

Equation (5.3) gives us a method to calculate the intensity matrix $V = (V^t), t = 1,2,3,4$, the elements of which represent the transition between states at time $n-t$ to states at time n. The following is an example of intensity matrix of V^2, or the transition between states at two-month ago to state at the current month.

The transition intensity matrix between two-month-lags is given in Figure 5.4:

$V^2 =$

	R^0_1	R^0_2	R^0_3	R^0_4	S^0_{-3}	S^0_{-2}	S^0_{-1}	S^0_0	S^0_1	S^0_2	S^0_3
R^2_1	1	0.	0	0	0	0	0	0	0	0	0
R^2_2	0	1	0	0	0	0	0	0	0	0	0
R^2_3	0	0	1	0	0	0	0	0	0	0	0
R^2_4	0	0	0	1	0	0	0	0	0	0	0
S^2_{-3}	0.0114	0	0	0	0.0145	0.2354	0	0.6874	0	0	0
S^2_{-2}	0	0	0.0142	0.0024	0	0	0.4571	0.1023	0.0100	0	0
S^2_{-1}	0	0	0	0	0.0001	0.0089	0.2001	0.4517	0.0045	0.0002	0
S^2_0	0.0001	0	0.0541	0	0.0003	0.0313	0.0065	0.7942	0.0504	0.0314	0.0055
S^2_1	0.0894	0	0	0	0	0	0	0.1247	0.4872	0.1245	0.4011
S^2_2	0.1021	0.1721	0	0	0	0	0	0.1148	0.3247	0.1055	0.5478
S^2_3	0.1011	0	0.0048	0	0	0	0	0	0.0033	0.1387	0.7245

Figure 5.4 Transition intensity matrix between two-month-lags.

Please note that if n is the number of available monthly data, we have $Mod(\frac{n-1}{t})$ of transition matrices between time $n-t$ and time n . We took the average over the corresponding elements to reach the matrix in Figure 5.4.

By the same token, we used only 15 time periods to build the model, and data in the last period were used to test the performance in subsection 5.2.4. The probability distribution vector was estimated to give in Figure 5.5:

$$F = (0.0131 \quad 0.0286 \quad 0.1025 \quad 0.7523 \quad 0.0246 \quad 0.0321 \quad 0.0125 \quad 0.0098 \quad 0.0078 \quad 0.0115 \quad 0.0051)^T$$

Figure 5.5 Probability distribution vector.

We can see that the $F = F_\alpha$ in Figure 5.3. Thus, by the linear programming of (5.11), one has the following scheme given in Figure 5.6:

$$V^1F = (0.0002, 0.0124, 0.1554, 0.0147, 0.7146, 0.0078, 0.0065, 0.0512, 0.0547, 0.0101, 0.0187)^T$$
$$V^2F = (0.0131, 0.0026, 0.1025, 0.7523, 0.0146, 0.0100, 0.0072, 0.0148, 0.0083, 0.0128, 0.0056)^T$$
$$V^3F = (0.0125, 0.0457, 0.1712, 0.0145, 0.0897, 0.4571, 0.2578, 0.0547, 0.0345, 0.0777, 0.1463)^T$$
$$V^4F = (0.0784, 0.0124, 0.1574, 0.1244, 0.1278, 0.4587, 0.2144, 0.2874, 0.0013, 0.0784, 0.0659)^T$$

Figure 5.6 Linear programming schemes.

$$Min_{\lambda_1,\lambda_2,\lambda_3,\lambda_4}(w_1+w_2+w_3+w_4+w_5+w_6+w_7)$$

Subject to:

$$\begin{cases} w_1 \ge 0.0131-0.0002\lambda_1-0.0131\lambda_2-0.0125\lambda_3-0.0784\lambda_4 \\ w_2 \ge 0.0286-0.0124\lambda_1-0.0026\lambda_2-0.0457\lambda_3-0.0124\lambda_4 \\ w_3 \ge 0.1025-0.1554\lambda_1-0.1025\lambda_2-0.1712\lambda_3-0.1574\lambda_4 \\ w_4 \ge 0.7523-0.0147\lambda_1-0.7523\lambda_2-0.0145\lambda_3-0.1244\lambda_4 \\ w_5 \ge 0.0246-0.7146\lambda_1-0.0146\lambda_2-0.0897\lambda_3-0.1278\lambda_4 \\ w_6 \ge 0.0321-0.0078\lambda_1-0.0100\lambda_2-0.4571\lambda_3-0.4587\lambda_4 \\ w_7 \ge 0.0125-0.0065\lambda_1-0.0072\lambda_2-0.2578\lambda_3-0.2144\lambda_4 \\ w_8 \ge 0.0098-0.0512\lambda_1-0.0148\lambda_2-0.0547\lambda_3-0.2874\lambda_4 \\ w_9 \ge 0.0078-0.0547\lambda_1-0.0083\lambda_2-0.0345\lambda_3-0.0013\lambda_4 \\ w_{10} \ge 0.0115-0.0101\lambda_1-0.0128\lambda_2-0.0777\lambda_3-0.0784\lambda_4 \\ w_{11} \ge 0.0051-0.0187\lambda_1-0.0056\lambda_2-0.1463\lambda_3-0.0659\lambda_4 \\ w_1 \ge -0.0131+0.0002\lambda_1+0.0131\lambda_2+0.0125\lambda_3+0.0784\lambda_4 \\ w_2 \ge -0.0286+0.0124\lambda_1+0.0026\lambda_2+0.0457\lambda_3+0.0124\lambda_4 \\ w_3 \ge -0.1025+0.1554\lambda_1+0.1025\lambda_2+0.1712\lambda_3+0.1574\lambda_4 \\ w_4 \ge -0.7523+0.0147\lambda_1+0.7523\lambda_2+0.0145\lambda_3+0.1244\lambda_4 \\ w_5 \ge -0.0246+0.7146\lambda_1+0.0146\lambda_2+0.0897\lambda_3+0.1278\lambda_4 \\ w_6 \ge -0.0321+0.0078\lambda_1+0.0100\lambda_2+0.4571\lambda_3+0.4587\lambda_4 \\ w_7 \ge -0.0125+0.0065\lambda_1+0.0072\lambda_2+0.2578\lambda_3+0.2144\lambda_4 \\ w_8 \ge -0.0098+0.0512\lambda_1+0.0148\lambda_2+0.0547\lambda_3+0.2874\lambda_4 \\ w_9 \ge -0.0078+0.0547\lambda_1+0.0083\lambda_2+0.0345\lambda_3+0.0013\lambda_4 \\ w_{10} \ge -0.0115+0.0101\lambda_1+0.0128\lambda_2+0.0777\lambda_3+0.0784\lambda_4 \\ w_{11} \ge -0.0051+0.0187\lambda_1+0.0056\lambda_2+0.1463\lambda_3+0.0659\lambda_4 \\ w_1,w_2,w_3,w_4,w_5,w_6,w_7,w_8,w_9,w_{10},w_{11} \ge 0, \\ \lambda_1+\lambda_2+\lambda_3+\lambda_4=1, \lambda_1,\lambda_2,\lambda_3,\lambda_4 \ge 0 \end{cases}$$

Figure 5.6 Continued.

Applying the above scheme to the Excel *Solver()*, the parameters and the higher-order Markov chain model are given as

$$\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (0.6387, 0.2356, 0.1023, 0.0234), \sum_{i=1}^{4} \lambda_i = 1 \qquad (5.18)$$

$$F^n = 0.6387V^1F^{n-1} + 0.2356V^2F^{n-2} + 0.1023V^3F^{n-3} + 0.0234V^4F^{n-4},$$

where $V^t, t = 1,2,3,4$ is given in Figure 5.4, and F^{n-t} is the probability distribution observed at time lag t. It is seen that the correlation decreases as the number of time lags increases.

5.2.3 Higher-Order Multivariate Model for Correlation and Prediction

Before the model is applied, one needs to clarify the transition intensities. Consider two data sequences, retail mortgages $\alpha_t, t = 1,2,...,16$ and credit cards $\beta_t, t = 1,2,...,16$. The transition patterns are given in Figure 5.7:

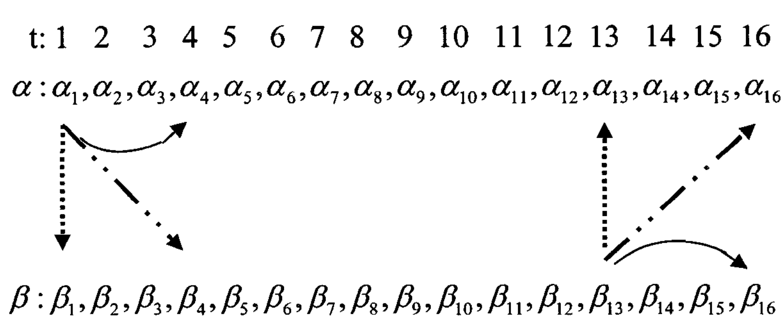

Figure 5.7 A high-order multivariate transition example.

(1) Multivariate transition: $V^{\alpha\beta}, V^{\beta\alpha}$

(2) Higher-Order transition: $V_t, t = 1,2,3,4$

(3) Higher-Order Multivariate transition: $V_t^{\alpha\beta}, V_t^{\beta\alpha} t = 1,2,3,4$

$\alpha_t, \beta_t \in \{S_i, R_k\}, i = -3, -2, -1, 0, 1, 2, 3, k = 1, 2, 3, 4, t = 1, 2, 3, 4$

The model is built based on (1) in 5.2.1, based on (2) in 5.2.2. In this subsection, the model will analyze the correlation and predict the next period probability distribution based on (3). Again, the data, provided by that Ohio local bank, will be used up to the first 15 period, and the last one will be used to compare and test the model in the next subsection.

Based on the example demonstrated in Figure 5.6, we define the transition intensity with $t = 1, 2, 3, 4$ time lags between states in retail mortgages α and credit cards β as $V_t^{\alpha\beta}, V_t^{\beta\alpha} t = 1, 2, 3, 4$. We let $V_t^{\alpha\alpha}, V_t^{\beta\beta} t = 1, 2, 3, 4$ be the higher-order transition within retail mortgages and credit cards, respectively. As a result, there is a total of $4 \times 4 = 16$ transition intensity matrices. Here, in Figure 5.8, we present $V_3^{\alpha\beta}$, the transition from retail mortgages states to credit cards states with time lags equal to 3.

