A numerical method for solving the elliptic and elasticity interface problems

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A NUMERICAL METHOD FOR SOLVING THE ELLIPTIC AND ELASTICITY INTERFACE PROBLEMS

by

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ABSTRACT

Interface problems arise when dealing with physical problems composed of different materials or of the same material at different states. Because of the irregularity along interfaces, many common numerical methods do not work, or work poorly, for interface problems. Matrix-coefficient elliptic and elasticity equations with oscillatory solutions and sharp-edged interfaces are especially complicated and challenging for most existing methods. An accurate and efficient method is desired.

In 1999, the boundary condition capturing method was proposed to deal with Poisson equations with interfaces whose variable coefficients and solutions may be discontinuous. In 2003, a weak formulation was derived. Built on previous work that solves elliptic interface problems with two domains in two dimensions, this dissertation improves the accuracy in the presence of sharp-edged interfaces and extends to elasticity interface problems with two domains in two dimensions, elliptic interface problems with three domains in two dimensions, and elliptic interface problems with two domains in three dimensions.

The method used in this dissertation is a non-traditional finite element method. The test function basis is chosen to be the standard finite element basis independent of the interface, and the solution basis is chosen to be piecewise linear, satisfying the jump conditions across the interface. These two bases are different, which leads to the non-symmetric matrix generated by this method, but the resulting linear system...
of equations is shown to be positive definite under certain assumptions in all the four topics mentioned in this dissertation. This method has matrix coefficients and lower-order terms, and uses the non-body-fitting grid, which makes it easy to deal with different kinds of interfaces, like the examples “Star”, “Happy face”, “Chess board”, to name a few.

The methods used in this dissertation solve the non-smooth interface case and promise results for oscillatory solutions. Numerical experiments show that this method is second-order accurate in the $L^\infty$ norm for piecewise smooth solutions.
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NOMENCLATURE

\( \Omega \) Whole domain

\( \overline{\Omega} \) Closure of the domain

\( \Omega^\pm \) Subdomain

\( \partial \Omega \) Boundary of the domain

\( \Gamma \) Interface

\((u, v)\) or \(u \ v\) \(u \ v = \sum_{i=1}^{n}(u_i v_i)\)

\(\nabla u\) \(\nabla u = (\partial_1 u, \partial_2 u, \ldots, \partial_n u)^T\)

\(\nabla u\) \(\nabla u = \sum_{i=1}^{n}(\partial_i u_i)\)

\(L^2(\Omega)\) \(\{u \ u \text{ is defined on } \Omega, \text{ and } \int_\Omega u^2 \, dx < \infty\}\)

\(H^1(\Omega)\) \(\{u \ u \text{ and } \nabla u \text{ belong to } L^2(\Omega)\}\)

\(H^1_0(\Omega)\) \(\{u \in H^1(\Omega) \ u = 0 \text{ on } \partial\Omega\}\)

\(L^\infty \text{ norm}\) \(||x||_\infty = \max\{|x_1|, |x_2|, \ldots, |x_n|\}\)

\(\chi_\Omega\) \(\chi_\Omega = \begin{cases} 1 & \text{in } \Omega \\ 0 & \text{otherwise} \end{cases}\)

\(\Gamma^h_K\) Interface segment in two dimensions

\(\Gamma^h_L\) Interface segment in three dimensions

\(\triangle_k\) Interface cell in two dimensions
\[ \Delta_k \] Interface cell in three dimensions

\( \phi \) level-set function

\( n = \frac{\nabla \phi}{|\nabla \phi|} \) is a unit normal vector

\( h \) is the grid size
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CHAPTER 1

INTRODUCTION

1.1 Problems and Formulations

In the physical world, there are many problems whose solutions are separated by interfaces. Determining the flow pattern of blood in the heart that is separated by heart valves, or finding the electric potential of a macromolecule that is infused into an ionic solvent (e.g., water) are two examples of such problems [7]. This kind of problem is called an interface problem. Interface problems have wide application in fluid dynamics, biomathematics, and material science among other fields.

In this dissertation, the focus is on elliptic and elasticity interface problems. For elliptic problems, the partial differential equation is

\[- \nabla (\beta(x) \nabla u(x)) = f(x), \quad x \in \Omega \setminus \Gamma,\]

with jump conditions

\[
\begin{aligned}
[u]_{\Gamma}(x) &\equiv u^+(x) - u^-(x) = a(x), \\
[(\beta \nabla u) \cdot n]_{\Gamma}(x) &\equiv n \cdot (\beta^+(x) \nabla u^+(x)) - n \cdot (\beta^-(x) \nabla u^-(x)) = b(x),
\end{aligned}
\]

and boundary conditions

\[u(x) = g(x), \quad x \in \partial\Omega\]
For elasticity problems, the partial differential equation is

\[
\begin{cases}
-\nabla \left( \beta_1(x) \nabla u_1(x) \right) - \nabla \left( \beta_2(x) \nabla u_2(x) \right) = f_1(x), \\
-\nabla \left( \beta_3(x) \nabla u_1(x) \right) - \nabla \left( \beta_4(x) \nabla u_2(x) \right) = f_2(x),
\end{cases}
\]

with jump conditions

\[
\begin{cases}
[u_1]_\Gamma(x) \equiv u_1^+(x) - u_1^-(x) = a_1(x), \\
[u_2]_\Gamma(x) \equiv u_2^+(x) - u_2^-(x) = a_2(x), \\
n \left( \beta_1^+(x) \nabla u_1^+(x) + \beta_2^+(x) \nabla u_2^+(x) \right) - \\
n \left( \beta_1^-(x) \nabla u_1^-(x) + \beta_2^-(x) \nabla u_2^-(x) \right) = b_1(x), \\
n \left( \beta_3^+(x) \nabla u_1^+(x) + \beta_4^+(x) \nabla u_2^+(x) \right) - \\
n \left( \beta_3^-(x) \nabla u_1^-(x) + \beta_4^-(x) \nabla u_2^-(x) \right) = b_2(x),
\end{cases}
\]

and boundary conditions

\[
\begin{cases}
 u_1(x) = g_1(x), \\
u_2(x) = g_2(x),
\end{cases}
\]

In electrostatics, for example, \( \beta \) represents the dielectric coefficient. It is about 2 in a macromolecule, 80 in water. \( f \) represents the charge density. Solving the interface problem gives the electric potential \( u \). In material science, \( u \) represents the potential or the pressure, and \( \beta \) is about 1 for air, 12 - 13 for silicon. Usually, the balance laws across interfaces bring out the jump conditions [7].

Since an irregular domain can be embedded into a regular domain, the original boundary condition can be changed to jump conditions, and a boundary value problem for an irregular domain can be converted into an interface problem for a regular domain [7].
12 The Current Method

This dissertation further generalizes the method introduced in [15, 16]. A finite element formulation was used to solve the elliptic and elasticity interface problems. The theorems in [15] are generalized in this dissertation and proofs are provided. It was also proved that the resulting linear system is (unsymmetric) positive definite if \( \beta \) is positive definite and lower-order terms are not present. The numerical results show that this method is second-order accurate in the \( L^\infty \) norm for piecewise smooth solutions.

The idea of solving elliptic and elasticity interface problems is shown in the following steps:

(1) Set up the partition of the domain. In two-dimensional models, the whole domain is cut into right triangles. In three-dimensional models, the whole domain is cut into similar tetrahedrons.

(2) On the interface cells, locate the end points of the interface segment. In two dimensions, for the case of two domains, the interface segment is a straight line, for the case of three domains, the interface segment can either be one straight line or three straight lines connected at one point. The interface segment is denoted by \( \Gamma^K_h \). In three dimensions, the interface segment would be a triangle or a polygon, and is denoted by \( \Gamma^L_h \). The locations of the interface segments can be calculated from the level-set function \( \phi = \phi(x_1, y) \). The jump condition \( a \) is defined at these end points, and another jump condition \( b \) is defined at the center point of the interface segment.
(3) Use the jump conditions \(a\) and \(b\) to calculate the numerical solution at end points on the interface segment. For elliptic interface problems, the numerical solution at end points should be the linear combination of the jump condition values mentioned above and the values of interface cell vertices. For elasticity interface problems, it is a little more complicated than the elliptic case. Because there are two solutions defined on each interface cell, the number of jump conditions and the number of vertices would double.

(4) Calculate the integration on the left hand side of Equations 1.1 and 1.4 on each cell. For a regular cell, it would be easy to integrate because all the functions are supposed to be continuous on this cell. For an interface cell, if it is separated into two different subdomains by the interface, the integration consists of two different functional integrations. If the interface cell is separated into three different subdomains by the interfaces, the integration consists of three different functional integrations. In order to make this method more accurate, the Gaussian quadrature rule is used for integration in this dissertation.

(5) Set up the system matrix.

(6) Calculate the integration on the right hand side of Equations 1.1 and 1.4 on each cell. Use the same technique as above.

(7) Solve the linear system of equations. Because the system matrix is non-symmetric, the biconjugate gradient stabilized method is used in this dissertation.

(8) Draw the figure and analyze the result.
1.3 Outline of This Dissertation

The study of elliptic and elasticity interface problems has a long history. In Chapter 2, the main previous work in this field is introduced.

Chapter 3 builds on the method in [15]. A more accurate finite element method is proposed to solve elliptic equations with sharp-edged interfaces with $\beta$ being uniformly elliptic (therefore positive definite) and lower-order terms present. Experimental results show that the order of accuracy for sharp-edged interfaces was improved from 0.8th to close to second order.

In Chapter 4, the numerical method in [16] is extended to solve the elasticity problem with sharp-edged interfaces. The method is simpler compared to that developed in [12] and it can be applied for more general problems since the $\beta_i$ are allowed to be matrices. Also, the proof of the positive definite property of the system matrix is provided, and numerical results are second-order accurate.

Solving the elliptic problem with three domains is a new and challenging work. In Chapter 5, this method is used to deal with three-domain problems. The appearance of the triple junction point is a new challenge. The method is extended and numerical results demonstrate near second-order accuracy for piecewise smooth solutions.

In Chapter 6, this method is extended to solve the three-dimensional elliptic problem with two domains. Three-dimensional problems are always more complicated, and solving it accurately would be a big challenge. However, this method can deal with three dimensions simply and accurately. All the results can achieve second-order accuracy.
CHAPTER 2

PREVIOUS WORK

Although the importance of elliptic and elasticity interface problems has been well recognized in a variety of disciplines, designing highly efficient methods for these problems is a difficult job because of the low global regularity of the solution. Since 1977, after the pioneering work of Peskin [30], much attention has been paid to the numerical solution of elliptic interface equations on regular Cartesian grids. In many studies, simple Cartesian grids are preferred. In this way, the complicated procedure of generating an unstructured grid can be bypassed, and well-developed fast algebraic solvers can be used.

In [30, 31], in order to simulate the flow pattern of blood in the heart, Peskin proposed the “immersed boundary” method, which used an improved numerical approximation of the δ-function. In [32], in order to compute two-phase flow, a level-set method was combined with the “immersed boundary” method. The level-set method was used to “capture” the interface between two fluids. This method can get first-order accuracy even in multiple spatial dimensions.

In [25, 26], the interface is smooth but irregular. They extend the solution to a rectangular region by using Fredholm integral equations. This equation can deal with interface conditions \([u] \neq 0\) and \([u_n] = 0\). The discrete Laplacian was evaluated using
these jump conditions. When a fast Poisson solver is used to compute the extended solution, it can achieve second or higher-order accuracy.

In [6], second-order elliptic problems with two-dimensional convex polygonal domains are solved with a finite element method. It can achieve second-order accuracy in the energy norm and nearly second-order accuracy in the $L^2$ norm when the interfaces are smooth but of arbitrary shape, and it can be extended to solve self-adjoint elliptic problems.

The "immersed interface" method was proposed in [17]. This method incorporates the interface conditions into the finite difference stencil, preserving that neither of the two jump conditions are zero. It can get second-order accuracy. The corresponding linear system is neither positive definite nor symmetric. Various applications and extensions of the "immersed interface" method are provided in [21].

In [18], on the basis of the "immersed interface" method, a fast iterative method was proposed to solve constant coefficient problems with the interface conditions $[u] = 0$ and $[\beta u_n] \neq 0$. Before using the immersed interface method, the differential equation is preconditioned. The discretization can guarantee second-order accuracy. A GMRES iteration is used to solve the Schur complement system. The number of iterations is independent of the jump in the coefficients and the mesh size.

In [19, 20], the immersed finite element methods (IFEM) were developed using non-body-fitted Cartesian meshes for homogeneous jump conditions. The idea is to modify the basis functions so that the homogeneous jump conditions are satisfied. Both non-conforming and conforming IFEM were developed in [20] for two-dimensional problems.
The boundary condition capturing method [22] was proposed on basis of the Ghost fluid method [10]. Both methods are robust and simple to implement. In [33], they improved the boundary condition capturing method with a multi-grid method. The weak formulation provided in [23] was discretized to achieve this method. Elliptic problems with interface conditions \([u] \neq 0\) and \([\beta u_n] \neq 0\) in two dimensions and three dimensions can be solved by this method. However, the method in [22] can only get first-order accuracy. It is in recent work [24] that for smooth interfaces the result was improved to second-order accuracy.