$V_3^{\alpha\beta} =$

	R_1	R_2	R_3	R_4	S_{-3}	S_{-2}	S_{-1}	S_0	S_1	S_2	S_3
R_1	0	0.	0.0014	0	0	0	0	0.0124	0.4514	0.0014	0.4012
R_2	0	0.1922	0	0.1475	0	0	0.0001	0	0.0045	0.0145	0.6214
R_3	0	0	0.7812	0	0	0	0	0.0004	0.0056	0.0014	0.1024
R_4	0.5214	0	0.0001	0	0	0.0001	0	0.0047	0.0789	0.1247	0.1245
S_{-3}	0.0002	0	0	0	0.0111	0.0042	0.0072	0.0105	0.0078	0	0
S_{-2}	0	0	0	0.0034	0.0169	0.1254	0.2487	0.2347	0.0149	0.0021	0
S_{-1}	0	0	0.0002	0.0005	0	0.0231	0.3645	0.1247	0.0524	0.0007	0
S_0	0.0005	0	0.0457	0	0.0087	0.0987	0.1032	0.9045	0.0124	0.0241	0.0001
S_1	0.0547	0	0	0.0002	0	0	0.0087	0.0657	0.3578	0.1187	0.3454
S_2	0.2014	0.0009	0	0.0065	0	0	0	0.0032	0.1008	0.0008	0.6111
S_3	0.1024	0	0.0007	0.1125	0	0	0	0	0.0148	0	0.5487

Figure 5.8 $V_3^{\alpha\beta}$ is the higher—order inter transition intensity matrix.

The construction of the above matrix follows equation (5.3). Also, we note that there are no absorbing states in $V_t^{\alpha\beta}, \alpha \neq \beta, t = 1,2,3,4$. This phenomenon has been explained in subsection 5.2.1.

The model in (5.14) is too difficult to solve by linear programming. We will apply the MathCAD's *Minerr()* method to solve this problem. Due to the MathCAD's maximum limit of the elements a matrix could have, we need to decompose the model in (5.14) to smaller systems of linear equations:

$$\begin{aligned} F_1^{\alpha} &= \lambda_{\alpha\alpha}^1 V_1^{\alpha\alpha} F_1^{\alpha} + \lambda_{\alpha\alpha}^2 V_2^{\alpha\alpha} F_2^{\alpha} + \lambda_{\alpha\alpha}^3 V_3^{\alpha\alpha} F_3^{\alpha} + \lambda_{\alpha\alpha}^4 V_4^{\alpha\alpha} F_4^{\alpha} \\ &\quad + \lambda_{\alpha\beta}^1 V_1^{\alpha\beta} F_1^{\beta} + \lambda_{\alpha\beta}^2 V_2^{\alpha\beta} F_2^{\beta} + \lambda_{\alpha\beta}^3 V_3^{\alpha\beta} F_3^{\beta} + \lambda_{\alpha\beta}^4 V_4^{\alpha\beta} F_4^{\beta} \\ F_1^{\beta} &= \lambda_{\beta\beta}^1 V_1^{\beta\beta} F_1^{\alpha} + \lambda_{\beta\beta}^2 V_2^{\beta\beta} F_2^{\alpha} + \lambda_{\beta\beta}^3 V_3^{\beta\beta} F_3^{\alpha} + \lambda_{\beta\beta}^4 V_4^{\beta\beta} F_4^{\alpha} \\ &\quad + \lambda_{\beta\alpha}^1 V_1^{\beta\alpha} F_1^{\beta} + \lambda_{\beta\alpha}^2 V_2^{\beta\alpha} F_2^{\beta} + \lambda_{\beta\alpha}^3 V_3^{\beta\alpha} F_3^{\beta} + \lambda_{\beta\alpha}^4 V_4^{\beta\alpha} F_4^{\beta} \end{aligned} \tag{5.19}$$

where $V \in (V_t^{\alpha\alpha}, V_t^{\alpha\beta}, V_t^{\beta\alpha}, V_t^{\beta\beta}), t = 1,2,3,4, \sum_{\alpha,\beta} \sum_{t=1}^{4} V_t^{\alpha\beta} = 1$ are the 11 by 11 transition matrices given by Figure 5.7, and $F_t^{\alpha,\beta}, t = 1,2,3,4$ are the 4 consecutive observed probability distribution vector of retail mortgages and credit cards.

From the MathCAD analysis, we obtained the following equations:

$$\begin{aligned} \lambda &= (\lambda_{\alpha\alpha}^1, \lambda_{\alpha\alpha}^2, \lambda_{\alpha\alpha}^3, \lambda_{\alpha\alpha}^4, \lambda_{\alpha\beta}^1, \lambda_{\alpha\beta}^2, \lambda_{\alpha\beta}^3, \lambda_{\alpha\beta}^4, \lambda_{\beta\beta}^1, \lambda_{\beta\beta}^2, \lambda_{\beta\beta}^3, \lambda_{\beta\beta}^4, \lambda_{\beta\alpha}^1, \lambda_{\beta\alpha}^2, \lambda_{\beta\alpha}^3, \lambda_{\beta\alpha}^4)^T \\ &= (0.1278, 0.0914, 0.0311, 0.0154, 0.3209, 0.2365, 0.1398, 0.0371, \\ &\quad 0.2355, 0.1165, 0.0977, 0.0211, 0.0098, 0.3871, 0.0403, 0.0920)^T \end{aligned}$$

$$\begin{aligned} F^{\alpha}_{1,t+1} &= 0.1278V_1^{\alpha\alpha}F^{\alpha}_{1,t} + 0.0914V_2^{\alpha\alpha}F^{\alpha}_{2,t} + 0.0311V_3^{\alpha\alpha}F^{\alpha}_{3,t} + 0.0154V_4^{\alpha\alpha}F^{\alpha}_{4,t} \\ &+ 0.3209V_1^{\alpha\beta}F^{\beta}_{1,t} + 0.2365V_2^{\alpha\beta}F^{\beta}_{2,t} + 0.1398V_3^{\alpha\beta}F^{\beta}_{3,t} + 0.0371V_4^{\alpha\beta}F^{\beta}_{4,t} \\ F^{\beta}_{1,t+1} &= 0.2355V_1^{\beta\beta}F^{\alpha}_{1,t} + 0.1165V_2^{\beta\beta}F^{\alpha}_{2,t} + 0.0977V_3^{\beta\beta}F^{\alpha}_{3,t} + 0.0211V_4^{\beta\beta}F^{\alpha}_{4,t} \\ &+ 0.0098V_1^{\beta\alpha}F^{\beta}_{1,t} + 0.3871V_2^{\beta\alpha}F^{\beta}_{2,t} + 0.0403V_3^{\beta\alpha}F^{\beta}_{3,t} + 0.0920V_4^{\beta\alpha}F^{\beta}_{4,t} \end{aligned} \quad (5.20)$$

Here, $\sum_{\alpha,\beta}\sum_{t=1}^{4} V_t^{\alpha\beta} = 1$. As we can see from the parameters, the correlations within Mortgages are less significant than those within credit cards, while the correlations between retails and cards are not symmetric as confirmed by the first-order multivariate model in subsection 5.2.1. The performance of this model is compared with the other two models in the previous subsections.

5.2.4 Summary of Model Performance

For the multivariate model, the data set observed in periods 1-15 (Figure 5.3) was used. For the higher-order model, the dataset with 4 consecutive months observations (Figure 5.5) was required. For the higher-order multivariate model, the dataset is more complicated and could be represented in Figure 5.9:

$$F = \begin{pmatrix} F_t^{\alpha} \\ F_t^{\beta} \end{pmatrix}$$
$$F_t^{\alpha} = (F^{\alpha}_{1,t}, F^{\alpha}_{2,t}, F^{\alpha}_{3,t}, F^{\alpha}_{4,t})^T$$
$$F_t^{\beta} = (F^{\beta}_{1,t}, F^{\beta}_{2,t}, F^{\beta}_{3,t}, F^{\beta}_{4,t})^T$$

Figure 5.9 Observed probability distributions for model (3).

Criteria used for measuring the prediction error was the normalized error:

$$E_{S_i} = \left| \frac{F_{S_i,16} - \hat{F}_{S_i}}{F_{S_i,16}} \right|, E_R = \frac{\left| \sum_{k=1}^{4} F_{R_k,16} - \hat{F}_{R_k} \right|}{\sum_{k=1}^{4} F_{R_k,16}} \tag{5.21}$$

$$i = -3,-2,-1,0,1,2,3, k = 1,2,3,4$$

where E_{S_i} and E_{R_k} are the normalized error for transient state S_i and absorbing states R_k, respectively. $F_{S_i,16}$ is the observed probability distribution of transient states at period 16, while $\hat{F}_{S_i}$ is the predicted probability distribution for the same states. $F_{R_k,16}$ and $\hat{F}_{R_k}$ follow the same notation rules. For predictions of transient states, the normalized errors are calculated individually, while the normalized error for absorbing states are measured as a whole because the different types of charge-off sometime are actually at the arbitration of the bank management. Equation (5.21) gives the percentage of errors in the observed dataset. Small normalized errors are expected for good model prediction performances. Comparisons of percent prediction errors among the three models are presented in Table 5.1.

Table 5.1 Comparisons of percent prediction errors among the three models.

	S_{-3}	S_{-2}	S_{-1}	S_0	S_1	S_2	S_3	$\sum_{k=1}^{4} R_k$
Model (1)	23.11%	35.47%	22.70%	12.55%	16.32%	20.02%	27.21%	38.09%
Model (2)	22.98%	36.98%	21.77%	10.37%	17.87%	19.68%	26.97%	47.51%
Model (3)	24.57%	33.98%	14.78%	9.54%	15.87%	16.40%	29.41%	50.87%

Also, the Data in Table 5.1 are presented in Figure 5.10.

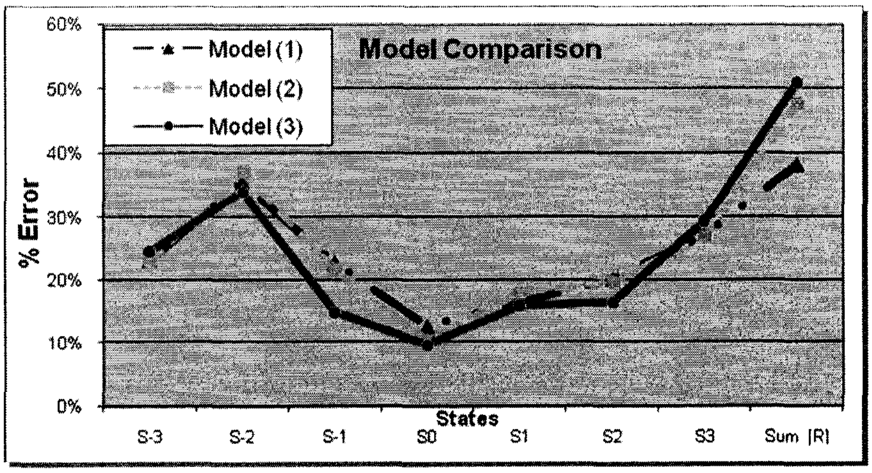

Figure 5.10 Model comparison.

Generally speaking, model (3) is more accurate in the normal state, S_0, and is better than the other two models in most other cases. Not surprisingly, the best model to predict the absorbing states is simply the higher-order model. This result could be due to the fact that the charge-off decisions for retail mortgages have been made independently of the decisions for credit cards. This result is crucial information for the credit asset management. In other word, the bank management failed to take this correlation information into account when they made the charge-off decisions. By charge-off decisions, we means the bank took one the approaches, mentioned in Table 2.1, to charge the assets off from the system. Also, although model (2) is not necessarily better than model (1), there is still a conceivable difference in the prediction performance based on this dataset.

5.3 Conclusion

The tested models in section 5.2 offer bank management quantitative methods to analyze and predict its loans' behavior, which is required by the Federal Reserve Bank. This result could help bank management in making strategic decisions. Furthermore, the measurement of correlation offered by a higher-order Markov chains model offers a simple and reliable method to analyze data for small-to-medium size local commercial banks, which, in most case, do not have adequate resources for implementing comprehensive large computation systems. In the next chapter, models based on hidden theory of Markov chains will be used to analyze unobservable forces behind observable behavior.

CHAPTER 6

A HIGHER-ORDER INTERACTIVE HIDDEN MARKOV CHAIN MODEL FOR RETAIL MORTGAGE

This chapter is concerned with Hidden Markov Models or HMM. HMM is very useful in decoding the unobservable forces affecting the retail mortgages loans by analyzing the observable state transition behaviors of the loans. Also, a fourth-order HMM, solved by the Heuristic Method introduced by Ching and Ng (2006), is presented based on the assumption that the past several periods of payment behavior have an effect on current behavior. Finally, an Interactive Hidden Markov Model (IMMM) is also presented in order to capture the interaction between the observable states, loan transition behavior, and unobservable underlying local macro-economic factors.

6.1 Models

Following MacDonald and Zucchini (1997), a standard HMM has the following elements: (1) N, the number of hidden states, $H = \{H_1, H_2, ..., H_N\}$, (2) S, the number of observable states, $O = \{o_1, o_2, ..., o_S\}, S \in (S_i, R_k), i = -3, -2, -1, 0, 1, 2, 3, k = 1, 2, 3, 4$, (3) A, the transition probability matrix within the hidden states, $A = \{a_{ij}\}, a_{ij} =$

$P(H_{j,t=n} \mid H_{i,t=n-1}), 1 \le i, j \le N$, (4) B , the emission probabilities matrix, $B = \{b_{S|j}\}$, where $b_{S|j} = P(O_S \mid H_j), 1 \le j \le N, S \in (S_i, R_k), i = -3,-2,-1,0,1,2,3, k = 1,2,3,4$, (5) Π , the initial state distribution, $\Pi = \{\pi_i\}, \pi_i = P(O_S)$. Thus, an HMM could be completely specified by $\Lambda = (A, B, \Pi)$.

The ultimate purpose of the HMM is to better understand and predict the transition probabilities between the observable states by analyzing the underlying forces that have influence on the observable behavior. Generally speaking, what people are really interested in are the observable states. However, to better simulate or estimate the true pattern of the state transition under different prevailing underlying situations, underlying forces must be taken into account in the model. Empirically speaking, as more information is built into the model, more accurate results could be expected, which is the general idea of the higher-order HMM. From the linear programming scheme proposed by Raftery and Travare (1994), which was extended by Ching and Ng (2006) by allowing for non-stationary transition intensity ($Q_i, i = 1,2,...,T$) overtime, one can avoid the problem of having to estimate too many parameters in a higher-order Markov model. In addition, the higher-order model could be further improved by assuming that the observable states could also have influences on the unobservable or hidden states. As a result, an HMM will allow for the interaction between these two types of states and might produce even more accurate prediction results.

6.1.1 Hidden Markov Model (HMM)

In most cases, the observable phenomenon is veiled by invisible forces which sometimes make physical sense. In this case, these hidden forces are crucial to understanding the perceivable pattern. In this subsection, a simple Hidden Markov Model is introduced to track and predict the transition probabilities of payment states in retail mortgage loans by taking local macroeconomic situations into consideration. The macro-economic environment is the main factor influencing business development. In chapter 3, we also showed that macroeconomic factors affect the retail mortgage health index. It is desirable to have a measurement which could track hidden macroeconomic transition processes that have a close relationship with the financial industry. One good candidate is the state space model concerning the business industry industrial production index by Liu (2005). The model is given as:

$$\begin{aligned} y_t = {} & 0.4096y_{t-2} + 0.0835Ir_{t-2} - 0.6258Un_{t-2} - 0.0619In_{t-2} - 0.0236Dp_{t-2} \\ & - 0.987529Ir_{t-1} + 0.26377In_{t-1} + 0.002143Dp_{t-1} \end{aligned} \tag{6.1}$$

where, y_t is the industrial production index at time t, Ir_t is interest rate, Un_t is unemployment, In_t is inflation, and Dp_t is disposable personal income at time lags. We define an economic environment to be positive if the industrial production index is at least 100 at that period and negative otherwise. Thus, we have 2 hidden states. From time to time, the hidden state transits from good to bad or from bad to good. Without loss of generality, we assume that the probability of the industrial production index being positive is α, and the probability of it being negative is $1-\alpha$. Also, we follow the definition of observable retail mortgage states. That is, $S \in (S_i, R_k), i = -3, -2, -1, 0, 1, 2, 3,$ $k = 1, 2, 3, 4$ (Table 2.1). By the definition of hidden states, we can observe the steady state probability distribution (under different hidden states), $O_{i,S}, i = 1, 2,$ $S \in (S_i, R_k), i = -3, -2, -1, 0, 1, 2, 3,$ which are defined as

$$O_{S|i} = \begin{cases} O_{S|1}, \text{ if observed under a positive economic environment} \\ \\ O_{S|2}, \text{ if observed under a negative economic environment} \end{cases} \quad (6.2)$$

A new way for estimating of parameter α has been introduced by Ching and Ng (2006). Following their method, we need to define a probability distribution at steady state. Unfortunately, in this dynamic economic environment, there is no such thing as a steady state. The way we can bypass this dilemma is as follows: Let X_S be the $S \cdot th$ element of the steady probability distribution vector X, $S \in (S_i, R_k), i = -3, -2, -1, 0, 1, 2, 3, k = 1, 2, 3, 4$, we have

$$X_S = \frac{\sum_{i=1}^{2} \sum_{t=1}^{n} O_{i,S}^{t_i}}{n}, S \in (S_i, R_k), i = -3, -2, -1, 0, 1, 2, 3, k = 1, 2, 3, 4, n = 16 \quad (6.3)$$

where, $O_{i,S}^{t_i}$ is the S element of the observed probability distribution at the intersection of two hidden states. Thus, the steady probability distribution is approximated by averaging all the observed distributions over the intersections, where n is the number of intersections in the available time series of data. Thus, to estimate α in the hidden Markov chain, we use Eq (6.4) as suggested by Ching and Ng (2006). Eq (6.4) minimizes the sum of squared deviations between $\hat{P}_S$ and X_S.

$$\begin{aligned} &Min_{0 \le \alpha \le 1} \{\psi\} = \{\|\hat{P}_S - X_S\|_2\} \\ &S \in (S_i, R_k), i = -3, -2, -1, 0, 1, 2, 3, k = 1, 2, 3, 4 \end{aligned} \quad (6.4)$$

$\hat{P}_S$ is given by the following matrix manipulation. Let P be so defined such that

$$P = \begin{pmatrix} 0 & H_{2\times 11} \\ P'_{11\times 2} & 0 \end{pmatrix}_{13\times 13} \tag{6.5}$$

where, $H_{2\times 11} = \begin{pmatrix} \alpha & \dots & \alpha \\ 1-\alpha & \dots & 1-\alpha \end{pmatrix}_{2\times 11}$, and $P'_{11\times 2} = \begin{pmatrix} O^T_{S|1} & O^T_{S|2} \end{pmatrix}_{11\times 2}$. $O^T_{S|i}, i = 1,2,$

$S \in (S_i, R_k), i = -3,-2,-1,0,1,2,3, k = 1,2,3,4$ are defined in Equation (6.2). Thus,

$$P^2 = \begin{pmatrix} 0 & H_{2\times 11} \\ P'_{11\times 2} & 0 \end{pmatrix} \times \begin{pmatrix} 0 & H_{2\times 11} \\ P'_{11\times 2} & 0 \end{pmatrix} = \begin{pmatrix} H_{2\times 11} \times P'_{11\times 2} & 0 \\ 0 & P'_{11\times 2} \times H_{2\times 11} \end{pmatrix}_{13\times 13} \tag{6.6}$$

Therefore, $\hat{P}_S$, the probability distribution taking hidden states into consideration with α known, is defined as

$$\hat{P}_S = P'_{11\times 2} \times H_{2\times 11} \times 1_{11\times 1}, \tag{6.7}$$

Where $1_{1x11} = (1,1,\dots,1)^T$.

Based on the assumption that $\hat{P}_S$ is a stationary probability distribution, we can build a Markov prediction model to approximate the probability distribution in the next period under the consideration of a hidden process. The model is given as:

$$\begin{cases} Min_\lambda \{\psi\} = \{\left\| \lambda V_S \hat{P}^t_S - \hat{P}^{t+1}_S \right\|_l\}, l = 1,2,\infty \\ \text{subject to } \lambda > 0 \\ S \in (S_i, R_k), i = -3,-2,-1,0,1,2,3, k = 1,2,3,4 \end{cases} \tag{6.8}$$

where V_S is the transition intensities given in equation 6.3 in chapter 6. Once we find the parameter λ, we can use the observable probability distribution observed at time $t-1$ to predict that at time t. The model in equation (6.8), listed here only for the completeness of the theory, uses a similar idea of equation (6.3) and, therefore, would not be tested in the application section. A higher-order Markov prediction model for hidden processes will be presented in the next subsection.