In [14], a discontinuous Galerkin (DG) method is proposed to solve elliptic interface problems. The matrix generated by this method is symmetric, and can be efficiently solved with standard algorithms. Numerical experiments show that this method is optimally convergent in the \(L^2\) norm for \(C^2\) interfaces.

In [15], a non-traditional finite element formulation for solving elliptic equations with smooth or sharp-edged interfaces was proposed with non-body-fitting grids for \([u] \neq 0\) and \([\beta u_n] \neq 0\). It achieved second-order accuracy in the \(L^\infty\) norm for smooth interfaces and about 0.8th order for sharp-edged interfaces. In [40], the matched interface and boundary (MIB) method was proposed to solve elliptic equations with smooth interfaces. In [39], the MIB method was generalized to treat sharp-edged interfaces. In [38], the three-dimensional generalization of the MIB method was developed for solving elliptic equations with discontinuous coefficients and non-smooth interfaces. In [34], they developed MIB method based schemes for solving two-dimensional elliptic PDEs with geometric singularities of multi-material interfaces. With an elegant treatment, second-order accuracy was achieved in the \(L^\infty\) norm.
However, for oscillatory solutions, the errors degenerated. Also, there has been a large body of work from the finite volume perspective for developing high order methods for elliptic equations in complex domains, such as [8, 28] for two-dimensional problems and [29] for three-dimensional problems. Another recent work in this area is a class of kernel-free boundary integral (KFI) methods for solving elliptic BVPs, presented in [37].

There are some other approaches to solve the elliptic interface problems. In particular, the recent work in [2] can handle sharp-edged interfaces. However, these approaches have not been developed to solve elasticity interface problems. Designing highly efficient methods for these problems is a difficult job, especially when the interface is not smooth.

An elasticity system can be solved by both the finite difference and the finite element method. Due to the cross derivative term, usually the linear system of equations using the finite element formulation is better conditioned compared with that obtained using a finite difference discretization.

To solve the interface problem, first a mesh must be generated. One approach is to use a body-fitted mesh coupled with a finite element discretization [1, 3, 4, 5] for scalar elliptic partial differential equations (PDEs). Recently, Cartesian meshes have become popular, especially for moving interface problems to overcome the cost in the grid generation at every or every other time step.

Finite difference methods are proposed in [35, 36] with non-homogeneous jump conditions. While second-order accuracy was achieved, the condition number of the discrete system is quite large, especially in the nearly incompressible case ($\lambda$ is large).
compared with that obtained from finite element formulations. In [35, 36], a first-order immersed interface finite element method (IIIFEM) was proposed using Cartesian meshes for the elasticity problem with homogeneous jump conditions. In general, the discretization using a finite element discretization has a better conditioned system of equations compared with that obtained from the finite difference method. The Sobolev space theory provides strong theoretical foundations for convergence analysis of finite element methods.

In [11], an immersed-interface finite element method was proposed for scalar elliptic interface problems with non-homogeneous jump conditions. In [12], a class of new immersed-interface finite element methods (IIIFEM) was proposed to solve elasticity interface problems with homogeneous and non-homogeneous jump conditions in two dimensions.
CHAPTER 3

2-D ELLIPTIC PROBLEM WITH TWO DOMAINS

In this chapter, a finite element formulation is used to solve elliptic equations with sharp-edged interfaces with \( \beta \) being uniformly elliptic (therefore positive definite) and lower-order terms present. The resulting linear system of equations is shown to be positive definite under certain assumptions. Extensive numerical experiments are also provided. Compared with the previous work in [15], the order of accuracy for sharp-edged interfaces is improved from 0.8th to close to second order. Compared with the results in [39], the more oscillatory the solution is, the more advantageous the current method is. The orders of accuracy for different regularities of solutions and different regularities of interfaces are listed in Table 3.11

3.1 Equations and Weak Formulations

Let \( \Omega \subset \mathbb{R}^d \) be an open bounded domain and let \( \Gamma \) be an interface. \( \Gamma \) divides \( \Omega \) into two disjoint open subdomains \( \Omega^- \) and \( \Omega^+ \), \( \Omega = \Omega^- \cup \Omega^+ \cup \Gamma \). Let \( \partial \Omega \) be the boundary of \( \Omega \), \( \partial \Omega^\pm \) be the boundary of each subdomain. We assume that \( \partial \Omega \) and \( \partial \Omega^\pm \) are Lipschitz continuous and so is \( \Gamma \). A unit normal vector of \( \Gamma \) can be defined almost everywhere on \( \Gamma \).
The variable coefficient elliptic interface problem is given by

\[- \nabla \left( \beta(x) \nabla u(x) \right) + p(x) \nabla u(x) + q(x) u(x) = f(x), \quad x \in \Omega \setminus \Gamma, \tag{3.1}\]

where \( x = (x_1, \ldots, x_d) \) are the spatial variables, \( \beta(x) \) is defined to be a \( d \times d \) matrix that is uniformly elliptic on \( \Omega^- \) and \( \Omega^+ \), and its components are continuously differentiable on \( \Omega^- \) and \( \Omega^+ \), but they might be discontinuous across \( \Gamma \). \( f(x) \) is in \( L^2(\Omega) \.

The jump conditions are prescribed

\[
\begin{aligned}
[u]_\Gamma(x) &\equiv u^+(x) - u^-(x) = a(x), \\
[(\beta \nabla u) \cdot n]_\Gamma(x) &\equiv n \left( \beta^+(x) \nabla u^+(x) \right) - n \left( \beta^-(x) \nabla u^-(x) \right) = b(x),
\end{aligned}
\tag{3.2}
\]

\( a \) and \( b \) are given functions along the interface \( \Gamma \), "±" denote limits taken within \( \Omega^\pm \)

The boundary conditions are prescribed by a function \( g \), given on \( \partial \Omega \)

\[ u(x) = g(x), \quad x \in \partial \Omega \tag{3.3} \]

The weak formulation in [15] is generalized for the elliptic equation with matrix coefficients and lower-order terms present. The usual Sobolev space \( H^1(\Omega) \) is used. For \( H^1_0(\Omega) \), an inner product is chosen as

\[
B[u, v] = \int_{\Omega^+} \beta \nabla u \cdot \nabla v + \int_{\Omega^-} \beta \nabla u \cdot \nabla v + \\
\int_{\Omega^+} (p \nabla u)v + \int_{\Omega^-} (p \nabla u)v + \int_{\Omega^+} quv + \int_{\Omega^-} quv \tag{3.4}
\]

Remark 1 For general second-order elliptic equations with lower-order \( p, q \) terms, one of the hypotheses of the Lax-Milgram Theorem is not guaranteed. For detailed discussion about the energy estimates and a first existence theorem for weak solutions,
Although a numerical example with \( p \neq 0, q \neq 0 \) in Section 3.3 is provided, for ease of theoretical discussion, it is assumed that \( p = 0, q = 0 \) for the rest of this section as well as in Section 3.2.

Equation 3.4 without the \( p, q \) terms induces a norm on \( H^1_0(\Omega) \), which is equivalent to the usual one, thanks to the Poincare inequality and the uniformly ellipticity and boundedness of \( \beta(x) \) on \( \Omega \).

Let \( R \) be the restriction operator from \( H^1(\Omega) \) to \( L^2(\partial\Omega^-) \). \( R \) is closed Lipschitz continuous (see Theorem 2.4.2 in [27]) on \( C^1(\Omega) \) and because \( C^1(\Omega) \) is dense in \( H^1(\Omega) \), it is well defined and bounded. For functions \( \tilde{a}, \tilde{b} \in H^1(\Omega) \), the restrictions to \( \partial\Omega^- \) are

\[
a = R_{\partial\Omega^-}(\tilde{a}), \quad b = R_{\partial\Omega^-}(\tilde{b})
\]

(3.5)

Throughout, we assume a function \( \tilde{c} \in H^1(\Omega) \) exists so that the boundary condition on \( \partial\Omega \) is

\[
g = \begin{cases} 
R_{\partial\Omega}(\tilde{c} - \tilde{a}), & \text{on } \partial\Omega \cap \partial\Omega^-, \\
R_{\partial\Omega}(\tilde{c}), & \text{on } \partial\Omega \setminus \partial\Omega^-
\end{cases}
\]

(3.6)

For simplicity, the tildes are dropped in this dissertation.

A unique solution of the problem is constructed in the space

\[
H(a, c) = \{ u \mid u - c + a\chi(\Omega^-) \in H^1_0(\Omega) \}
\]

(3.7)

If \( u \in H(a, c) \), then \( [u]_{\Gamma} = a, \ u|_{\partial\Omega} = g \) \( \quad H^1_0(\Omega) \) can be written as \( H(0, 0) \). A similar idea is also used in [15, 16].
Definition 3.1.1 \( u \in H(a, c) \) is called a weak solution of Equations 3.1-3.3, if
\[ v = u - c + a\chi(\Omega^-) \in H^1_0(\Omega) \]
satisfies
\[ B[v, \psi] = F(\psi), \quad (3.8) \]
for all \( \psi \in H^1_0(\Omega) \), where
\[ B[v, \psi] = \int_{\Omega^+} \beta \nabla v \cdot \nabla \psi + \int_{\Omega^-} \beta \nabla v \cdot \nabla \psi, \quad (3.9) \]
\[ F(\psi) = \int_{\Omega} f\psi + \int_{\Omega} \beta \nabla c \cdot \nabla \psi + \int_{\Omega^-} \beta \nabla a \cdot \nabla \psi + \int_{\Gamma} b\psi \quad (3.10) \]
Or equivalently

Definition 3.1.2 \( u \in H(a, c) \) is called a weak solution of Equations 3.1-3.3, if it satisfies, for all \( \psi \in H^1_0(\Omega) \),
\[ \int_{\Omega^+} \beta \nabla u \cdot \nabla \psi + \int_{\Omega^-} \beta \nabla u \cdot \nabla \psi = \int_{\Omega} f\psi + \int_{\Gamma} b\psi \quad (3.11) \]

Theorem 3.1.3 If \( f \in L^2(\Omega) \), and \( a, b, c \in H^1(\Omega) \), then there exists a unique weak solution of Equations 3.1-3.3 in \( H(a, c) \).

Proof See Theorem 2.1 in [15]. \( \square \)

3.2 Numerical Method

For simplicity, assume \( a, b \) and \( c \) are smooth on \( \overline{\Omega} \), \( \beta \) and \( f \) are smooth on \( \Omega^+ \) and \( \Omega^- \), but might be discontinuous across \( \Gamma \), \( \partial \Omega^+ \) and \( \partial \Omega^- \) are Lipschitz continuous. \( \phi \) is a level-set function on \( \Omega \), where \( \Gamma = \{ \phi = 0 \}, \Omega^- = \{ \phi < 0 \} \) and \( \Omega^+ = \{ \phi > 0 \} \). \( n = \frac{\nabla \phi}{|\nabla \phi|} \) is a unit normal vector of \( \Gamma \) pointing from \( \Omega^- \) to \( \Omega^+ \).

The setup is restricted to a rectangular domain \( \Omega = (x_{\text{min}}, x_{\text{max}}) \times (y_{\text{min}}, y_{\text{max}}) \) in the plane, and \( \beta \) is a \( 2 \times 2 \) matrix that is uniformly elliptic in each subdomain. Let \( I \)
and $J$ be positive integers, set $\Delta x = (x_{\text{max}} - x_{\text{min}})/I$ and $\Delta y = (y_{\text{max}} - y_{\text{min}})/J$. A uniform Cartesian grid is defined as $(x_i, y_j) = (x_{\text{min}} + \Delta x \cdot i, y_{\text{min}} + \Delta y \cdot j)$ for $i = 0, \ldots, I$ and $j = 0, \ldots, J$. $h = \max(\Delta x, \Delta y) > 0$ is the grid size.

Two grid function sets will be used

$$H^{1,h} = \{ \omega^h = (\omega_{i,j}) \mid 0 \leq i \leq I, 0 \leq j \leq J \},$$

and

$$H_0^{1,h} = \{ \omega^h = (\omega_{i,j}) \in H^{1,h} \mid \omega_{i,j} = 0 \text{ if } i = 0, I \text{ or } j = 0, J \}.$$ 

Every rectangular region $[x_i, x_{i+1}] \times [y_j, y_{j+1}]$ is cut into two right triangular regions. When all those triangular regions are collected, a uniform triangulation $T^h \bigcup_{K \in T^h} K$ is obtained, see Figure 3.1.

![Figure 3.1 A uniform triangulation](image)

If $\phi(x_i, y_j) \leq 0$, the grid point $(x_i, y_j)$ is counted as in $\Omega^-$, otherwise it is counted as in $\Omega^+$.
A cell $\triangle_k$ with corners $k_1, k_2, k_3$ belongs to one of two different sets

$$\Lambda_1 = \{ \triangle_k \subset \Omega \hspace{1em} k_1, k_2, k_3 \text{ are in the same domain among } \Omega^\pm \},$$

$$\Lambda_2 = \{ \triangle_k \subset \Omega \hspace{1em} k_1, k_2, k_3 \text{ are in two different domains among } \Omega^\pm \}$$

If a cell belongs to $\Lambda_1$, it is a regular cell, otherwise, it is an interface cell. The interface segment $\Gamma_h^k$ separates the interface cell into $K^+$ and $K^-$

In this dissertation, two extension operators are needed

$T^h : H^{1,h} \rightarrow H_0^1(\Omega)$ For any $\psi^h \in H_0^{1,h}$, $T^h(\psi^h)$ is a standard continuous piecewise linear function in every triangular cell matching $\psi^h$ on grid points. The function set is a subspace of $H_0^1(\Omega)$, which can be written as $H_0^{1,h}$

$U^h$ For any $u^h \in H^{1,h}$, $u^h = g^h$ at boundary points, $U^h(u^h)$ is a piecewise linear function in every triangular cell matching $u^h$ on grid points. In a regular cell, $U^h(u^h) = T^h(u^h)$ is a linear function. In an interface cell, $U^h(u^h)$ is one linear function on $K^+$ and another linear function on $K^-$. A similar extension is also used in [15, 16, 20, 22]. In order to use this extension, the following theorem is needed

**Theorem 3.2.1** For all $u^h \in H^{1,h}$, $U^h(u^h)$ can be constructed uniquely, if $T^h, \phi, a$ and $b$ are given

**Proof** There are three typical cases for $U^h(u^h)$

**Case 0** As is shown in Figure 3-2, if $K$ is a regular cell, $U^h(u^h) = T^h(u^h)$, i.e.

$$U^h(u^h) = u(p_1) + \frac{u(p_2) - u(p_1)}{\Delta x}(x - x_i) + \frac{u(p_3) - u(p_1)}{\Delta y}(y - y_i) \tag{3.12}$$
Case 1 As is shown in Figure 3.3, if $K$ is an interface cell with $\Gamma$ cutting through two legs of $K$, then

$$U^h(u^h) = \begin{cases} 
  u(p_1) + u_x^+(x-x_i) + u_y^+(y-y_i) & (x,y) \in K^+, \\
  u(p_2) + u_x^-(x-x_i - \Delta x) + u_y^-(y-y_i) & (x,y) \in K^-,
\end{cases}$$

(3.13)

here $u_y^- = \frac{u(p_3)-u(p_2)}{\Delta y} + \frac{\Delta x}{\Delta y} u_x^-$.
In Figure 3.3, \( \mathbf{n} = (-\frac{dy}{\sqrt{dx^2 + dy^2}}, -\frac{dx}{\sqrt{dx^2 + dy^2}}) \)

\[
\begin{align*}
\left\{ 
\begin{array}{ll}
\frac{u_x^+}{u} = u(p_4) + a - u(p_1), \\
\frac{u_y^+}{u} = u(p_5) + a - u(p_1),
\end{array}
\right.
\end{align*}
\tag{3.14}
\]

In Figure 3.4, it is assumed that the extensions of \( p_3 p_5 \) and \( p_2 p_4 \) intersect at a ghost point called \( p_1^G \), therefore

\[
\begin{align*}
(x, y + \Delta y) &= p_1, \\
(x, y + dy) &= p_3 \\
(x, y) &= p_1^G, \\
(x + dx, y) &= p_4 \\
(x + \Delta x, y) &= p_2
\end{align*}
\]

Figure 3.4 The ghost point

\[
\begin{align*}
\left\{ 
\begin{array}{ll}
\frac{u(p_4^G) - u(p_4)}{dx} = \frac{u(p_4) - u(p_2)}{\Delta x - dx}, \\
\frac{u(p_5^G) - u(p_5)}{dy} = \frac{u(p_5^G) - u(p_3)}{\Delta y}
\end{array}
\right.
\end{align*}
\tag{3.15}
\]

and

\[
\begin{align*}
\left\{ 
\begin{array}{ll}
\frac{u_x^-}{u} = \frac{u(p_2) - u(p_4)}{\Delta x - dx}, \\
\frac{u_y^-}{u} = \frac{u(p_3) - u(p_5)}{\Delta y - dy},
\end{array}
\right.
\end{align*}
\tag{3.16}
\]

From Equation 3.15 and Equation 3.16

\[
\begin{align*}
u(p_1^G) &= \frac{dx}{\Delta x - dx}(u(p_4) - u(p_2)) + u(p_4), \\
u(p_5) &= u(p_1^G) - \frac{dy}{\Delta y}(u(p_1^G) - u(p_3))
\end{align*}
\tag{3.17}
\]

\tag{3.18}
Let

$$\beta = \begin{pmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{pmatrix} \quad (3.19)$$

From Equations 3.14-3.19, note that $u_x^-$, $u_y^-$, $u_x^+$ and $u_y^+$ can all be written as linear functions of $u(p_1), u(p_2), u(p_3)$ and $u(p_4)$. Since $b = \beta \nabla u \bar{n}$, then

$$b = \beta^+ \nabla u^+ \bar{n} - \beta^- \nabla u^- \bar{n}$$

$$= \beta_{11} u_x^+ n_1 + \beta_{12} u_y^+ n_1 + \beta_{21} u_x^- n_2 + \beta_{22} u_y^+ n_2 -$$

$$(\beta_{11} u_x^- n_1 + \beta_{12} u_y^- n_1 + \beta_{21} u_x^- n_2 + \beta_{22} u_y^+ n_2) \quad (3.20)$$

From Equations 3.14-3.20, the value of $u(p_4)$ can be obtained. It is a linear function of $u(p_1), u(p_2), u(p_3)$. Hence $u_x^-$, $u_y^-$, $u_x^+$ and $u_y^+$ can be written in the following form

$$\begin{cases} 
  u_x^+ = c_{x,1} u(p_1) + c_{x,2} u(p_2) + c_{x,3} u(p_3) + c_{x,4} a(p_4) + c_{x,5} a(p_5) + c_{x,6} b(p_6), \\
  u_y^+ = c_{y,1} u(p_1) + c_{y,2} u(p_2) + c_{y,3} u(p_3) + c_{y,4} a(p_4) + c_{y,5} a(p_5) + c_{y,6} b(p_6), \\
  u_x^- = c_{x,1} u(p_1) + c_{x,2} u(p_2) + c_{x,3} u(p_3) + c_{x,4} a(p_4) + c_{x,5} a(p_5) + c_{x,6} b(p_6), \\
  u_y^- = c_{y,1} u(p_1) + c_{y,2} u(p_2) + c_{y,3} u(p_3) + c_{y,4} a(p_4) + c_{y,5} a(p_5) + c_{y,6} b(p_6), \\
\end{cases} \quad (3.21)$$

To complete the proof for Case 1, the following lemma is needed

**Lemma 3.2.2** All coefficients $c$ in Equation 3.21 are independent of $u^h, a$ and $b$

For simplicity, $c_{x,3}^+$ is taken as an example. The claim for the other coefficients can be proved similarly

$$c_{i,3}^+ = \alpha \left[ -(\beta_{12}^+ dy + \beta_{22}^+ dx) dy (\Delta x - dx) + (\beta_{12}^- dy + \beta_{22}^- dx) dy (\Delta x - dx) \right], \quad (3.22)$$

where $\frac{1}{\alpha} = (\beta_{11}^+ dy + \beta_{21}^+ dx) \Delta y (\Delta x - dx) dy + (\beta_{12}^+ dy + \beta_{22}^+ dx) \Delta x (\Delta y - dy) dx$ 

$$+ (\beta_{11}^- dy + \beta_{21}^- dx) \Delta y dx dy + (\beta_{12}^- dy + \beta_{22}^- dx) \Delta x dy dx$$
From Equation 3.22, it is easy to tell that $c_{x,3}^+$ is independent of $u^h, a$ and $b$

**Case 2** As is shown in Figure 3.5, if $K$ is an interface cell with $\Gamma$ cutting through the hypotenuse and one leg of $K$, then

\[
\begin{align*}
(x, y) &= p_1 \\
(x, y + \Delta y) &= p_3 \\
(x + \Delta x, y) &= p_5 \\
(x + \Delta x - dx, y_j) &= p_4 \\
(x + \Delta x, y_j) &= p_2
\end{align*}
\]

Figure 3.5 The interface cell Case 2

\[
U^h(u^h) = \begin{cases} 
  u(p_2) + u_x^+(x - x_i - \Delta x) + u_y^+(y - y_i) & (x, y) \in K^+ , \\
  u(p_1) + u_x^-(x - x_i) + \frac{u(p_3) - u(p_1)}{\Delta y} (y - y_i) & (x, y) \in K^- 
\end{cases}
\]

(3.23)

Similar derivation as in Case 1 gives

\[
\begin{align*}
  u_x^+ &= d_{x,1}^+ u(p_1) + d_{x,2}^+ u(p_2) + d_{x,3}^+ u(p_3) + d_{x,4}^+ a(p_4) + d_{x,5}^+ a(p_5) + d_{x,6}^+ b(p_6) , \\
  u_y^+ &= d_{y,1}^+ u(p_1) + d_{y,2}^+ u(p_2) + d_{y,3}^+ u(p_3) + d_{y,4}^+ a(p_4) + d_{y,5}^+ a(p_5) + d_{y,6}^+ b(p_6) , \\
  u_x^- &= d_{x,1}^- u(p_1) + d_{x,2}^- u(p_2) + d_{x,3}^- u(p_3) + d_{x,4}^- a(p_4) + d_{x,5}^- a(p_5) + d_{x,6}^- b(p_6) , \\
  u_y^- &= d_{y,1}^- u(p_1) + d_{y,2}^- u(p_2) + d_{y,3}^- u(p_3) + d_{y,4}^- a(p_4) + d_{y,5}^- a(p_5) + d_{y,6}^- b(p_6)
\end{align*}
\]

(3.24)

To complete the proof for Case 2, the following lemma is needed

**Lemma 3.2.3** All coefficients $d$ in Equation 3.24 are independent of $u^h, a$ and $b$

Same idea as Lemma 3.2.2, details are skipped here

Therefore, Theorem 3.2.1 has been completely proved

\[
\square
\]

Based on the above discussion, the following method is proposed
**Method 1** Find a discrete function $u^h \in H^{1,h}$ such that $u^h = g^h$ on the boundary points and so that for all $\psi^h \in H_0^{1,h}$, there is

$$\sum_{K \in T^h} \left( \int_{K^+} \beta \nabla U^h(u^h) \cdot \nabla T^h(\psi^h) + \int_{K^-} \beta \nabla U^h(u^h) \cdot \nabla T^h(\psi^h) \right) = \sum_{K \in T^h} \left( \int_{K^+} f T^h(\psi^h) + \int_{K^-} f T^h(\psi^h) + \int_{\Gamma_k} b T^h(\psi^h) \right)$$

(3.25)

On the boundary $u = g$ is equivalent to $u - c + a \chi(\Omega^-) = 0$

For the general case with $p \neq 0, q \neq 0$, the integral for these lower-order terms could be added to the above weak formulation.

To implement the above method, the Gaussian quadrature rule for integrals is used. The idea is illustrated in Figure 3.6. If $T$ is separated into two pieces by the interface $\overline{u_4u_5}$, $u_3$ and $u_4$ are connected, then three triangles are the result: $T_1 = \Delta u_1u_4u_5$, and $T_2 = \Delta u_2u_3u_4$, $T_3 = \Delta u_3u_4u_5$. For each triangle, the center point $p_{ij}$ is labeled for each edge $\overline{u_iu_j}$. In numerical computation, the average of three $f(p_{ij})$ is applied in each triangle. Numerical results show an improvement over [15], where fewer sample points were used.
Since the solution bases and test function bases are different, the matrix $A$ for the linear system generated by Method 1 is not symmetric in the presence of an interface. However, it can be proved that it is positive definite.

**Theorem 3.2.4** If $\beta$ is positive definite, and $p = q = 0$, then the $n \times n$ matrix $A$ for the linear system generated by Method 1 is positive definite.

**Proof** For any vector $c \in \mathbb{R}^n$,

$$c^T Ac = \sum_{i,j=1}^{n} a_{ij} c_i c_j = B \left[ \sum_{i=1}^{n} c_i u^i, \sum_{i=1}^{n} c_i \psi^i \right],$$

where $u^i$ and $\psi^i$ are basis functions for the solution and the test function, respectively. Note that they have compact support and have nonzero values only inside the six triangles around the $i$th grid point. For ease of discussion, each of $u^i$ and $\psi^i$ is decomposed into six parts, so that each part has nonzero values only inside one triangle. Now the summation over $i$ is equivalent to a summation over all the triangles, and there are three terms, $c_1 u_1 + c_2 u_2 + c_3 u_3$, $c_1 \psi_1 + c_2 \psi_2 + c_3 \psi_3$ for each triangle, where $u_1, u_2, u_3, \psi_1, \psi_2, \psi_3$ equals 1 on one vertex of a triangle and zero on two other vertices. The difference between $u_i$ and $\psi_i$ is, $u_i$ depends on the location of the interface and $\psi_i$ does not. $c_1 u_1 + c_2 u_2 + c_3 u_3$ is a piecewise linear function satisfying the jump conditions, and $c_1 \psi_1 + c_2 \psi_2 + c_3 \psi_3$ is a linear function. At the three vertices, the two functions coincide. Now the jump conditions can be set at $a = 0$ and $b$ can be set to have the value in the triangle such that $c_1 u_1 + c_2 u_2 + c_3 u_3 = c_1 \psi_1 + c_2 \psi_2 + c_3 \psi_3$ everywhere. In other words, compensation is made for the jump in $\beta$ by using $b$ to make sure the gradients on both sides of the interface coincide. Since Lemma 3.2.2 and Lemma 3.2.3 imply that the matrix $A$ is independent of $a, b$, choosing the above...
$a, b$ would not change the matrix $A$ and would only change the constant term, i.e., the right hand side of the linear system. Now the triangles are summed overall and the result is

$$
\sum_{i=1}^{n} c_i u^i = \sum_{i=1}^{n} c_i \psi^i
$$

It now follows from the positive definiteness of $\beta$ that

$$
c^T A c = B \left[ \sum_{i=1}^{n} c_i u^i, \sum_{i=1}^{n} c_i \psi^i \right] > 0
$$

Therefore $A$ is positive definite.