6.1.2 Heuristic Method for the Higher-Order HMM (HHMM)

Given observed states, a higher-order HMM is believed able to solve the following three problems: (1) the prediction of the probability distribution of observed states $P(O \mid \Lambda), \Lambda = (A, B, \Pi)$, (2) the optimal hidden states that best explain the observed behaviors, (3) the model parameters, $\Lambda = (A, B, \Pi)$. In the real economic world, we seldom have the capability to choose underlying factors affecting the observable behavior of a process. Thus, problem (2) is irrelevant to our case. To solve problems (1) and (3) by conventional methods require tedious recursive algorithms. As is the case for the forward algorithm for problem (1), and for the EM algorithm for problem (3). Detailed discussion of the forward and EM algorithms could be found in MacDonald and Zucchini (1997).

In this subsection, we will present the Heuristic method proposed by Ching and Ng (2006) for a fourth-order HMM based on the assumption that the emission probabilities matrix, $B = \{b_{S|j}\}$, where $b_{jk} = P(S_k \mid H_j), 1 \le j \le N, 1 \le k \le i$ could be observed, which is generally the case. Let $\{\hat{h}_i\} \in \hat{H}, i = 1,2$ be the stationary probability distribution for the hidden states, and $\{\hat{v}_{i,t}\} \in \hat{V}_t, t = 1,2,3,4, i = 1,2$ be the transition intensities between the hidden states with different time lags. An equation for estimating λ_i in a fourth-order hidden Markov model is given as

$$\begin{cases} Min_{\lambda_i}\{\left\|\sum_{j=1}^{k}\lambda_j \hat{V}_k \hat{H} - \hat{H}_i\right\|_l\}, i = 1,2, k = 1,2,3,4, \\ \text{subject to} \sum_{i=1}^{k}\lambda_i = 1, \lambda_i \geq 0 \end{cases} \quad (6.9)$$

For practical reasons, we choose $l = 1$ in the vector norm $\|\square\|_l$. Thus, the more applicable version of equation (6.9) that could be solved by the Excel ***Solve()*** function is

$$\begin{cases} Min_{\lambda}\sum_{l=1}^{4} w_l, \text{subject to} \\ \begin{pmatrix} w_1 \\ w_2 \\ \dots \\ w_i \end{pmatrix} \geq H - [V_1H \mid V_2H \dots \mid V_kH]\begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \dots \\ \lambda_k \end{pmatrix} \\ \begin{pmatrix} w_1 \\ w_2 \\ \dots \\ w_i \end{pmatrix} \geq H + [V_1H \mid V_2H \dots \mid V_kH]\begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \dots \\ \lambda_k \end{pmatrix} \end{cases} \quad (6.10)$$

Here, $\hat{H}_i$, the hidden stationary probability distribution, needs to be approximated since it cannot be observed directly. Ching and Ng (2006) proposed a method to calculate $\hat{H}_i$ from the observed probability distribution, $O_{S,i}$:

$$\begin{aligned} &\left\|\hat{O}_S - B\hat{H}_i\right\|_l, l = 1,2,\infty, i = 1,2 \\ &S \in (S_i, R_k), i = -3,-2,-1,0,1,2,3, k = 1,2,3,4 \end{aligned}, \quad (6.11)$$

where B is the emission probability matrix, $B=\{b_{S|i}\}$, $b_{S|i}=P(S_S \mid H_i), i=1,2$, and $\hat{O}_S$ is the observed probability distribution. For the accuracy of the model, we choose $l=2$ and equation (6.11) given in matrix form become

$$Min\left\|\{o_S\}_{11\times1}-\{b_{S|i}\}_{11\times2}\{h_i\}_{2\times1}\right\|_2, i=1,2 \tag{6.12}$$

$$S\in(S_i,R_k), i=-3,-2,-1,0,1,2,3, k=1,2,3,4$$

Also, in need of estimation are the transition intensities among the hidden states, $\{\hat{v}_{i,t}\}\in\hat{V}_t, t=1,2,3,4, i=1,2$. As pointed by Ching and Ng (2006), $\hat{H}_i$, the hidden stationary probability distribution estimated by equation (6.12) could be used to estimate the first-order transition intensity matrix for hidden states, $\hat{H}_{2\times1}=\begin{pmatrix}\hat{h}_1\\ \hat{h}_2\end{pmatrix}\Rightarrow\hat{V}_1=\begin{pmatrix}\hat{h}_1 & \hat{h}_2\\ \hat{h}_2 & \hat{h}_1\end{pmatrix}$.

Thus, as the transition intensity matrix is assumed to be stationary, the second, third, and fourth order could be estimated by the following procedures:

$$\begin{aligned}\hat{V}_2&=\hat{V}_1\times\hat{V}_1,\\ \hat{V}_3&=\hat{V}_1\times\hat{V}_1\times\hat{V}_1,\\ \hat{V}_4&=\hat{V}_1\times\hat{V}_1\times\hat{V}_1\times\hat{V}_1,\end{aligned} \tag{6.13}$$

As such, the above estimation provides us a stable method to approximate different orders of transition intensities.

The following is a summary of the above steps for a higher-order HMM. **Step 1**: Use equation (6.12) to find the stationary probability distribution for the hidden states, where $b_{S|i}$ is the emission transition from hidden states to observed states given by

$b_{S|i} = P(S_{i,k} \mid H_i), i = 1,2$. **Step 2**: find the transition intensities for various orders by equation (6.13). **Step 3**: Use equation (6.12) to estimate model parameters $\lambda_i, i = 1,2,3,4$ for a fourth-order HMM.

6.1.3 An Interactive Higher-Order Hidden Markov Model (IHHMM)

The interactive HMM is different from the regular HMM in the sense that hidden states of an interactive HMM are affected by previous hidden states and by observable states. In case of retail mortgage analysis, not only local macro-economic factors can affect the mortgage payments, but the payment behavior also determine the collection policy deployed by the banks such as high mortgage rate to cover the foreseeable credit risks of the unusual payment patterns, which, in turn, affect the local businesses in many ways. Therefore, an interactive higher-order HMM seems to be a good candidate for capturing the mechanism in this system. Let $O_{S,i}$ be the observed probability distributions under different hidden states such that:

$$O_{S,i} = \begin{cases} O_{S,1}, \text{ if observed under a positive economic environment} \\ \\ O_{S,2}, \text{ if observed under a negative economic environment} \end{cases} \tag{6.14}$$

We define $\alpha_{S,} S \in (S_i, R_k), i = -3,-2,-1,0,1,2,3, k = 1,2,3,4$ to be the probability with which the hidden state is positive, given the observable states in S . Thus, the transition matrix is given as

$$P=\left(\begin{array}{c|c} 0 & O_{2\times 11} \\ \hline A_{11\times 2} & 0 \end{array}\right)_{13\times 13}, A_{11\times 2}=\{a_{i,s}\} \tag{6.15}$$

where, $A=\begin{pmatrix} \alpha_1 & \cdots & \alpha_{11} \\ 1-\alpha_1 & \cdots & 1-\alpha_{11} \end{pmatrix}^T, O=\begin{pmatrix} o_{1,1} & \cdots & o_{1,11} \\ o_{2,1} & \cdots & o_{2,11} \end{pmatrix}$. $S\in(S_i,R_k), i=-3,-2,-1,0,1,$ $2,3, k=1,2,3,4$. Thus,

$$P^2=\begin{pmatrix} 0 & O_{2\times 11} \\ A_{11\times 2} & 0 \end{pmatrix}\times\begin{pmatrix} 0 & O_{2\times 11} \\ A_{11\times 2} & 0 \end{pmatrix}=\begin{pmatrix} O_{2\times 11}\times A_{11\times 2} & 0 \\ 0 & A_{11\times 2}\times O_{2\times 11} \end{pmatrix}_{13\times 13} \tag{6.16}$$

Therefore, $\hat{P}_S$, the probability distribution under hidden states, is defined as

$$\hat{P}_S = A_{11\times 2}\times O_{2\times 11}1_{11\times 1}, \tag{6.17}$$

where $1_{1x11}=(1,1,\ldots,1)^T$.

To estimate the parameters α_S, we need the steady one-step transition probability matrix which could be approximated by $\tilde{P}_{11\times 11}=\{\tilde{p}_S\}_{11\times 11}, S\in(S_i,R_k)$, $i=-3,-2,-1,$ $0,1,2,3, k=1,2,3,4$. Letting $c_{ik}, i=-3,-2,-1,0,1,2,4, k=1,2,3,4$ be the transition frequency between state i and state k, the calculation of $\tilde{p}_S$ is given as

$$C_{ik}=\begin{pmatrix} c_{-3,-3} & \cdots & c_{-3,4} \\ \vdots & \ddots & \vdots \\ c_{4,-3} & \cdots & c_{4,-3} \end{pmatrix}_{11\times 11} \quad \tilde{P}_S=\begin{pmatrix} \tilde{p}_{-3,-3} & \cdots & \tilde{p}_{-3,4} \\ \vdots & \ddots & \vdots \\ \tilde{p}_{4,-3} & \cdots & \tilde{p}_{4,-3} \end{pmatrix}_{11\times 11} \tag{6.18}$$

$$\tilde{p}_{i,k} = \begin{cases} \dfrac{c_{i,k}}{\sum_{i=S_{-3}}^{R_4} c_{i,k}}, & \text{if } \sum_{i=S_{-3}}^{R_4} c_{i,k} \neq 0 \\ 0, & \text{Otherwise} \end{cases}$$

We define the Frobenius norm as $\|A_{n\times n}\|_F^2 = \sum_{j=1}^{n}\sum_{i=1}^{n} A_{ij}^2$. Thus, the parameters α_S could be approximated by minimizing the Frobenius norm given as

$$Min_{\alpha_i} \left\| \hat{P}_S - \tilde{P}_S \right\|_F^2 \qquad (6.19)$$

Therefore, the above minimizing algorithm could also be expressed as

$$\begin{array}{lll} (1)\alpha_1 : Min_{0\le\alpha_1\le 1}\{(\tilde{p}_{-3,-3} - \hat{p}_{-3,-3})^2 + \ldots + (\tilde{p}_{-3,4} - \hat{p}_{-3,4})^2\}; \\ (2)\alpha_2 : Min_{0\le\alpha_2\le 1}\{(\tilde{p}_{-2,-3} - \hat{p}_{-2,-3})^2 + \ldots + (\tilde{p}_{-2,4} - \hat{p}_{-2,4})^2\}; \\ \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ (11)\alpha_{11} : Min_{0\le\alpha_{11}\le 1}\{(\tilde{p}_{4,-3} - \hat{p}_{4,-3})^2 + \ldots + (\tilde{p}_{4,4} - \hat{p}_{4,4})^2\}; \end{array} \qquad (6.20)$$

The equation to estimate λ_i in a fourth order hidden Markov model is given as

$$\begin{cases} Min_{\lambda_i} \{ \left\| \sum_{j=1}^{k} \lambda_j V_j \hat{P}_S - \hat{P}_S \right\|_l \}, i = 1, 2, \\ \text{subject to} \sum_{i=1}^{k} \lambda_i = 1, \lambda_i \geq 0, \end{cases} \qquad (6.21)$$

where $\hat{P}_S$, the hidden stationary probability distribution, is given by equation (6.17). Finally, the transition intensities among hidden states could be estimated by exactly the same idea of equation (6.13). The only difference is the fact that the transition intensities are 11×11 matrices to capture the effects between observed processes and hidden processes. Thus, from Ching and Ng (2006), the higher-order interactive transition intensities can be calculated as follows: Let $\{\hat{p}_S\}\in\hat{P}_S,\ S\in(S_i,R_k), i=-3,$

$-2,-1,0,1,2,3,k=1,2,3,4$:

$$\begin{aligned}
\hat{V}_1 &= \begin{pmatrix} \hat{p}_{-3} & \hat{p}_{-2} & \cdots & \hat{p}_4 \\ \hat{p}_{-2} & \hat{p}_{-3} & \hat{p}_4 & \vdots \\ \vdots & \hat{p}_4 & \ddots & \hat{p}_{-2} \\ \hat{p}_4 & \cdots & \hat{p}_{-2} & \hat{p}_{-3} \end{pmatrix}_{11\times 11} \\
\hat{V}_2 &= \hat{V}_1\times\hat{V}_1, \\
\hat{V}_3 &= \hat{V}_1\times\hat{V}_1\times\hat{V}_1, \\
\hat{V}_4 &= \hat{V}_1\times\hat{V}_1\times\hat{V}_1\times\hat{V}_1,
\end{aligned} \tag{6.22}$$

The whole algorithm for an Interactive Higher-Order HMM can be as follows:

Step 1: Use equation (6.20) to find the stationary probability distribution for hidden states, where $b_{S|i}$ is the emission transition from hidden states to observed states given by: $b_{S|i}=P(S_{i,k}\mid H_i), i=1,2, S\in(S_i,R_k), i=-3,-2,-1,0,1,2,3,k=1,2,3,4$. **Step 2**: determine the transition intensities by equation (6.22). **Step 3**: Use equation (6.21) to estimate model parameters $\lambda_i, i=1,2,3,4$ for a fourth-order HMM.

6.2 Application of HMMs

A bank, providing the retail mortgage services, never operates in a vacuum environment because the transitions of mortgages payment behaviors and its credit asset quality are affected by many macroeconomic factors. In a general case, the transition pattern of the mortgage payment behavior varies under different macro-economic environments which, in turn, are presented by a group of indices or factors. HMMs, however, could provide a way to unveil more accurate transition processes and therefore provides a probability distribution for mortgage payment states closer to the real prevailing macro-economic situation.

In this section, 18 consecutive months of monthly paid retail mortgage data, provided by an Ohio local bank, will be analyzed by the hidden Markov model in section 6.1. That is, a basic first-order HMM given in equation (6.8), a higher-order HMM solved by the Heuristic method given in equation (6.10), and finally, an interactive HMM in equation (6.20).

6.2.1 HMM for Unobservable Factors in Retail Mortgages

In this subsection, a basic HMM is used to analyze and predict the probability distribution among states considering the effects of underlying macro-economic factors. Due to the lack of an industrial production index in the local Ohio area where the bank data were obtained, we estimated the index from Equation (6.1) by using macro-economic data for Ohio from February 2005 to September 2006. The macro-economic data for Ohio from Feb 2005 to Sep 2006 are presented in Table 6.1.

Table 6.1 Macro-economic data and Index for Ohio.

Year Month	*Un*	*Ir*	*In*	*Dp*	Index from Eq.[(6.1)]
2005 02	5.78	5.93	3.52	5.23	
2005 03	5.80	5.87	4.20	5.08	
2005 04	5.90	5.86	5.74	4.93	11.48

Table 6.1 Continued.

2005 05	5.60	5.72	4.62	4.78	12.26
2005 06	6.10	5.58	5.62	4.63	14.47
2005 07	5.80	5.70	7.69	4.48	10.51
2005 08	5.50	5.82	6.98	4.30	14.88
2005 09	5.60	5.77	5.71	4.20	11.93
2005 10	5.30	6.07	3.06	3.90	12.13
2005 11	5.60	6.33	7.56	3.80	13.54
2005 12	5.50	6.27	6.05	4.30	16.30
2006 01	6.10	6.15	8.18	3.30	14.48
2006 02	6.10	6.25	0.61	3.50	13.57
2006 03	5.30	6.32	4.30	3.20	16.23
2006 04	5.40	6.51	7.45	2.70	15.17
2006 05	4.90	6.60	5.51	2.50	13.07
2006 06	5.20	6.68	2.40	2.50	14.18
2006 07	5.80	6.76	5.47	2.50	16.64
2006 08	5.40	6.52	2.99	2.10	13.94
2006 09	5.00	6.40	5.74	2.10	15.10

Here, Ir is interest rate, Un is unemployment, In is inflation, Dp is disposable personal income at different times. For the purpose of this analysis, we refer to the industrial production index from the model in equation (6.1) as the macro-economic situation in Ohio. The hidden Markov index sequence is presented in Figure (6.1). The average index from Table (6.1) is 14.023. If we let a year takes a value of 1 or 0 depending on whether the index for that year is larger or smaller than 14.023, respectively, we obtain the hidden transition sequence in Table (6.2).

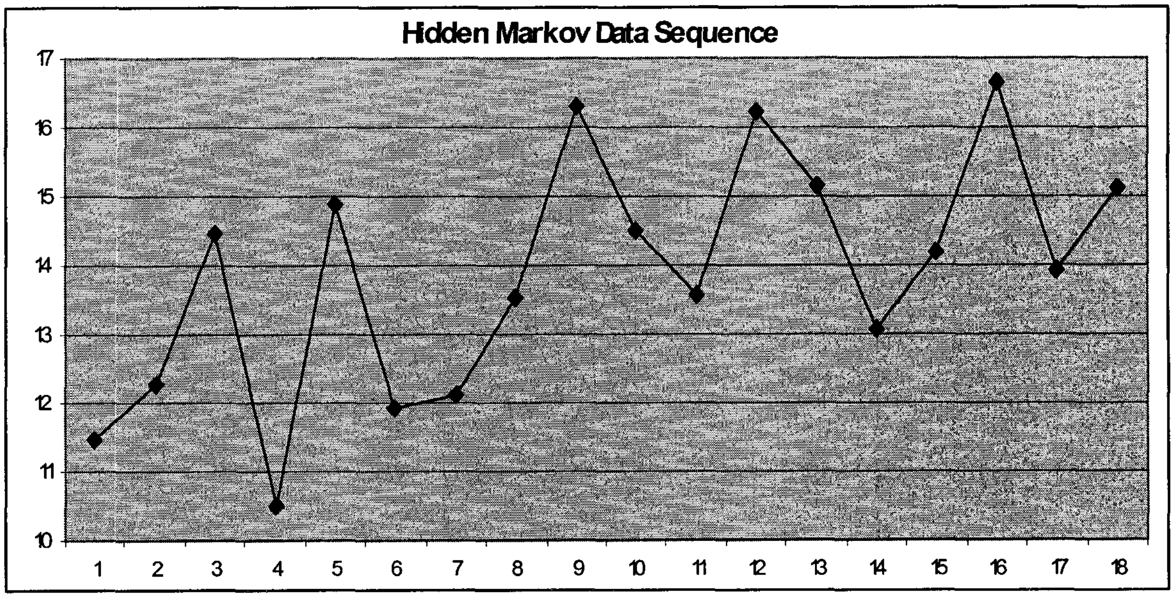

Figure 6.1 Hidden Markov Data Sequence.

Table 6.2 Hidden transition sequence.

t:	1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17
H_t:	0, 0, 1, 0, 1, 0, 0, 0, 1, 1, 0, 1, 1, 0, 1, 1, 0

From the data sequence in Table (6.2), one can estimate the emission probability matrix, $B = \{b_{S|j}\}$. We define the steady state probability distribution for the positive hidden states (1's in Table 6.2) as: $O_{S|1} = \frac{\sum_t o_S}{8}, t = 3,5,9,10,12,13,15,16$. Similarly, the steady state probability distribution for negative (0's) hidden states as: $O_{S|2} = \frac{\sum_t o_S}{9}$, $t = 1,2,4,6,7,8,11,14,17$. From MathCAD, we obtained the probability distribution as shown in Figure 6.2:

$$O_{S|1} = (0.0052 \quad 0.0094 \quad 0.0578 \quad 0.9452 \quad 0.0547 \quad 0.0412 \quad 0.0224 \quad 0.0001 \quad 0.0378 \quad 0.0028 \quad 0.0014)^T$$
$$O_{S|2} = (0.0021 \quad 0.0023 \quad 0.0098 \quad 0.7380 \quad 0.0531 \quad 0.1078 \quad 0.0009 \quad 0.0300 \quad 0.0424 \quad 0.0015 \quad 0.0021)^T$$
$$X_S = (0.0038 \quad 0.0087 \quad 0.0187 \quad 0.8012 \quad 0.0947 \quad 0.0094 \quad 0.0145 \quad 0.0300 \quad 0.0147 \quad 0.0024 \quad 0.0019)^T$$

Figure 6.2 Steady state probability distributions.