**Remark 2** A positive definite matrix $A$ has positive determinant, and is therefore invertible. It also has an $LDMT$ factorization where $D = \text{diag}(d_i)$ and $d_i > 0$, and $L, M$ are lower triangular. The linear system $Ax = b$ can be solved efficiently.

**Remark 3** For ease of discussion, both the $p, q$ terms have been dropped. However, the Lax-Milgram Theorem, the current Theorem 3.1.3, and Theorem 3.2.4 work for the case $p = 0$ and $q > 0$ as well. For the case with nonzero $p$ or negative $q$, the positive definiteness of $A$ is no longer guaranteed, nor is one of the hypotheses of the Lax-Milgram Theorem.

### 3.3 Numerical Experiments

Consider the problem

$$
-\nabla \ (\beta \nabla u) + p \ \nabla u + qu = f, \ \text{in} \ \Omega^\pm,
$$

$$
[u] = a, \ \text{on} \ \Gamma,
$$

$$
[(\beta \nabla u) \ n] = b, \ \text{on} \ \Gamma,
$$

$$
u = g, \ \text{on} \ \partial \Omega,
$$

(3.26)  

(3.27)  

(3.28)  

(3.29)
on the rectangular domain $\Omega = (x_{\text{min}}, x_{\text{max}}) \times (y_{\text{min}}, y_{\text{max}})$ The interface $\Gamma$ is prescribed by a level-set function $\phi(x, y)$ $\mathbf{n} = \frac{\nabla \phi}{|\nabla \phi|}$ is the unit normal vector of $\Gamma$ pointing from $\Omega^-$ to $\Omega^+$

In all examples of this section, given $\phi(x, y), \beta^\pm(x, y), p^\pm(x, y), q^\pm(x, y)$ and

\begin{align}
\quad u &= u^+(x, y), \quad \text{in } \Omega^+, \\
\quad u &= u^-(x, y), \quad \text{in } \Omega^-
\end{align}

Hence

\begin{align}
f &= -\nabla \cdot (\beta \nabla u) + p \nabla u + qu, \\
a &= u^+ - u^-, \\
b &= (\beta^+ \nabla u^+) \cdot \mathbf{n} - (\beta^- \nabla u^-) \cdot \mathbf{n},
\end{align}

on $\Omega$ $g$ is obtained from the given solutions as a proper Dirichlet boundary condition

All errors in solutions are measured in the $L^\infty$ norm in the whole domain $\Omega$ All errors in the gradients of solutions are measured in the $L^\infty$ norm away from interfaces

For Examples 1, 2, 3 and 4, let $p(x, y) = q(x, y) = 0$ and let $\beta^\pm$ be scalars Method 1 was implemented For Example 6, $\beta^\pm$ are symmetric positive definite matrices, and Method 1 was modified by adding the integrals for lower-order $p, q$ terms As discussed in Section 3.1, in this general case, one of the hypotheses of the Lax-Milgram Theorem is not guaranteed However, since the true solution was constructed first, the existence of a weak solution is automatically guaranteed The numerical result is promising
Example 1  This example is taken from [39]  $\phi, \beta^\pm$ are

\[
\phi(r, \theta) = \frac{R \sin(\theta_t/2)}{\sin(\theta_t/2 + \theta - \theta_r - 2\pi(t - 1)/5)} - r
\]

\[
\theta_r + \pi(2t - 2)/5 \leq \theta < \theta_r + \pi(2t - 1)/5,
\]

\[
\phi(r, \theta) = \frac{R \sin(\theta_t/2)}{\sin(\theta_t/2 - \theta + \theta_r - 2\pi(t - 1)/5)} - r
\]

\[
\theta_r + \pi(2t - 3)/5 \leq \theta < \theta_r + \pi(2t - 2)/5,
\]

(3.35)

(3.36)

with $\theta_t = \pi/5$, $\theta_r = \pi/7$, $R = 6/7$ and $t = 1, 2, 3, 4, 5$

\[
\beta^+(x, y) = 1,
\]

(3.37)

\[
\beta^-(x, y) = 2 + \sin(x + y)
\]

(3.38)

When the solutions $u^\pm$ are given as

\[
u^+(x, y) = 5 + 5(x^2 + y^2),
\]

(3.39)

\[
u^-(x, y) = x^2 + y^2 + \sin(x + y)
\]

(3.40)

The computed solution with the current method using a $40 \times 40$ grid is shown in

Figure 3.7

When the solutions $u^\pm$ are given as

\[
u^+(x, y) = 6 + \sin(2\pi x)\sin(2\pi y),
\]

(3.41)

\[
u^-(x, y) = x^2 + y^2 + \sin(x + y)
\]

(3.42)

The computed solution with the current method using a $40 \times 40$ grid is shown in

Figure 3.8

When the solutions $u^\pm$ are given as

\[
u^+(x, y) = 6 + \sin(6\pi x)\sin(6\pi y),
\]

(3.43)
Figure 3.7  Star shape interface  Case a

\[ u^-(r, y) = r^2 + y^2 + \sin(r + y) \quad (3.44) \]

The computed solution with the current method using a 40 \times 40 grid is shown in Figure 3.9. Table 3.1 shows the error of these three cases with the current method on different grids. Table 3.2 shows the error of these three cases using the method in [39] on different grids. These two tables show that as the solution gets more oscillatory, the current method is superior as better results were obtained than those presented in Table 3.2.

Example 2  This example comes from [22]. \( \phi(x, y) \), \( \beta^\pm(x, y) \) and \( u^\pm(x, y) \) are

\[ \phi(x, y) = x^2 + y^2 - 0.25, \quad (3.45) \]
\[ \beta^+(x, y) = 1, \quad (3.46) \]
\[ \beta^-(x, y) = 1, \quad (3.47) \]
\[ u^+(x, y) = 0, \quad (3.48) \]
\[ u^-(x, y) = \exp(x) \cos(y) \quad (3.49) \]
Figure 3.8 Star shape interface Case b

Figure 3.9 Star shape interface Case c

Figure 3.10 shows the computed solution with the current method using a 40 x 40 grid. Table 3.3 shows the error on different grids for the new developed method and the method in [22]. Comparing the results, it is easy to see that the method in [22] is first-order accurate, while the new developed method in this dissertation is second-order accurate.
Table 3.1 Star Results of the new developed method

<table>
<thead>
<tr>
<th>$n_x \times n_y$</th>
<th>Case(a) Error in $U$</th>
<th>Order</th>
<th>Case(b) Error in $U$</th>
<th>Order</th>
<th>Case(c) Error in $U$</th>
<th>Order</th>
</tr>
</thead>
<tbody>
<tr>
<td>20 x 20</td>
<td>7.70e-3</td>
<td></td>
<td>4.05e-2</td>
<td></td>
<td>3.40e-1</td>
<td></td>
</tr>
<tr>
<td>40 x 40</td>
<td>1.76e-3</td>
<td>2.13</td>
<td>1.06e-2</td>
<td>1.94</td>
<td>8.88e-2</td>
<td>1.94</td>
</tr>
<tr>
<td>80 x 80</td>
<td>5.49e-4</td>
<td>1.68</td>
<td>2.50e-3</td>
<td>2.08</td>
<td>2.33e-2</td>
<td>1.93</td>
</tr>
<tr>
<td>160 x 160</td>
<td>1.41e-4</td>
<td>1.96</td>
<td>6.31e-4</td>
<td>1.98</td>
<td>5.68e-3</td>
<td>2.04</td>
</tr>
</tbody>
</table>

Table 3.2 Star Results using the method described in [39]

<table>
<thead>
<tr>
<th>$n_x \times n_y$</th>
<th>Case(a) Error in $U$</th>
<th>Order</th>
<th>Case(b) Error in $U$</th>
<th>Order</th>
<th>Case(c) Error in $U$</th>
<th>Order</th>
</tr>
</thead>
<tbody>
<tr>
<td>20 x 20</td>
<td>6.11e-4</td>
<td></td>
<td>5.26e-2</td>
<td></td>
<td>9.72e-1</td>
<td></td>
</tr>
<tr>
<td>40 x 40</td>
<td>6.07e-5</td>
<td>3.33</td>
<td>8.51e-3</td>
<td>2.62</td>
<td>1.94e-2</td>
<td>2.32</td>
</tr>
<tr>
<td>80 x 80</td>
<td>1.34e-5</td>
<td>2.18</td>
<td>2.39e-3</td>
<td>1.83</td>
<td>5.49e-2</td>
<td>1.82</td>
</tr>
<tr>
<td>160 x 160</td>
<td>4.15e-6</td>
<td>1.69</td>
<td>6.64e-4</td>
<td>1.85</td>
<td>1.48e-2</td>
<td>1.89</td>
</tr>
</tbody>
</table>

Example 3 This example comes from [17] $\phi(x, y)$, $\beta^\pm(r, y)$ and $u^\pm(r, y)$ are

\[
\phi(x, y) = x^2 + y^2 - 0.25, \tag{3.50}
\]

\[
\beta^+(r, y) = 1, \tag{3.51}
\]

\[
\beta^-(x, y) = 1, \tag{3.52}
\]

\[
u^+(x, y) = 1 + \log(2\sqrt{x^2 + y^2}), \tag{3.53}
\]
Table 3.3  Example taken from [22]

<table>
<thead>
<tr>
<th>Method</th>
<th>The new developed Method</th>
<th>Method in [22]</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n_x \times n_y$</td>
<td>Error in $U$</td>
<td>Order</td>
</tr>
<tr>
<td>$20 \times 20$</td>
<td>8.9972e-4</td>
<td>0.0153</td>
</tr>
<tr>
<td>$40 \times 40$</td>
<td>2.4524e-4</td>
<td>1.8753</td>
</tr>
<tr>
<td>$80 \times 80$</td>
<td>6.0982e-5</td>
<td>2.0077</td>
</tr>
<tr>
<td>$160 \times 160$</td>
<td>1.2886e-5</td>
<td>2.2425</td>
</tr>
</tbody>
</table>

$$u^-(x, y) = 1$$  \hspace{1cm} (3.54)

Figure 3.11 shows the computed solution with the current method using a $40 \times 40$ grid. Table 3.4 shows the error on different grids for the new developed method and the method in [17]. Because the interface is smooth, both of these two methods can get to second-order accuracy.
Example 4  This example is from [15] $\phi(x, y)$, $\beta^\pm(x, y)$ and $u^\pm(x, y)$ are

\[
\phi(x, y) = (\sin(5\pi x) - y)(-\sin(5\pi y) - x), \quad (3.55)
\]

\[
\beta^+(x, y) = xy + 2, \quad (3.56)
\]

\[
\beta^-(x, y) = x^2 - y^2 + 3, \quad (3.57)
\]

\[
u^+(x, y) = 4 - x^2 - y^2, \quad (3.58)
\]

\[
u^-(x, y) = x^2 + y^2 \quad (3.59)
\]

![Figure 3.11 Example taken from [17]](image)

The computed solution with the current method using a 40 x 40 grid is shown in Figure 3.12. Table 3.5 shows the error on different grids. Compared with the results of [15], shown in Table 3.6, the current solution is more accurate than the previous work due to the quadrature rule discussed in Section 3.2.