where X_S is given by equation (6.3). We let $H_{2\times 11} = \begin{pmatrix} \alpha & \cdots & \alpha \\ 1-\alpha & \cdots & 1-\alpha \end{pmatrix}_{2\times 11}$, and $P'_{11\times 2} = \begin{pmatrix} O_{S|1}^T & O_{S|2}^T \end{pmatrix}_{11\times 2}$, $O_{S,i}^T, i = 1,2$, α is the probability of the hidden state being positive and 1-α the probability of being negative. Thus, the parameter α, could be calculated by equation (6.4) or the following algorithm by letting $l = 2$:

$$\begin{cases} Min_\alpha \{\sum_S (\hat{P}_S - X_S)^2\} \\ subject \text{ to } 0 \le \alpha \le 1 \end{cases} \tag{6.23}$$

where $\hat{P}_S$ is given by: $\hat{P}_S = (0.0073\alpha + 0.0021 \quad 0.0117\alpha + 0.0023 \quad \ldots \quad 0.0035\alpha + 0.0021)_{1\times 11}^T$

By the Excel ***Solver()*** function, we estimate α to be 0.9143, which means that 91.43% of the time between Apr 2005 to Sep 2006 the macro-economic environment would stay in a positive state. As a result, the estimated probability distribution affected by the hidden macro-economic factors is given as

$$\hat{P}_S = (0.0045 \quad 0.0083 \quad 0.0520 \quad 0.8010 \quad 0.0455 \quad 0.0284 \quad 0.0204 \quad 0.00248 \quad 0.0031 \quad 0.0021 \quad 0.0011)_{1\times 11}^{T}$$

In the next subsection, we will apply a Higher-order HMM to test the retail mortgage data. Figure 6.3 presents the Excel ***Solver()*** function interface for solving the above model:

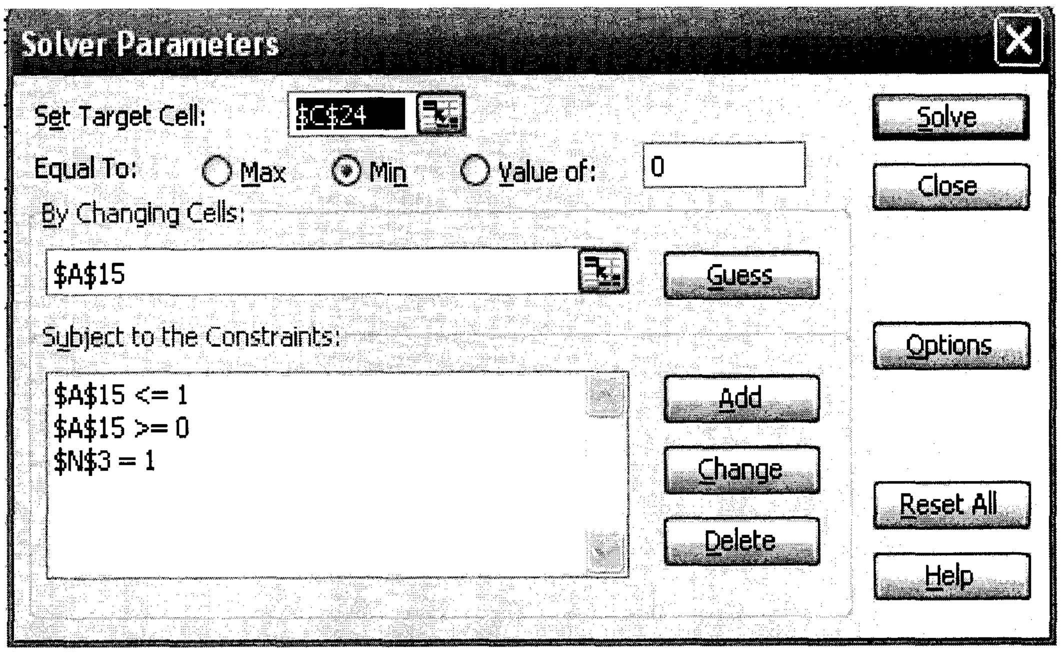

Figure 6.3 Excel *Solver()* interface.

6.2.2 A Higher-Order HMM

In this section, we will use a higher-order HMM to track and predict the hidden transition process. Following the procedures specified at the end of subsection 6.1.2, we first need to approximate the steady state hidden probability distribution by equation (6.12) or a more practical version that could be solved directly by the Excel ***Solver()***. For the solution using excel, we modified equation (6.12) to give

$$\begin{cases} Min_{h_i}\{\sum_S(\{o_S\}_{11\times1}-\{b_{S|i}\}_{11\times2}\{h_i\}_{2\times1})^2\}, i=1,2 \\ subject\ \text{to}\ 0\le h_i\le 1, \sum_{i=1}^{2}h_i=1 \\ S\in(S_i,R_k), i=-3,-2,-1,0,1,2,3, k=1,2,3,4 \end{cases} \tag{6.24}$$

where $\{b_{S|i}\}\in B_{S|i}$, the emission probabilities, are actually the probability distribution vectors under hidden states 1 and 2, respectively, which are given as in Figure 6.4:

$$b_{S|1}=(0.0052\ \ 0.0094\ \ 0.0578\ \ 0.9452\ \ 0.0547\ \ 0.0412\ \ 0.0224\ \ 0.0001\ \ 0.0378\ \ 0.0028\ \ 0.0014)^T$$

$$b_{S|2}=(0.0021\ \ 0.0023\ \ 0.0098\ \ 0.7380\ \ 0.0531\ \ 0.1078\ \ 0.0009\ \ 0.0300\ \ 0.0424\ \ 0.0015\ \ 0.0021)^T$$

$$O_S=(0.0038\ \ 0.0087\ \ 0.0187\ \ 0.8012\ \ 0.0947\ \ 0.0094\ \ 0.0145\ \ 0.0300\ \ 0.0147\ \ 0.0024\ \ 0.0019)^T$$

Figure 6.4 Input variables for Equation (6.12).

The Excel *Solver()* gives us $\hat{H}=\{\hat{h}_1,\hat{h}_2\}=\{0.4033, 0.5967\}$ with the following report in Figure 6.5:

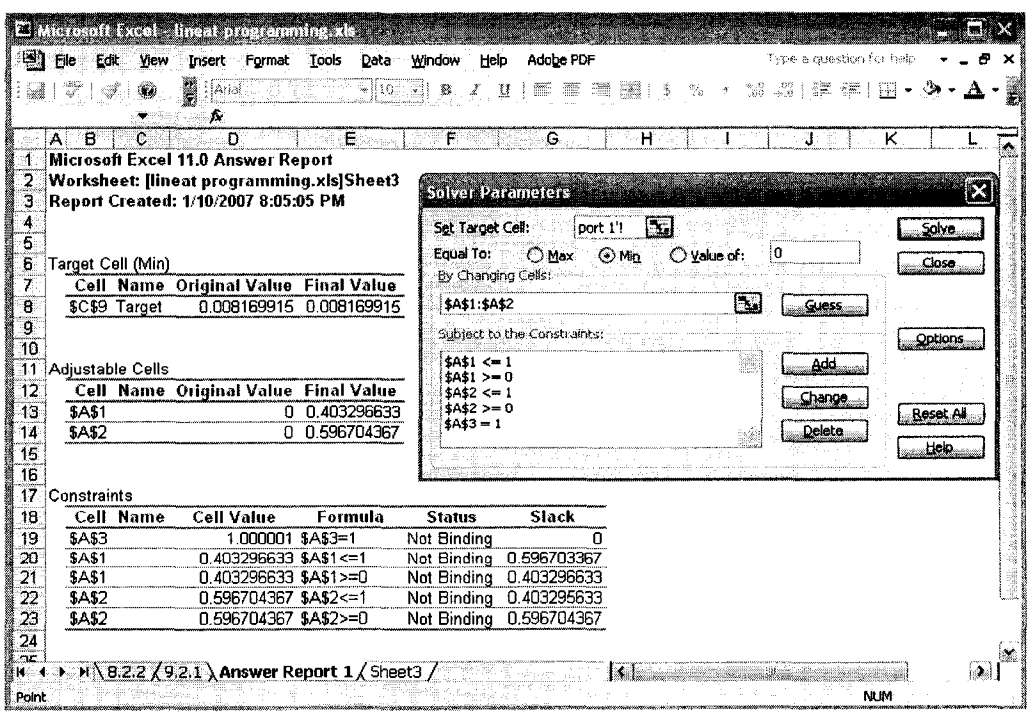

Figure 6.5 Excel *Solver ()* report.

As the above figure indicates, each part of the report is corresponding to each parameters input controls in the ***Solver ()*** function interface. In the next step, we will approximate the transition intensities for different orders by equation (6.13). Please note that the first-order transition intensity matrix is given by: $\hat{H}_{2\times1} = \begin{pmatrix} \hat{h}_1 \\ \hat{h}_2 \end{pmatrix} \Rightarrow \hat{V}_1 = \begin{pmatrix} \hat{h}_1 & \hat{h}_2 \\ \hat{h}_2 & \hat{h}_1 \end{pmatrix}$.

Thus, the transition intensities for four orders are estimated from MathCAD to give

$$\hat{V}_1 = \begin{pmatrix} 0.4033 & 0.5967 \\ 0.5967 & 0.4033 \end{pmatrix} \quad \hat{V}_2 = \begin{pmatrix} 0.5187 & 0.4813 \\ 0.4813 & 0.5187 \end{pmatrix}$$
$$\hat{V}_3 = \begin{pmatrix} 0.5007 & 0.4993 \\ 0.4993 & 0.5007 \end{pmatrix} \quad \hat{V}_4 = \begin{pmatrix} 0.4964 & 0.5036 \\ 0.5036 & 0.4964 \end{pmatrix} \tag{6.25}$$

The method to estimate the parameters $\lambda_i, i = 1,2,3,4$ for the higher-order HMM is given by equation (6.10). The linear programming scheme is as follows:

$$\begin{aligned}
\hat{H} &= (0.4033 \quad 0.5967)^T \\
V_1\hat{H} &= (0.5187 \quad 0.4813)^T \\
V_2\hat{H} &= (0.4964 \quad 0.5036)^T \\
V_3\hat{H} &= (0.5007 \quad 0.4993)^T \\
V_4\hat{H} &= (0.4999 \quad 0.5001)^T
\end{aligned}$$

$$\begin{cases}
Min_{\lambda_1,\lambda_2,\lambda_3,\lambda_4} (w_1 + w_2 + w_3 + w_4) \\
\\
w_1 \geq 0.4033 - 0.5187\lambda_1 - 0.4964\lambda_2 - 0.5007\lambda_3 - 0.4999\lambda_4 \\
w_2 \geq 0.5967 - 0.4813\lambda_1 - 0.5036\lambda_2 - 0.4993\lambda_3 - 0.5001\lambda_4 \\
w_1 \geq -0.4033 + 0.5187\lambda_1 + 0.4964\lambda_2 + 0.5007\lambda_3 + 0.4999\lambda_4 \\
w_2 \geq -0.5967 + 0.4813\lambda_1 + 0.5036\lambda_2 + 0.4993\lambda_3 + 0.5001\lambda_4 \\
w_1, w_2, w_3, w_4 \geq 0, \\
\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 1, \lambda_1, \lambda_2, \lambda_3, \lambda_4 \geq 0
\end{cases}$$

Subject to:

Applying the above scheme to the Excel ***Solver()***, the parameters for the higher-order Markov chain model are given by $\lambda = (0.1876 \quad 0.8125 \quad 0 \quad 0)$. As a result, the HHMM is given as

$$\hat{H}_{t+1} = 0.1876 V_1 \hat{H}_t + 0.8125 V_2 \hat{H}_{t-1} \tag{6.26}$$

equation (6.26) implies that the probability distribution of the hidden states at $t = n+1$ are dependent on only those at $t = n$ and at $t = n-1$.

6.2.3 Interactive Effects Analysis for Retail Mortgages

The observable probability distributions, $O_{2\times 11}$, under both positive and negative states, are given as

$$O_S = \begin{pmatrix} 0.0052 & 0.0094 & 0.0578 & 0.9452 & 0.0547 & 0.0412 & 0.0224 & 0.0001 & 0.0378 & 0.0028 & 0.0014 \\ 0.0021 & 0.0023 & 0.0098 & 0.7380 & 0.0531 & 0.1078 & 0.0009 & 0.0300 & 0.0424 & 0.0015 & 0.0021 \end{pmatrix} \tag{6.27}$$

Therefore, $\hat{P}_S$, the probability distribution under hidden states, is given by equation (6.17) as $\hat{P}_S = A_{11\times 2} \times O_{2\times 11} 1_{11\times 1}, 1_{1\times 11} = (1,1,\ldots,1)^T$ where $A = \begin{pmatrix} \alpha_1 & \ldots & \alpha_{11} \\ 1-\alpha_1 & \ldots & 1-\alpha_{11} \end{pmatrix}^T$. Thus, $\hat{P}_S$ is given as

$$
\begin{pmatrix}
1+0.0031\alpha_1 & 1+0.0071\alpha_1 & 1+0.048\alpha_1 & 1+0.2072\alpha_1 & 1+0.0016\alpha_1 & 1-0.0666\alpha_1 & 1+0.0215\alpha_1 & 1-0.0299\alpha_1 & 1-0.0046\alpha_1 & 1+0.0013\alpha_1 & 1-0.0007\alpha_1 \\
1+0.0031\alpha_2 & 1+0.0071\alpha_2 & 1+0.048\alpha_2 & 1+0.2072\alpha_2 & 1+0.0016\alpha_2 & 1-0.0666\alpha_2 & 1+0.0215\alpha_2 & 1-0.0299\alpha_2 & 1-0.0046\alpha_2 & 1+0.0013\alpha_2 & 1-0.0007\alpha_2 \\
1+0.0031\alpha_3 & 1+0.0071\alpha_3 & 1+0.048\alpha_3 & 1+0.2072\alpha_3 & 1+0.0016\alpha_3 & 1-0.0666\alpha_3 & 1+0.0215\alpha_3 & 1-0.0299\alpha_3 & 1-0.0046\alpha_3 & 1+0.0013\alpha_3 & 1-0.0007\alpha_3 \\
1+0.0031\alpha_4 & 1+0.0071\alpha_4 & 1+0.048\alpha_4 & 1+0.2072\alpha_4 & 1+0.0016\alpha_4 & 1-0.0666\alpha_4 & 1+0.0215\alpha_4 & 1-0.0299\alpha_4 & 1-0.0046\alpha_4 & 1+0.0013\alpha_4 & 1-0.0007\alpha_4 \\
1+0.0031\alpha_5 & 1+0.0071\alpha_5 & 1+0.048\alpha_5 & 1+0.2072\alpha_5 & 1+0.0016\alpha_5 & 1-0.0666\alpha_5 & 1+0.0215\alpha_5 & 1-0.0299\alpha_5 & 1-0.0046\alpha_5 & 1+0.0013\alpha_5 & 1-0.0007\alpha_5 \\
1+0.0031\alpha_6 & 1+0.0071\alpha_6 & 1+0.048\alpha_6 & 1+0.2072\alpha_6 & 1+0.0016\alpha_6 & 1-0.0666\alpha_6 & 1+0.0215\alpha_6 & 1-0.0299\alpha_6 & 1-0.0046\alpha_6 & 1+0.0013\alpha_6 & 1-0.0007\alpha_6 \\
1+0.0031\alpha_7 & 1+0.0071\alpha_7 & 1+0.048\alpha_7 & 1+0.2072\alpha_7 & 1+0.0016\alpha_7 & 1-0.0666\alpha_7 & 1+0.0215\alpha_7 & 1-0.0299\alpha_7 & 1-0.0046\alpha_7 & 1+0.0013\alpha_7 & 1-0.0007\alpha_7 \\
1+0.0031\alpha_8 & 1+0.0071\alpha_8 & 1+0.048\alpha_8 & 1+0.2072\alpha_8 & 1+0.0016\alpha_8 & 1-0.0666\alpha_8 & 1+0.0215\alpha_8 & 1-0.0299\alpha_8 & 1-0.0046\alpha_8 & 1+0.0013\alpha_8 & 1-0.0007\alpha_8 \\
1+0.0031\alpha_9 & 1+0.0071\alpha_9 & 1+0.048\alpha_9 & 1+0.2072\alpha_9 & 1+0.0016\alpha_9 & 1-0.0666\alpha_9 & 1+0.0215\alpha_9 & 1-0.0299\alpha_9 & 1-0.0046\alpha_9 & 1+0.0013\alpha_9 & 1-0.0007\alpha_9 \\
1+0.0031\alpha_{10} & 1+0.0071\alpha_{10} & 1+0.048\alpha_{10} & 1+0.2072\alpha_{10} & 1+0.0016\alpha_{10} & 1-0.0666\alpha_{10} & 1+0.0215\alpha_{10} & 1-0.0299\alpha_{10} & 1-0.0046\alpha_{10} & 1+0.0013\alpha_{10} & 1-0.0007\alpha_{10} \\
1+0.0031\alpha_{11} & 1+0.0071\alpha_{11} & 1+0.048\alpha_{11} & 1+0.2072\alpha_{11} & 1+0.0016\alpha_{11} & 1-0.0666\alpha_{11} & 1+0.0215\alpha_{11} & 1-0.0299\alpha_{11} & 1-0.0046\alpha_{11} & 1+0.0013\alpha_{11} & 1-0.0007\alpha_{11}
\end{pmatrix}
$$

Also, the observed one-step transition intensity matrix, calculated from equation (6.18) is

$\tilde{P}_S =$

	R_1	R_2	R_3	R_4	S_{-3}	S_{-2}	S_{-1}	S_0	S_1	S_2	S_3
R_1	1	0.	0	0	0	0	0	0	0	0	0
R_2	0	1	0	0	0	0	0	0	0	0	0
R_3	0	0	1	0	0	0	0	0	0	0	0
R_4	0	0	0	1	0	0	0	0	0	0	0
S_{-3}	0.0195	0	0	0	0.5759	0.0996	0.1646	0.1329	0	0	0
S_{-2}	0.0105	0	0	0	0.0365	0.6137	0.1563	0.1825	0.0005	0	0
S_{-1}	0.0101	0	0	0	0.02756	0.0341	0.6774	0.2399	0.0015	0	0
S_0	0.02901	0	0	0	0.0041	0.0097	0.0491	0.8920	0.0109	0	0
S_1	0.0133	0.1523	0	0.2090	0.0091	0.0089	0.0140	0.2078	0.2063	0.0552	0
S_2	0	0.0905	0.1456	0.1221	0	0.0031	0.0237	0.0347	0.0853	0.1747	0.3184
S_3	0	0.1053	0.1505	0.1305	0	0	0.0021	0.0632	0.0952	0.1510	0.2931

By the Frobenius norm defined in equation (6.19), the 11 linear programming schemes are given as

$$
\begin{cases}
Min_{\alpha_1}\{(1+0.0031\alpha_1-1)^2+(1+0.0071\alpha_1)^2+(1+0.2072\alpha_1)^2+\dots+(1-0.0007\alpha_1)^2\} \\
subject\ to: 0 \le \alpha_1 \le 1
\end{cases}
$$

$$
\begin{cases}
Min_{\alpha_2}\{(1+0.0031\alpha_2)^2+(1+0.0071\alpha_2-1)^2+(1+0.2072\alpha_2)^2+\dots+(1-0.0007\alpha_2)^2\} \\
subject\ to: 0 \le \alpha_2 \le 1
\end{cases}
$$

$\vdots$

$$
\begin{cases}
Min_{\alpha_{11}}\{(1+0.0031\alpha_{11})^2+(1+0.0071\alpha_{11}-0.1053)^2+(1+0.2072\alpha_{11}-0.1505)^2 \\
\qquad +\dots+(1-0.0007\alpha_{11}-2931)^2\}
\end{cases}
$$
$$
subject\ to: 0 \le \alpha_{11} \le 1
$$