Example 5 is taken from [15]. This example is used to investigate the order of the error in $u$ and $\nabla u$ on solutions and interfaces with different regularity.
Table 3.4 Example taken from [17]

<table>
<thead>
<tr>
<th>Method</th>
<th>The new developed Method</th>
<th>Method in [17]</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n_x \times n_y$</td>
<td>Error in $U$</td>
<td>Order</td>
</tr>
<tr>
<td>20 $\times$ 20</td>
<td>3.2039e-3</td>
<td>1.8555</td>
</tr>
<tr>
<td>40 $\times$ 40</td>
<td>8.8536e-4</td>
<td>1.9014</td>
</tr>
<tr>
<td>80 $\times$ 80</td>
<td>2.3700e-4</td>
<td></td>
</tr>
<tr>
<td>160 $\times$ 160</td>
<td>5.8734e-5</td>
<td>2.0126</td>
</tr>
</tbody>
</table>

Figure 3.12 Interface with the shape of a chess board

Example 5 $\phi(x, y)$, $\beta^\pm(x, y)$ and $u^\pm(x, y)$ are given as follows. The interface is Lipschitz continuous but has a sharp corner at $(0, 0)$, $u$ is piecewise $H^2$

$$\phi(x, y) = y - 2x, \quad x + y > 0, \quad (3.60)$$

$$\phi(x, y) = y + r/2, \quad r + y \leq 0, \quad (3.61)$$

$$\beta^+(x, y) = 1, \quad (3.62)$$
Table 3.5 Chess board Results of the new developed method

<table>
<thead>
<tr>
<th>$n_x \times n_y$</th>
<th>Error in $U$</th>
<th>Order</th>
<th>Error in $\nabla U$</th>
<th>Order</th>
</tr>
</thead>
<tbody>
<tr>
<td>40 x 40</td>
<td>9.74e-4</td>
<td></td>
<td>4.650e-3</td>
<td></td>
</tr>
<tr>
<td>80 x 80</td>
<td>2.71e-4</td>
<td>18051</td>
<td>3.454e-3</td>
<td>0.4290</td>
</tr>
<tr>
<td>160 x 160</td>
<td>9.4e-5</td>
<td>15276</td>
<td>1.433e-3</td>
<td>1.2692</td>
</tr>
<tr>
<td>320 x 320</td>
<td>2.6e-5</td>
<td>18541</td>
<td>6.89e-4</td>
<td>1.0565</td>
</tr>
<tr>
<td>41 x 39</td>
<td>9.36e-4</td>
<td></td>
<td>5.356e-3</td>
<td></td>
</tr>
<tr>
<td>81 x 79</td>
<td>2.58e-4</td>
<td>18591</td>
<td>3.144e-3</td>
<td>0.7686</td>
</tr>
<tr>
<td>161 x 159</td>
<td>7.7e-5</td>
<td>17444</td>
<td>1.390e-3</td>
<td>1.1775</td>
</tr>
<tr>
<td>321 x 319</td>
<td>2.2e-5</td>
<td>18074</td>
<td>6.47e-4</td>
<td>1.1032</td>
</tr>
</tbody>
</table>

$$\beta^-(x, y) = 2 + \sin(x + y),$$  \hspace{1cm} (3.63)

$$u^+(x, y) = 8,$$  \hspace{1cm} (3.64)

$$u^-(x, y) = (x^2 + y^2)^{5/6} + \sin(x + y)$$  \hspace{1cm} (3.65)

Figure 3.13 shows the computed solution with the current method using an 81 x 41 grid. Table 3.7 shows the error on different grids.

**Example 6** This example has a “happy face” interface and matrix form $\beta^\pm$, with lower-order terms $p, q$ present. $\phi(x, y), \beta^\pm(x, y)$ and $u^\pm(x, y)$ are

$$\phi(x, y) = \max(\min(\phi_1, \phi_2, \phi_3), \phi_4, \phi_5, \phi_6, \min(\phi_7, \phi_8)),$$  \hspace{1cm} (3.66)

$$\phi_1(x, y) = x^2 + y^2 - 0.75^2 - 0.15^2,$$  \hspace{1cm} (3.67)

$$\phi_2(x, y) = (x - 0.75)^2 + y^2 - 0.15^2,$$  \hspace{1cm} (3.68)

$$\phi_3(x, y) = (x + 0.75)^2 + y^2 - 0.15^2,$$  \hspace{1cm} (3.69)
Table 3.6 Chess board Results using the method described in [15]

<table>
<thead>
<tr>
<th>( n_x \times n_y )</th>
<th>Error in ( U )</th>
<th>Order</th>
</tr>
</thead>
<tbody>
<tr>
<td>40 ( \times ) 40</td>
<td>2.38e-1</td>
<td></td>
</tr>
<tr>
<td>80 ( \times ) 80</td>
<td>7.88e-2</td>
<td>1.59</td>
</tr>
<tr>
<td>160 ( \times ) 160</td>
<td>5.43e-2</td>
<td>0.54</td>
</tr>
<tr>
<td>320 ( \times ) 320</td>
<td>2.57e-2</td>
<td>1.08</td>
</tr>
<tr>
<td>41 ( \times ) 39</td>
<td>1.24e-1</td>
<td></td>
</tr>
<tr>
<td>81 ( \times ) 79</td>
<td>6.75e-2</td>
<td>0.88</td>
</tr>
<tr>
<td>161 ( \times ) 159</td>
<td>4.56e-2</td>
<td>0.57</td>
</tr>
<tr>
<td>321 ( \times ) 319</td>
<td>2.25e-2</td>
<td>1.02</td>
</tr>
</tbody>
</table>

Figure 3.13 A singular point at \((0, 0)\)

\[
\phi_4(x, y) = -\frac{0.1}{0.12} (x - 0.2)^2 - \frac{0.12}{0.1} (y - 0.22)^2 + 0.12 \times 0.1, \quad (3.70)
\]

\[
\phi_5(x, y) = -\frac{0.1}{0.12} (x + 0.2)^2 - \frac{0.12}{0.1} (y - 0.22)^2 + 0.12 \times 0.1, \quad (3.71)
\]
Table 3.7 Singular point on the interface in two dimensions

<table>
<thead>
<tr>
<th>$n_x \times n_y$</th>
<th>Error in $U$</th>
<th>Order</th>
<th>Error in $\nabla U$</th>
<th>Order</th>
</tr>
</thead>
<tbody>
<tr>
<td>41 $\times$ 21</td>
<td>4.940e-3</td>
<td></td>
<td>4.698e-2</td>
<td></td>
</tr>
<tr>
<td>81 $\times$ 41</td>
<td>1.745e-3</td>
<td>15013</td>
<td>2.978e-2</td>
<td>0.6577</td>
</tr>
<tr>
<td>161 $\times$ 81</td>
<td>6.06e-4</td>
<td>15258</td>
<td>1.886e-2</td>
<td>0.6590</td>
</tr>
<tr>
<td>321 $\times$ 161</td>
<td>2.09e-4</td>
<td>15358</td>
<td>1.194e-2</td>
<td>0.6595</td>
</tr>
</tbody>
</table>

\[
\phi_0(r, y) = -r^2 - (y + 0.08)^2 + 0.12^2, \quad (3.72)
\]
\[
\phi_1(x, y) = -x^2 - (y + 0.625)^2 + 0.425^2, \quad (3.73)
\]
\[
\phi_2(x, y) = -x^2 - (y + 0.25)^2 + 0.2^2, \quad (3.74)
\]
\[
\beta^+(x, y) = \begin{pmatrix}
(r y + 2)/5 & 0 \\
0 & (x y + 2)/5
\end{pmatrix}, \quad (3.75)
\]
\[
\beta^-(x, y) = \begin{pmatrix}
(x^2 - y^2 + 3)/7 & 0 \\
0 & (x^2 - y^2 + 3)/7
\end{pmatrix}, \quad (3.76)
\]
\[
u^+(x, y) = 5 - 5x^2 - 5y^2, \quad (3.77)
\]
\[
u^-(x, y) = 7x^2 + 7y^2 + 1 \quad (3.78)
\]

The computed solution with the current method using a 40 $\times$ 40 grid is shown in Figure 3.14. Table 3.8 shows the error on different grids using the current method. Table 3.9 shows the error on different grids in [15]. These two tables show that the accuracy is significantly improved. The numerical result shows second-order accuracy in the $L^\infty$ norm for the solution.
Figure 3.14 Happy face without lower-order terms

Table 3.8 Happy face without lower-order terms

<table>
<thead>
<tr>
<th>$n_x \times n_y$</th>
<th>Error in $U$</th>
<th>Order</th>
</tr>
</thead>
<tbody>
<tr>
<td>40 x 40</td>
<td>3.2575e-3</td>
<td></td>
</tr>
<tr>
<td>80 x 80</td>
<td>8.1030e-4</td>
<td>2.0072</td>
</tr>
<tr>
<td>160 x 160</td>
<td>2.1751e-4</td>
<td>1.8974</td>
</tr>
<tr>
<td>320 x 320</td>
<td>6.4081e-5</td>
<td>1.7631</td>
</tr>
</tbody>
</table>

When the coefficients $\beta^{\pm}(x,y)$, $p^{\pm}(x,y)$ and $q^{\pm}(r,y)$ are

$$\beta^{+}(x,y) = \begin{pmatrix} xy + 2 & xy + 1 \\ xy + 1 & xy + 3 \end{pmatrix}$$

$$\beta^{-}(x,y) = \begin{pmatrix} x^2 - y^2 + 3 & x^2 - y^2 + 1 \\ x^2 - y^2 + 1 & x^2 - y^2 + 4 \end{pmatrix}$$

$$p^{+}(x,y) = \begin{pmatrix} xy \\ r^2 - y^2 - 1 \end{pmatrix}$$
Table 3.9 Happy face without lower-order terms in [15]

<table>
<thead>
<tr>
<th>$n_x \times n_y$</th>
<th>Error in $U$</th>
<th>Order</th>
</tr>
</thead>
<tbody>
<tr>
<td>40 x 40</td>
<td>6.06e-2</td>
<td></td>
</tr>
<tr>
<td>80 x 80</td>
<td>1.64e-2</td>
<td>1.89</td>
</tr>
<tr>
<td>160 x 160</td>
<td>4.34e-3</td>
<td>1.92</td>
</tr>
<tr>
<td>320 x 320</td>
<td>1.15e-3</td>
<td>1.92</td>
</tr>
</tbody>
</table>

$$p^-(x, y) = \begin{pmatrix} x^2 - y^2 \\ 2xy - 1 \end{pmatrix}, \quad (3.82)$$
$$q^+(x, y) = x^2 + y^2 - 2, \quad (3.83)$$
$$q^-(x, y) = xy + 1 \quad (3.84)$$

The computed solution with the current method using a 40 x 40 grid is shown in Figure 3.15. Table 3.10 shows the error on different grids. The numerical result shows second-order accuracy for the solution and first-order accuracy for the gradient in the $L^\infty$ norm.

From Table 3.5 and Table 3.7, the orders of the errors in $u$ and $\nabla u$ are listed in Table 3.11.

Compared with [15], when $\Gamma$ is $C^1$, the current order of accuracy is consistent with [15], and when $\Gamma$ is Lipschitz continuous, the current order of accuracy is higher than [15]. Besides, for the same grid size, the current error is consistently smaller than [15], thanks to the more elegant quadrature formula discussed in Section 3.2.
Figure 3.15 Happy face with lower-order terms

Table 3.10 Happy face with lower-order terms

<table>
<thead>
<tr>
<th>$n_x \times n_y$</th>
<th>Error in $U$</th>
<th>Order</th>
<th>Error in $\nabla U$</th>
<th>Order</th>
</tr>
</thead>
<tbody>
<tr>
<td>$40 \times 40$</td>
<td>5.931e-3</td>
<td>5.121e-2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$80 \times 80$</td>
<td>1.669e-3</td>
<td>1.8293</td>
<td>2.757e-2</td>
<td>0.8933</td>
</tr>
<tr>
<td>$160 \times 160$</td>
<td>4.51e-4</td>
<td>1.8878</td>
<td>1.686e-2</td>
<td>0.7095</td>
</tr>
<tr>
<td>$320 \times 320$</td>
<td>1.24e-4</td>
<td>1.8628</td>
<td>8.940e-3</td>
<td>0.9153</td>
</tr>
</tbody>
</table>

Table 3.11 Conclusion of numerical experiments

<table>
<thead>
<tr>
<th>$u$ is $C^2$</th>
<th>$\Gamma$ is $C^1$</th>
<th>$\Gamma$ is Lipschitz continuous</th>
</tr>
</thead>
<tbody>
<tr>
<td>2nd order in $u$, 1st order in $\nabla u$</td>
<td>2nd order in $u$, 1st order in $\nabla u$</td>
<td></td>
</tr>
<tr>
<td>1st order in $u$, 0th order in $\nabla u$</td>
<td>1st order in $u$, 0th order in $\nabla u$</td>
<td></td>
</tr>
<tr>
<td>16th order in $u$, 0th order in $\nabla u$</td>
<td>15th order in $u$, 0th order in $\nabla u$</td>
<td></td>
</tr>
</tbody>
</table>
CHAPTER 4

2-D ELASTICITY PROBLEM WITH TWO DOMAINS

In this chapter, based on the method in Chapter 3, a numerical method is proposed for solving the elasticity problem with sharp-edged interfaces. It was proved that the resulting linear system is non-symmetric but positive definite under certain assumptions. The method is simpler compared with that developed in [12] and can be applied for more general problems since the $\beta_i$ are allowed to be matrices.

4.1 The Weak Formulations

The variable coefficient elasticity interface problem is given by

$$\begin{cases} -\nabla \left( \beta_1(x) \nabla u_1(x) \right) - \nabla \left( \beta_2(x) \nabla u_2(x) \right) = f_1(x), \\ -\nabla \left( \beta_3(x) \nabla u_1(x) \right) - \nabla \left( \beta_4(x) \nabla u_2(x) \right) = f_2(x), \end{cases} \quad x \in \Omega \setminus \Gamma,$$

where $x = (x_1, \ldots, x_d)$ is the spatial variables, $\beta_i(x), i = 1, 2, 3, 4$ are assumed to be $d \times d$ matrices that are uniformly elliptic on $\Omega^-$ and $\Omega^+$, $f_i(x), i = 1, 2$ is in $L^2(\Omega)$. 

38
The jump conditions are prescribed

\[
\begin{align*}
[u_1]_\Gamma (x) &\equiv u_1^+(x) - u_1^-(x) = a_1(x), \\
[u_2]_\Gamma (x) &\equiv u_2^+(x) - u_2^-(x) = a_2(x), \\
n \left( \beta_1^+(x) \nabla u_1^+(x) + \beta_2^+(x) \nabla u_2^+(x) \right) - \\
n \left( \beta_1^-(x) \nabla u_1^-(x) + \beta_2^-(x) \nabla u_2^-(x) \right) &= b_1(x), \\
n \left( \beta_3^+(x) \nabla u_1^+(x) + \beta_4^+(x) \nabla u_2^+(x) \right) - \\
n \left( \beta_3^-(x) \nabla u_1^-(x) + \beta_4^-(x) \nabla u_2^-(x) \right) &= b_2(x),
\end{align*}
\]

(4.2)

\(a_{1,2}\) and \(b_{1,2}\) are given functions along the interface \(\Gamma\), "\(\pm\)" denote limits taken within \(\Omega^\pm\).

Functions \(g_{1,2}\) are given on \(\partial \Omega\), the boundary conditions are prescribed

\[
\begin{align*}
u_1(x) &= g_1(x), & x &\in \partial \Omega \\
u_2(x) &= g_2(x),
\end{align*}
\]

(4.3)

The setup of the problem is illustrated in Figure 4.1

The weak formulation in [15, 16] is modified. The usual Sobolev space \(H^1(\Omega)\) is used. For \(H^1_0(\Omega)\), an inner product is chosen as

\[
B[u, v] = \begin{cases} 
\int_{\Omega^+} (\beta_1 \nabla u_1 \cdot \nabla v_1 + \beta_2 \nabla u_2 \cdot \nabla v_1) + \int_{\Omega^-} (\beta_1 \nabla u_1 \cdot \nabla v_1 + \beta_2 \nabla u_2 \cdot \nabla v_1), \\
\int_{\Omega^+} (\beta_3 \nabla u_1 \cdot \nabla v_2 + \beta_4 \nabla u_2 \cdot \nabla v_2) + \int_{\Omega^-} (\beta_3 \nabla u_1 \cdot \nabla v_2 + \beta_4 \nabla u_2 \cdot \nabla v_2)
\end{cases}
\]

(4.4)

The weak formulation in [15, 16] is generalized for the elliptic equation with matrix coefficient

\[
B[v, \psi] = \int_{\Omega^+} \beta \nabla v \cdot \nabla \psi + \int_{\Omega^-} \beta \nabla v \cdot \nabla \psi
\]

(4.5)
Figure 4.1 Setup of the problem with a uniform triangulation

**Definition 4.1.1** $u \in H(a, c)$ is called a weak solution of Equations 4.1-4.3, if it satisfies, for all $\psi \in H^1_0(\Omega)$,

$$
\begin{align*}
&\int_{\Omega^+} (\beta_1 \nabla u_1 \cdot \nabla \psi_1 + \beta_2 \nabla u_2 \cdot \nabla \psi_1) + \int_{\Omega^-} (\beta_3 \nabla u_1 \cdot \nabla \psi_1 + \beta_4 \nabla u_2 \cdot \nabla \psi_1) \\
&= \int_{\Omega} f_1 \psi_1 + \int_{\Gamma} b_1 \psi_1,
\end{align*}
$$

(4.6)

**Theorem 4.1.2** If $f \in L^2(\Omega)$, $a$, $b$ and $c \in H^1(\Omega)$, then there exists a unique weak solution of Equations 4.1-4.3 in $H(a, c)$

**Proof** See Theorem 2.1 in [15] \qed
4.2 Numerical Method

Define

\[
\begin{align*}
\mathbf{u} & = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, & \mathbf{f} & = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, & \mathbf{g} & = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}, \\
\end{align*}
\]

(4.7)

\[
\begin{align*}
\mathbf{a} & = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, & \mathbf{b} & = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}, & \mathbf{\beta} & = \begin{bmatrix} \beta_1 & \beta_2 \\ \beta_3 & \beta_4 \end{bmatrix},
\end{align*}
\]

and choose a test function

\[
\psi = \begin{bmatrix} \psi^1 \\ 0 \\ 0 \\ \psi^2 \end{bmatrix}
\]

(4.8)

and redefine the gradient and divergence operator

\[
\begin{align*}
\nabla & = \begin{bmatrix}
\frac{\partial}{\partial x} & 0 \\
\frac{\partial}{\partial y} & 0 \\
0 & \frac{\partial}{\partial r} \\
0 & \frac{\partial}{\partial y}
\end{bmatrix}, & \nabla & = \begin{bmatrix}
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & 0 & 0 \\
0 & 0 & \frac{\partial}{\partial x} & \frac{\partial}{\partial y}
\end{bmatrix}
\end{align*}
\]

(4.9)

Then Equation 4.1 can be written as

\[
-\nabla \left( \beta(x) \nabla u(x) \right) = f(x), \ x \in \Omega \setminus \Gamma,
\]

(4.10)

the jump condition Equation 4.2 can be reformulated as

\[
\begin{align*}
[u]_{\Gamma}(x) & = u^+(x) - u^-(x) = a(x), \\
n \cdot (\beta^+(x) \nabla u^+(x)) - n \cdot (\beta^-(x) \nabla u^-(x)) & = b(x),
\end{align*}
\]

(4.11)

and the boundary condition is

\[
u(x) = g(x) \quad x \in \partial\Omega
\]

(4.12)
For simplicity, the following properties are discussed under the form of Equations 4.10, 4.11, and 4.12.

A cell $\triangle_k$ with corners $k_1, k_2, k_3$ belongs to one of two different sets

$$\Lambda_1 = \{\triangle_k \subset \Omega \mid k_1, k_2, k_3 \text{ are in the same domain among } \Omega^\pm\},$$

$$\Lambda_2 = \{\triangle_k \subset \Omega \mid k_1, k_2, k_3 \text{ are in two different domains among } \Omega^\pm\}$$

If a cell belongs to $\Lambda_1$, it is a regular cell, otherwise it is an interface cell. An interface cell is separated by a straight line segment, denoted by $\Gamma^h_k$.

**Theorem 4.2.1** If $\beta$ is positive definite, then the matrix $A$ for the linear system generated by the current method is positive definite.

*Proof* See proof of Theorem 3.2.4 in Chapter 3.

In some applications in [12], the matrix $\beta$ is only semi-positive definite with zero determinant. The above theorem does not apply. Below is the proof that when the matrix $\beta$ is of a certain form frequently appearing in applications and semi-positive definite, then the matrix $A$ generated by the current method is still positive definite.

**Theorem 4.2.2** If $\lambda > 0$, $\mu > 0$ and $\beta_1 = \begin{bmatrix} \lambda + 2\mu & 0 \\ 0 & \mu \end{bmatrix}$, $\beta_2 = \begin{bmatrix} 0 & \lambda \\ \mu & 0 \end{bmatrix}$, $\beta_3 = \begin{bmatrix} 0 & \mu \\ \lambda & 0 \end{bmatrix}$, $\beta_4 = \begin{bmatrix} \mu & 0 \\ 0 & \lambda + 2\mu \end{bmatrix}$, then the matrix $A$ for the linear system generated by the current method is positive definite.

*Proof* Suppose for a contradiction that $A$ is not positive definite. Then there is a vector $c \in \mathbb{R}^{2n}$ and $c \neq 0$ such that $c^T A c \leq 0$. Let

$$w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \sum_{i=1}^{2n} c_i \psi_i = \sum_{i=1}^{2n} c_i u_i,$$
then

\[ B[w, w] \leq 0, \]

\[ \Rightarrow \int_{\Omega} (\beta \nabla w(\bar{x}))^T \nabla w(x) d\bar{x} \leq 0, \]

\[ \Rightarrow \int_{\Omega} \begin{bmatrix} \frac{\partial w_1}{\partial x} & \frac{\partial w_1}{\partial y} & \frac{\partial w_2}{\partial x} & \frac{\partial w_2}{\partial y} \end{bmatrix} \begin{bmatrix} \lambda + 2\mu & 0 & 0 & \lambda \\ 0 & \mu & \mu & 0 \\ 0 & \mu & \mu & 0 \\ \lambda & 0 & 0 & \lambda + 2\mu \end{bmatrix} \begin{bmatrix} \frac{\partial w_1}{\partial x} \\ \frac{\partial w_1}{\partial y} \\ \frac{\partial w_2}{\partial x} \\ \frac{\partial w_2}{\partial y} \end{bmatrix} d\bar{x} \leq 0 \]

(4.13)

Since for all \( a = [a_1, a_2, a_3, a_4]^T \in \mathbb{R}^4, \)

\[ a^T \beta a = (a_1 + a_4)^2 \lambda + 2(a_1^2 + a_4^2)\mu + (a_2 + a_3)^2 \mu \geq 0 \]  

(4.14)

So \( a^T \beta a = 0 \) if and only if \( a_1 = a_4 = 0 \) and \( a_2 = -a_3 \). Then \( \frac{\partial w_1}{\partial x}(\bar{x}) = a_1 = 0, \forall \bar{x} \in \Omega \).

However, \( w_1 = \sum_{i=1}^{n} c_i \psi_i^1 \) implies \( \frac{\partial w_1}{\partial x} = \sum_{i=1}^{n} c_i \frac{\partial \psi_i^1}{\partial x} \). Since \( c = [c_1, c_2, \ldots, c_{2n}]^T \neq 0 \), without loss of generality, it is assumed that \( c_1 \neq 0 \). If a point \( \bar{x} \in \Omega \) is chosen such that \( \frac{\partial \psi_i^1(\bar{x})}{\partial x} \neq 0 \) and \( \frac{\partial \psi_i^1(\bar{x})}{\partial x} = 0, \forall i = 2, 3, \ldots, n, \) then \( \sum_{i=1}^{n} c_i \frac{\partial \psi_i^1}{\partial x} \neq 0 \), a contradiction.

Therefore \( c^T Ac > 0 \forall c \neq 0, \) that is, \( A \) is positive definite.

From Remark 2 in Chapter 3, it is known that a positive definite matrix has positive determinant, and is therefore invertible. The linear system \( Ax = b \) can be solved efficiently.
4.3 Numerical Experiments

Consider the problem

\[
\begin{align*}
-\nabla (\beta_1 \nabla u_1) - \nabla (\beta_2 \nabla u_2) &= f_1, \text{ in } \Omega^+, \\
-\nabla (\beta_3 \nabla u_1) - \nabla (\beta_4 \nabla u_2) &= f_2, \text{ in } \Omega^+
\end{align*}
\]  

(4.15)

The jump conditions and boundary conditions are given as

\[
\begin{align*}
[u_1] &= a_1, \text{ on } \Gamma, \\
[u_2] &= a_2, \text{ on } \Gamma, \\
[(\beta_1 \nabla u_1 + \beta_2 \nabla u_2) \cdot n] &= b_1, \text{ on } \Gamma, \\
[(\beta_3 \nabla u_1 + \beta_4 \nabla u_2) \cdot n] &= b_2, \text{ on } \Gamma, \\
u_1 &= g_1, \text{ on } \partial\Omega, \\
u_2 &= g_2, \text{ on } \partial\Omega,
\end{align*}
\]  

(4.16)

on the rectangular domain \( \Omega = (x_{\min}, x_{\max}) \times (y_{\min}, y_{\max}) \). The interface \( \Gamma \) is prescribed by a level-set function \( \phi(x, y) \) \( n = \frac{\nabla \phi}{|\nabla \phi|} \) is the unit normal vector pointing from \( \Omega^- \) to \( \Omega^+ \).

In all examples of this section, given \( \phi(x, y), \beta_{1,2,3,4}(x, y) \) and

\[
\begin{align*}
 u_1 &= u_1^+(x, y), \text{ in } \Omega^+, \\
 u_2 &= u_2^+(x, y), \text{ in } \Omega^+, \\
 u_1 &= u_1^-(x, y), \text{ in } \Omega^-, \\
 u_2 &= u_2^-(x, y), \text{ in } \Omega^-
\end{align*}
\]  

(4.17)
Hence, on $\Omega$,\begin{equation} \begin{aligned} f_1 &= -\nabla \left( \beta_1 \nabla u_1 \right) - \nabla \left( \beta_2 \nabla u_2 \right), \\ f_2 &= -\nabla \left( \beta_3 \nabla u_1 \right) - \nabla \left( \beta_4 \nabla u_2 \right), \\ a_1 &= u_1^+ - u_1^-, \\ a_2 &= u_2^+ - u_2^-, \\ b_1 &= \left( \beta_1^+ \nabla u_1^+ + \beta_2^+ \nabla u_2^+ \right) n - \left( \beta_1^- \nabla u_1^- + \beta_2^- \nabla u_2^- \right) n, \\ b_2 &= \left( \beta_3^+ \nabla u_1^+ + \beta_4^+ \nabla u_2^+ \right) n - \left( \beta_3^- \nabla u_1^- + \beta_4^- \nabla u_2^- \right) n, \end{aligned} \tag{4.18} \end{equation}

$g$ is obtained from the given solutions as a proper Dirichlet boundary condition.

All errors of solutions are measured in the $L^\infty$ norm in the whole domain $\Omega$.