Letting $\hat{P}_S = 1_{11\times11} - \{\hat{p}_S\}_{11\times11}$ to count for the overflow of each element, the probability distribution under the hidden states is given as

$$A = \{\alpha_S\} = (0.0001 \quad 0.0001 \quad 0.0001 \quad 0.0001 \quad 0.0047 \quad 0.0004 \quad 0.0001 \quad 0.0008 \quad 0.0002 \quad 0.0008 \quad 0.0001)^T$$

$$\hat{P}_S = \begin{array}{c} \\ R_1 \\ R_2 \\ R_3 \\ R_4 \\ S_{-3} \\ S_{-2} \\ S_{-1} \\ S_0 \\ S_1 \\ S_2 \\ S_3 \end{array} \begin{array}{ccccccccccc} R_1 & R_2 & R_3 & R_4 & S_{-3} & S_{-2} & S_{-1} & S_0 & S_1 & S_2 & S_3 \end{array}$$

$$\begin{bmatrix} 1 & 0. & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.0385 & 0 & 0 & 0 & 0.8202 & 0.1899 & 0.3019 & 0.2479 & 0 & 0 & 0 \\ 0.0208 & 0 & 0 & 0 & 0.0717 & 0.8508 & 0.2881 & 0.3316 & 0.0011 & 0 & 0 \\ 0.0218 & 0 & 0 & 0 & 0.0544 & 0.0671 & 0.8959 & 0.4222 & 0.0029 & 0 & 0 \\ 0.0572 & 0 & 0 & 0 & 0.0081 & 0.0195 & 0.0958 & 0.9883 & 0.0217 & 0 & 0 \\ 0.2483 & 0.2814 & 0 & 0.3743 & 0.0181 & 0.0179 & 0.0279 & 0.3724 & 0.3701 & 0.1074 & 0 \\ 0 & 0.1728 & 0.2700 & 0.2289 & 0 & 0.0064 & 0.0468 & 0.0682 & 0.1633 & 0.3189 & 0.5355 \\ 0 & 0.1993 & 0.2784 & 0.2440 & 0 & 0 & 0.0042 & 0.1224 & 0.1814 & 0.2793 & 0.5004 \end{bmatrix}$$

The above matrix is the transition intensities between observable states with the assumption of an interaction between the local macro-economic situation and retail mortgage payments. Because elements of the probability vector, $A = \{\alpha_S\}$, are small, we can conclude that retail mortgage payment behaviors of a single local bank have little to do with the local macroeconomic factors.

6.3 Conclusion

The models presented in section 6.2 are used to further analyze the relationship between local macro-economic factors and the payment pattern for a local bank's retail mortgages. From the analysis using MathCAD and the Excel ***Solver()***, we conclude the following:

(1) Based on a first-order HMM, the probability of stay in a positive macro-economic state is 0.9143 For definition of a positive macro-economic state, please refer to Table (6.2).

(2) For the period from April 2005 to September 2006, the estimated steady state probability distribution of the hidden macro-economic states is given as:

$$\hat{H} = \begin{pmatrix} \hat{h}_1 \\ \hat{h}_2 \end{pmatrix} = \begin{pmatrix} 0.4033 \\ 0.5967 \end{pmatrix} \tag{6.28}$$

(3) The effect of the macro-economic states on retail mortgage loans is strong as indicated by the relatively large differences between corresponding observation $O_{S,1}$ and $O_{S,2}$ in rows 1 and 2 of the O_S matrix in equation (6.27).

CHAPTER 7

SUMMARY AND FUTURE STUDY

7.1 Summary and Contributions

In this chapter, we present a summary of the models that have been applied to the banking data which include retail mortgages, credit cards, and local macroeconomic variables. The data have been provided by an Ohio local commercial bank under the condition that the data are strictly for academic usage only. The Table 7.1 summarizes the models that have been used in this study:

Table 7.1 Summary of the models Used in this Study.

Chapter	**Models**
Chapter 2	1. Discrete and Continuous Time Markov model for expected loan duration 2. Stochastic portfolio estimation model 3. A limiting probability model for expected proceedings from past due customers.
Chapter 3	1. A continuous Time Markov model for a loan Status 2. A multivariate regression model for analyzing the relation

Table 7.1 Continued

Chapter 4	1. A Markov decision model for a loan officer optimal compensation plan 2. A Markov decision model for optimal credit collection policies. 3. A Markov Model for Analyzing the Loan System Status
Chapter 5	1. A multivariate Markov model for analyzing the correlation between retail mortgages and credit cards. 2. A higher-order Markov model for retail mortgages 3. A higher-order multivariate Markov model
Chapter 6	1. A Hidden Markov model for retail mortgages 2. A Heuristic Method for the Higher-Order Hidden Markov Model 3. An Interactive Higher-Order Hidden Markov Model

The models presented in chapters 2 and chapters 3 are of practical importance with regard to credit risk management in a commercial bank. The models provide an estimate of the retail mortgage expected duration before the loan is charged off. 2. Also, the models in chapter 3 could be used by the management of the bank to track the loan's dynamic status over time. Furthermore, the multivariate regression model introduced in the same chapter presents a practical tool to estimate the retail mortgage in the presence of related macroeconomic data.

On the other hand, the two Markov decision models in chapter 4 are of interest to a bank's financial department as they could be used to design operation policies for daily decisions. A loan officer optimal compensation plan and optimal credit collection policies are two of the most important decisions the head of the financial department and the bank management have to make. These models provide bank officials with applicable tools in this regard. Also, the loan system status model is useful for estimating a retail mortgage portfolio. Combined with the health index model presented in chapter 3, this model could be used by the bank management as a tool for assessing financial performance.

Three stochastic models are compared with regard to percent prediction error. A higher-order multivariate Markov chain model is shown to be the best model for predicting the internal state, S_0. Figure 7.1 provides the model comparisons for the different states as defined in Table 2.1.

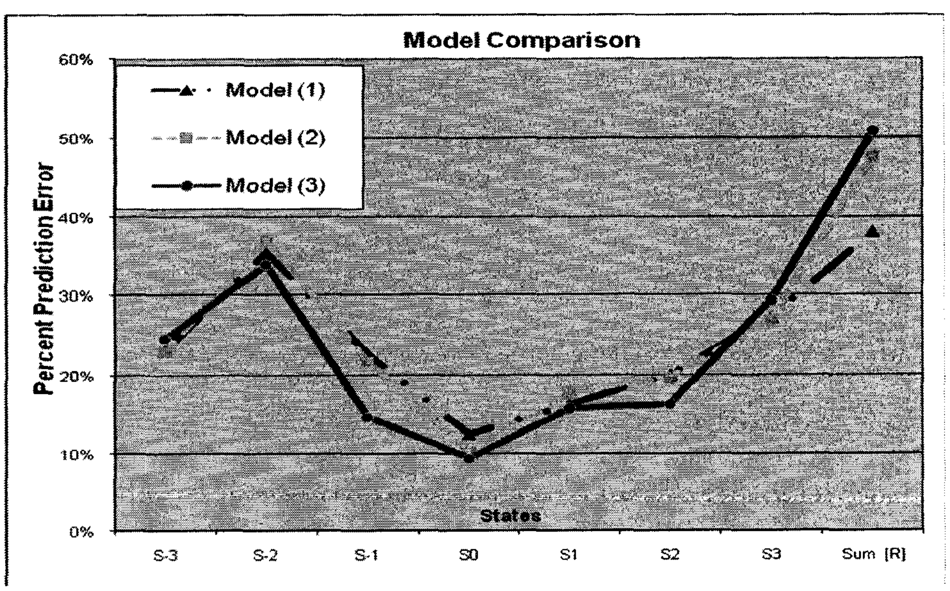

Figure 7.1 Comparisons of percent prediction errors among the three models.

Note that the lower the value in Figure (7.2), the better the model, since the value is the prediction error in percent.

A heuristic method has been used in chapter 6 to provide an estimate of the parameters of the interactive higher-order hidden Markov model, or IHHMM. This method is, in turn, applied to the ***solver()*** function, integrated in Excel®. The Figure 7.2 gives the detailed procedure to work with the Excel ***solver()*** function:

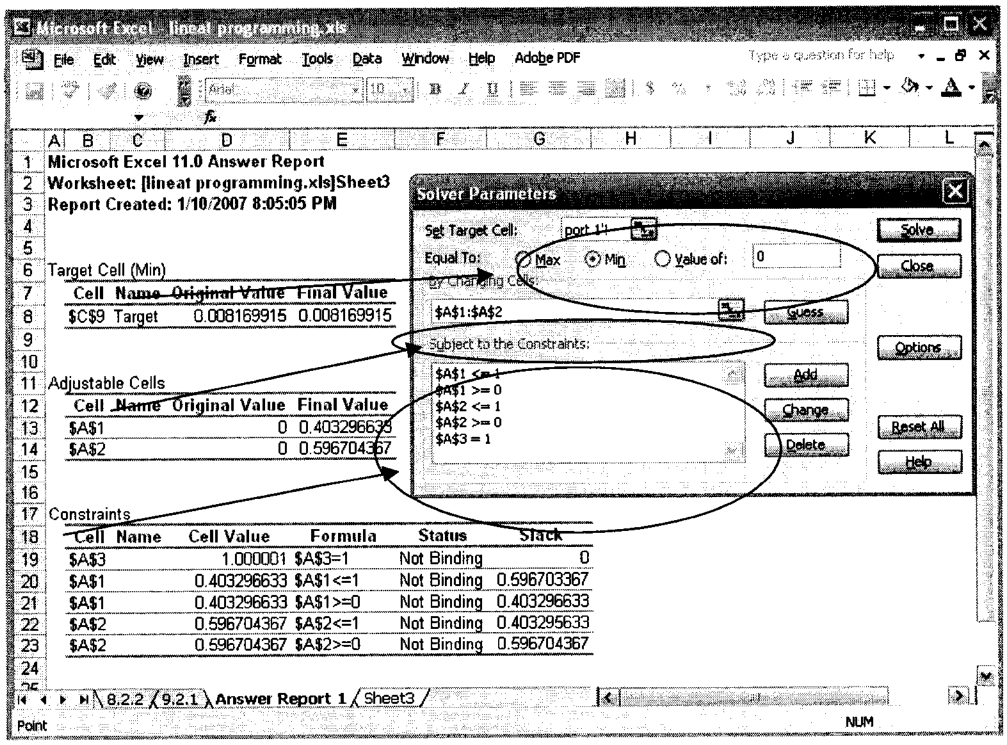

Figure 7.2 Excel *Solver ()* report.

Important conclusions drawn from the models are presented in chapter 6. These models are useful to a financial department in a bank for studying the effects of macroeconomic variables on retail mortgages. As a result, reports generated by these models would be of interest to the bank's management as well.

7.2 Future Study

The models presented in this study are by no mean comprehensive. The assumption of a discrete state process could be relaxed to give rise to a diffusion process. Also, besides modeling the credit risk from a bank's management point of view, this research approach could be readily applied in investment. Thus, more sophisticated models utilizing Stochastic Differential Equations or Value at Risk methodology could be applied in this regard.

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