Four numerical examples are presented in this chapter to demonstrate the effectiveness of the method.

**Example 7** This example has a smooth interface $\phi(x, y), \beta_1^+(x, y), \beta_2^+(x, y), \beta_3^+(x, y), \beta_4^+(x, y)$ and $u_1^+(x, y), u_2^+(x, y)$ are
\begin{align*}
\phi(x, y) &= x^2 + y^2 - 0.25, \\
\beta_1^+(x, y) &= \begin{pmatrix}
\cos(x)^2 + 0.1 & (x + y)^2 + 2 \\
(x + y)^2 + 2 & 0.6 \cos(x) + 1
\end{pmatrix}, \\
\beta_1^-(x, y) &= \begin{pmatrix}
x^2 + y^2 + 3 & \sin(xy) + 1 \\
\sin(x + y) + 1 & y^2 + 4
\end{pmatrix}, \\
\beta_2^+(x, y) &= \begin{pmatrix}
\cos(x)^2 + 0.1 & (x + y)^2 + 2 \\
(x + y)^2 + 2 & 0.6 \cos(x) + 1
\end{pmatrix}, \\
\beta_2^-(x, y) &= \begin{pmatrix}
\cos(y) + 1 & (x + y)^2 + 1 \\
2x^2 + 1 & 0.5 \cos(x)^2
\end{pmatrix}.
\end{align*}
\[
\beta^+_3(x, y) = \begin{pmatrix}
\cos(x + y)^2 & 3x^2y^2 \\
x^2 + 1 & \cos(y) + 1
\end{pmatrix}, \quad (4.24)
\]

\[
\beta^-_3(x, y) = \begin{pmatrix}
2\cos(x + y)^2 & 3x^2y^2 + 01 \\
2x^2 & 2\cos(xy) + 2
\end{pmatrix}, \quad (4.25)
\]

\[
\beta^+_4(x, y) = \begin{pmatrix}
x^2y^2 + 5 & (\sin(x + 2y))^2 \\
\sin(x + 2y) + 1 & y^2 + x^2 + 3
\end{pmatrix}, \quad (4.26)
\]

\[
\beta^-_4(x, y) = \begin{pmatrix}
5x^2y^2 + 4 \sin(x) + 1 \\
\sin(x + y) + 1 & y^2 + x^2 + 4
\end{pmatrix}, \quad (4.27)
\]

\[
u^+_1(x, y) = x^2 + y^2 - \sin(x + y), \quad (4.28)
\]

\[
u^-_1(x, y) = (\sqrt{x^2 + y^2})^2, \quad (4.29)
\]

\[
u^+_2(x, y) = 2y(x^3) + y^2, \quad (4.30)
\]

\[
u^-_2(x, y) = (\sqrt{x^2 + y^2})^3 \quad (4.31)
\]

The computed solutions with the current method using a 48 x 48 grid are shown in Figures 4.2 and 4.3. Table 4.1 shows the error on different grids. The numerical result shows second-order accuracy in the \(L^\infty\) norm for the solution.

**Example 8** This example is a “happy face” interface with corners \(\phi(x, y), \beta^+_1(x, y), \beta^+_2(r, y), \beta^+_3(x, y), \beta^+_4(x, y)\) and \(u^+_1(r, y), u^+_2(r, y)\) are

\[
\phi(x, y) = \max(\min(\phi_1, \phi_2, \phi_3), \phi_4, \phi_5, \phi_6, \min(\phi_7, \phi_8)), \quad (4.32)
\]

\[
\phi_1(x, y) = x^2 + y^2 - 0.75^2 - 0.15^2, \quad (4.33)
\]

\[
\phi_2(x, y) = (x - 0.75)^2 + y^2 - 0.15^2, \quad (4.34)
\]

\[
\phi_3(x, y) = (x + 0.75)^2 + y^2 - 0.15^2, \quad (4.35)
\]
Figure 4.2 The solution $u_1$ with a smooth circular interface

Figure 4.3 The solution $u_2$ with a smooth circular interface

\[
\phi_4(x, y) = -\frac{0.1}{0.12} (x - 0.2)^2 - \frac{0.12}{0.1} (y - 0.22)^2 + 0.12 \ 0.1, \quad (4.36)
\]

\[
\phi_5(x, y) = -\frac{0.1}{0.12} (x + 0.2)^2 - \frac{0.12}{0.1} (y - 0.22)^2 + 0.12 \ 0.1, \quad (4.37)
\]

\[
\phi_6(x, y) = -x^2 - (y + 0.08)^2 + 0.12^2, \quad (4.38)
\]

\[
\phi_7(x, y) = -x^2 - (y + 0.625)^2 + 0.425^2, \quad (4.39)
\]
Table 4.1 Circle shape interface

<table>
<thead>
<tr>
<th>$n_x \times n_y$</th>
<th>Error in $U$</th>
<th>Order</th>
</tr>
</thead>
<tbody>
<tr>
<td>$24 \times 24$</td>
<td>0.00558</td>
<td></td>
</tr>
<tr>
<td>$48 \times 48$</td>
<td>0.00147</td>
<td>1.92</td>
</tr>
<tr>
<td>$96 \times 96$</td>
<td>3.76e-004</td>
<td>1.97</td>
</tr>
<tr>
<td>$192 \times 192$</td>
<td>9.48e-005</td>
<td>1.99</td>
</tr>
<tr>
<td>$384 \times 384$</td>
<td>2.39e-005</td>
<td>1.99</td>
</tr>
</tbody>
</table>

\[
\phi_8(x, y) = -x^2 - (y + 0.25)^2 + 0.2^2, \quad (4.40)
\]

\[
\beta_1^+(x, y) = \begin{pmatrix}
  x^2 + 3 & \sin(x + y) + 1 \\
  0.5\sin(x + y) + 0.7 & y^2 + 5
\end{pmatrix}, \quad (4.41)
\]

\[
\beta_1^-(x, y) = \begin{pmatrix}
  x^2 + y^2 + 3 & \sin(xy) + 1 \\
  \sin(x + y) + 1 & y^2 + 4
\end{pmatrix}, \quad (4.42)
\]

\[
\beta_2^+(r, y) = \begin{pmatrix}
  \cos(x)^2 + 0.1 & (x + y)^2 + 2 \\
  2x^2 & 0.6\cos(x) + 1
\end{pmatrix}, \quad (4.43)
\]

\[
\beta_2^-(x, y) = \begin{pmatrix}
  \cos(y) + 1 & (x + y)^2 + 1 \\
  2x^2 + 1 & 0.5\cos(x)^2
\end{pmatrix}, \quad (4.44)
\]

\[
\beta_3^+(x, y) = \begin{pmatrix}
  \cos(x + y)^2 & 3x^2y^2 \\
  x^2 + 1 & \cos(y) + 1
\end{pmatrix}, \quad (4.45)
\]

\[
\beta_3^-(x, y) = \begin{pmatrix}
  2\cos(x + y)^2 & 3x^2y^2 + 0.1 \\
  2x^2 & 2\cos(xy) + 2
\end{pmatrix}, \quad (4.46)
\]
\[
\beta_4^+(x, y) = \begin{pmatrix}
x^2y^2 + 5 & (\sin(x + 2y))^2 \\
\sin(x + 2y) + 1 & y^2 + x^2 + 3
\end{pmatrix}, \tag{4.47}
\]

\[
\beta_4^-(x, y) = \begin{pmatrix}
0 & 5x^2y^2 + 4 & \sin(x) + 1 \\
\sin(x + y) + 1 & y^2 + x^2 + 4
\end{pmatrix}, \tag{4.48}
\]

\[
u_1^+(x, y) = x^2 + y^2 - \sin(x + y), \tag{4.49}
\]

\[
u_1^-(x, y) = (\sqrt{x^2 + y^2})^2, \tag{4.50}
\]

\[
u_2^+(x, y) = 2y(x^3) + y^2, \tag{4.51}
\]

\[
u_2^-(x, y) = (\sqrt{x^2 + y^2})^3 \tag{4.52}
\]

The computed solutions with the current method using a 48 x 48 grid are shown in Figures 4.4 and 4.5. Table 4.2 shows the error on different grids. The numerical result shows second-order accuracy in the \(L^\infty\) norm for the solution and first-order accuracy in the \(L^\infty\) norm for the gradient.

![Figure 4.4](image)

Figure 4.4 The solution \(u_1\) with a “Happy face” interface
Figure 4.5 The solution $u_2$ with a "Happy face" interface

Table 4.2 Face shape interface

<table>
<thead>
<tr>
<th>$n_x \times n_y$</th>
<th>Error in $U$</th>
<th>Order</th>
</tr>
</thead>
<tbody>
<tr>
<td>24 x 24</td>
<td>0.00663</td>
<td></td>
</tr>
<tr>
<td>48 x 48</td>
<td>0.00178</td>
<td>1.89</td>
</tr>
<tr>
<td>96 x 96</td>
<td>4.71e-04</td>
<td>1.92</td>
</tr>
<tr>
<td>192 x 192</td>
<td>1.21e-04</td>
<td>1.96</td>
</tr>
<tr>
<td>384 x 384</td>
<td>3.16e-05</td>
<td>1.94</td>
</tr>
</tbody>
</table>

Example 9 This example is a "star" interface $\phi(x, y)$, $\beta_1^\pm(r, y)$, $\beta_2^\pm(r, y)$, $\beta_3^\pm(r, y)$, $\beta_4^\pm(x, y)$ and $u_1^\pm(x, y)$, $u_2^\pm(x, y)$ are

$$
\phi(r, \theta) = \frac{R \sin(\theta_t/2)}{\sin(\theta_t/2 + \theta - \theta_r - 2\pi(t - 1)/5)} - r
$$

$$
\theta_r + \pi(2t - 2)/5 \leq \theta < \theta_r + \pi(2t - 1)/5,
$$

(4.53)

$$
\phi(r, \theta) = \frac{R \sin(\theta_t/2)}{\sin(\theta_t/2 - \theta + \theta_r - 2\pi(t - 1)/5)} - r
$$
\[ \theta_r + \pi(2t - 3)/5 \leq \theta < \theta_r + \pi(2t - 2)/5, \quad (4.54) \]

with \( \theta_t = \pi/5, \theta_r = \pi/7, \ R = 6/7 \) and \( t = 1, 2, 3, 4, 5, \)

\[
\beta_1^+(x, y) = \begin{pmatrix}
x^2 + 3 & \sin(x + y) + 1 \\
0.5\sin(x + y) + 0.7 & y^2 + 5
\end{pmatrix}, \quad (4.55)
\]

\[
\beta_1^-(x, y) = \begin{pmatrix}
x^2 + y^2 + 3 & \sin(xy) + 1 \\
\sin(x + y) + 1 & y^2 + 4
\end{pmatrix}, \quad (4.56)
\]

\[
\beta_2^+(x, y) = \begin{pmatrix}
\cos(x)^2 + 0.1 & (x + y)^2 + 2 \\
2r^2 & 0.6\cos(r) + 1
\end{pmatrix}, \quad (4.57)
\]

\[
\beta_2^-(x, y) = \begin{pmatrix}
\cos(y) + 1 & (x + y)^2 + 1 \\
2x^2 + 1 & 0.5\cos(x)^2
\end{pmatrix}, \quad (4.58)
\]

\[
\beta_3^+(x, y) = \begin{pmatrix}
\cos(x + y)^2 & 3x^2y^2 \\
x^2 + 1 & \cos(y) + 1
\end{pmatrix}, \quad (4.59)
\]

\[
\beta_3^-(x, y) = \begin{pmatrix}
2\cos(x + y)^2 & 3x^2y^2 + 0.1 \\
2x^2 & 2\cos(xy) + 2
\end{pmatrix}, \quad (4.60)
\]

\[
\beta_4^+(x, y) = \begin{pmatrix}
x^2y^2 + 5 & (\sin(x + 2y))^2 \\
\sin(r + 2y) + 1 & y^2 + r^2 + 3
\end{pmatrix}, \quad (4.61)
\]

\[
\beta_4^-(x, y) = \begin{pmatrix}
0.5r^2y^2 + 4 & \sin(r) + 1 \\
\sin(x + y) + 1 & y^2 + x^2 + 4
\end{pmatrix}, \quad (4.62)
\]

\[
u_1^+(x, y) = x^2 + y^2 - \sin(x + y), \quad (4.63)
\]

\[
u_1^-(x, y) = (\sqrt{x^2 + y^2})^2, \quad (4.64)
\]

\[
u_2^+(x, y) = 2y(x^3) + y^2, \quad (4.65)
\]
\[ u_2^-(x, y) = \left( \sqrt{x^2 + y^2} \right)^3 \] (4.66)

The computed solutions with the current method using a 48 x 48 grid are shown in Figures 4.6 and 4.7. Table 4.3 shows the error on different grids. The numerical result shows second-order accuracy in the $L^\infty$ norm for the solution and first-order accuracy in the $L^\infty$ norm for the gradient.

![Figure 4.6 The solution $u_1$ with a “Star” interface](image)

**Example 10** The solutions in this example have a singularity on the interface corner $\phi(x, y), \beta_1^\pm(x, y), \beta_2^\pm(x, y), \beta_3^\pm(x, y), \beta_4^\pm(x, y)$ and $u_1^\pm(x, y), u_2^\pm(x, y)$ are

\[
\phi(x, y) = (x - 0.4)^2 + y^2 - 0.16, \quad (4.67)
\]

\[
\beta_1^+(x, y) = \begin{pmatrix} x^2 + 3 & \sin(x + y) + 1 \\ 0.5\sin(x + y) + 0.7 & y^2 + 5 \end{pmatrix}, \quad (4.68)
\]

\[
\beta_1^-(x, y) = \begin{pmatrix} x^2 + y^2 + 3 & \sin(xy) + 1 \\ \sin(x + y) + 1 & y^2 + 4 \end{pmatrix}, \quad (4.69)
\]
Figure 4.7 The solution \( u_2 \) with a "Star" interface

![Graph of the solution u2]

Table 4.3 Star shape interface

<table>
<thead>
<tr>
<th>( n_x \times n_y )</th>
<th>Error in ( U )</th>
<th>Order</th>
</tr>
</thead>
<tbody>
<tr>
<td>24 ( \times ) 24</td>
<td>0.00533</td>
<td></td>
</tr>
<tr>
<td>48 ( \times ) 48</td>
<td>0.00159</td>
<td>1.75</td>
</tr>
<tr>
<td>96 ( \times ) 96</td>
<td>4.22e-004</td>
<td>1.91</td>
</tr>
<tr>
<td>192 ( \times ) 192</td>
<td>1.10e-004</td>
<td>1.94</td>
</tr>
<tr>
<td>384 ( \times ) 384</td>
<td>2.90e-005</td>
<td>1.93</td>
</tr>
</tbody>
</table>

\[
\beta_2^+(x,y) = \begin{pmatrix}
\cos(x)^2 + 0.1 & (x + y)^2 + 2 \\
2x^2 & 0.6\cos(x) + 1
\end{pmatrix}, \quad (4.70)
\]

\[
\beta_2^-(x,y) = \begin{pmatrix}
\cos(y) + 1 & (x + y)^2 + 1 \\
2x^2 + 1 & 0.5\cos(x)^2
\end{pmatrix}, \quad (4.71)
\]
\[ \beta_3^+(x, y) = \begin{pmatrix} \cos(x + y)^2 & 3x^2y^2 \\ x^2 + 1 & \cos(y) + 1 \end{pmatrix}, \quad (4.72) \]

\[ \beta_3^-(x, y) = \begin{pmatrix} 2\cos(x + y)^2 & 3x^2y^2 + 1 \\ 2x^2 & 2\cos(xy) + 2 \end{pmatrix}, \quad (4.73) \]

\[ \beta_4^+(x, y) = \begin{pmatrix} x^2y^2 + 5 & (\sin(x + 2y))^2 \\ \sin(x + 2y) + 1 & y^2 + x^2 + 3 \end{pmatrix}, \quad (4.74) \]

\[ \beta_4^-(x, y) = \begin{pmatrix} 5x^2y^2 + 4 & \sin(x) + 1 \\ \sin(x + y) + 1 & y^2 + x^2 + 4 \end{pmatrix}, \quad (4.75) \]

\[ u_1^+(x, y) = (x^2 + y^2)^{5/6}, \quad (4.76) \]

\[ u_1^-(x, y) = 1, \quad (4.77) \]

\[ u_2^+(x, y) = x, \quad (4.78) \]

\[ u_2^-(x, y) = 0 \quad (4.79) \]

The computed solutions with the current method using a 48 × 48 grid are shown in Figures 4.8 and 4.9. Table 4.4 shows the error on different grids.

**Example 11** This example has the special type of coefficients that satisfies the hypothesis of Theorem 3.2. \( \phi(x, y), \beta_1^\pm(x, y), \beta_2^\pm(x, y), \beta_3^\pm(x, y), \beta_4^\pm(x, y) \) and \( u_1^\pm(x, y), u_2^\pm(x, y) \) are

\[ \phi(x, y) = r^2 + y^2 - 0.16, \quad (4.80) \]

\[ \beta_1^+(x, y) = \begin{pmatrix} 8 & 0 \\ 0 & 4 \end{pmatrix}, \quad (4.81) \]
Figure 4.8 The solution $u_1$ with a singular point on the interface

Figure 4.9 The solution $u_2$ with a singular point on the interface

$$
\beta_1^-(x,y) = \begin{pmatrix} 7 & 0 \\ 0 & 2 \end{pmatrix}, \quad (4.82)
$$

$$
\beta_2^+(x,y) = \begin{pmatrix} 0 & 2 \\ 4 & 0 \end{pmatrix}, \quad (4.83)
$$
Table 4.4 Singular point on the interface

<table>
<thead>
<tr>
<th>$n_x \times n_y$</th>
<th>Error in $U$</th>
<th>Order</th>
</tr>
</thead>
<tbody>
<tr>
<td>$24 \times 24$</td>
<td>0.00347</td>
<td></td>
</tr>
<tr>
<td>$48 \times 48$</td>
<td>0.00118</td>
<td>1.55</td>
</tr>
<tr>
<td>$96 \times 96$</td>
<td>4.05e-004</td>
<td>1.55</td>
</tr>
<tr>
<td>$192 \times 192$</td>
<td>1.39e-004</td>
<td>1.54</td>
</tr>
<tr>
<td>$384 \times 384$</td>
<td>4.78e-005</td>
<td>1.54</td>
</tr>
</tbody>
</table>

\[
\beta_2^-(x, y) = \begin{pmatrix} 0 \\ 3 \\ 2 \\ 0 \end{pmatrix}, \quad (4.84)
\]
\[
\beta_3^+(x, y) = \begin{pmatrix} 0 \\ 4 \\ 2 \\ 0 \end{pmatrix}, \quad (4.85)
\]
\[
\beta_3^-(x, y) = \begin{pmatrix} 0 \\ 2 \\ 3 \\ 0 \end{pmatrix}, \quad (4.86)
\]
\[
\beta_4^+(x, y) = \begin{pmatrix} 4 \\ 0 \\ 0 \\ 8 \end{pmatrix}, \quad (4.87)
\]
\[
\beta_4^-(x, y) = \begin{pmatrix} 2 \\ 0 \\ 0 \\ 7 \end{pmatrix}, \quad (4.88)
\]
\[
u_1^+(x, y) = \sin(x) \cos(y), \quad (4.89)
\]
\[
u_1^-(x, y) = x \sin(y), \quad (4.90)
\]
\[
u_2^+(x, y) = \cos(x) + y^2, \quad (4.91)
\]
\[ u_2(x, y) = xy \] (4.92)

The computed solutions with the current method using a $48 \times 48$ grid are shown in Figures 4.10 and 4.11. Table 4.5 shows the error on different grids.

Figure 4.10 The solution $u_1$ with coefficients of special form

Figure 4.11 The solution $u_2$ with coefficients of special form
Table 4.5 Special form of coefficients

<table>
<thead>
<tr>
<th>$n_x \times n_y$</th>
<th>Error in $U$</th>
<th>Order</th>
</tr>
</thead>
<tbody>
<tr>
<td>24 x 24</td>
<td>0.00151</td>
<td></td>
</tr>
<tr>
<td>48 x 48</td>
<td>4.44e-004</td>
<td>1.77</td>
</tr>
<tr>
<td>96 x 96</td>
<td>1.20e-004</td>
<td>1.89</td>
</tr>
<tr>
<td>192 x 192</td>
<td>3.30e-005</td>
<td>1.86</td>
</tr>
<tr>
<td>384 x 384</td>
<td>8.66e-006</td>
<td>1.93</td>
</tr>
</tbody>
</table>
CHAPTER 5

2-D ELLIPTIC PROBLEM WITH THREE DOMAINS

Based on the method in Chapter 3, this chapter proposes a numerical method for solving the elliptic problem with three domains. An accurate treatment for the triple junction point shown in Figure 5.2 is proposed. It has been proved that the resulting linear system is non-symmetric but positive definite if $A_i, i = 1, 2, 3$ are positive definite for the three domains. Numerical results demonstrate near second-order accuracy for the method for piecewise smooth solutions.

5.1 Equations and Weak Formulations

Let $\Omega \subset R^d$ be an open bounded domain, and let $\Gamma$ be an interface. $\Gamma$ divides $\Omega$ into $\Omega_1$, $\Omega_2$ and $\Omega_3$, hence $\Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3 \cup \Gamma$, see Figure 5.1. Assuming that $\partial \Omega$ and $\partial \Omega_{1,2,3}$ are Lipschitz continuous as submanifolds, so is $\Gamma$. A unit normal vector of $\Gamma$ can be defined almost everywhere on $\Gamma$ (see Section 1.5 in [13]).

The variable coefficient elliptic interface problem is given by

$$-\nabla \cdot (\beta(x) \nabla u(x)) = f(x), \quad x \in \Omega \setminus \Gamma, \quad (5.1)$$

where $x = (x_1, \ldots, x_d)$ is the spatial variable, $\beta(x)$ is a $d \times d$ matrix that is uniformly elliptic on each disjoint subdomain, $\Omega_1$, $\Omega_2$ and $\Omega_3$, $f(x)$ is in $L^2(\Omega)$.

Consider the problem on the rectangular domain $\Omega = (x_{\min}, x_{\max}) \times (y_{\min}, y_{\max}) = \Omega_1 \cup \Omega_2 \cup \Omega_3 \quad \Gamma_j, \quad j = 1, 2, 3$
The jump conditions are prescribed as

\[
\begin{cases}
[u]_{\Gamma_1} = u_2 - u_3 = a_1, \text{ on } \Gamma_1, \\
[u]_{\Gamma_2} = u_3 - u_1 = a_2, \text{ on } \Gamma_2, \\
[u]_{\Gamma_3} = u_1 - u_2 = a_3, \text{ on } \Gamma_3, \\
[\beta \nabla u]_{\Gamma_1} = (\beta_2 \nabla u_2 - \beta_3 \nabla u_3) \cdot n_1 = b_1, \text{ on } \Gamma_1, \\
[\beta \nabla u]_{\Gamma_2} = (\beta_3 \nabla u_3 - \beta_1 \nabla u_1) \cdot n_2 = b_2, \text{ on } \Gamma_2, \\
[\beta \nabla u]_{\Gamma_3} = (\beta_1 \nabla u_1 - \beta_2 \nabla u_2) \cdot n_3 = b_3, \text{ on } \Gamma_3
\end{cases}
\]

\( a \) and \( b \) are given functions along the interfaces \( \Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \), the "1, 2, 3" subscripts denote limits taken within \( \Omega_{1,2,3} \).
The boundary conditions are prescribed as

\[
\begin{align*}
    u_1 &= g_1, \text{ on } \partial \Omega \cap \partial \Omega_1, \\
    u_2 &= g_2, \text{ on } \partial \Omega \cap \partial \Omega_2, \\
    u_3 &= g_3, \text{ on } \partial \Omega \cap \partial \Omega_3
\end{align*}
\] (5.4)

The interfaces are prescribed by level-set functions \( \phi_j(x, y) \)

\[
\begin{align*}
    \phi_1(x, y) &= \begin{cases} 
        < 0, & (x, y) \in \Omega_3, \\
        = 0, & (x, y) \in \Gamma_1, \\
        > 0, & (x, y) \in \Omega_2
    \end{cases} \\
    \phi_2(x, y) &= \begin{cases} 
        < 0, & (x, y) \in \Omega_1, \\
        = 0, & (x, y) \in \Gamma_2, \\
        > 0, & (x, y) \in \Omega_3
    \end{cases} \\
    \phi_3(x, y) &= \begin{cases} 
        < 0, & (x, y) \in \Omega_2, \\
        = 0, & (x, y) \in \Gamma_3, \\
        > 0, & (x, y) \in \Omega_1
    \end{cases}
\end{align*}
\] (5.5) (5.6) (5.7)

The unit normal vector of \( \Gamma_j \) is \( n_j = \frac{\nabla \phi_j}{|\nabla \phi_j|} \) pointing from \( \Omega_j^- = \{(x, y) \in \Omega | \phi_j(x, y) \leq 0\} \) to \( \Omega_j^+ = \{(x, y) \in \Omega | \phi_j(x, y) \geq 0\} \) for \( j = 1, 2, 3 \)

The weak formulation is generalized in [15, 16] for the elliptic equation with matrix coefficients. The usual Sobolev space \( H^1(\Omega) \) is used. For \( H^1_0(\Omega) \), an inner product is chosen as

\[
B[u, v] = \int_{\Omega_1} \beta \nabla u \cdot \nabla v + \int_{\Omega_2} \beta \nabla u \cdot \nabla v + \int_{\Omega_3} \beta \nabla u \cdot \nabla v
\] (5.8)
Definition 5.1.1: $u \in H(a, c)$ is called a weak solution of equations 5.1-5.4, if it satisfies, for all $\psi \in H^1_0(\Omega)$,

$$
\int_{\Omega} \beta \nabla u \cdot \nabla \psi + \int_{\Omega_2} \frac{\beta}{\nu} u \cdot \nabla \psi + \int_{\Omega_3} \frac{\beta}{\nu} u \cdot \nabla \psi = \int_{\Omega} f\psi + \int_{\Gamma} b\psi \quad (5.9)
$$

Theorem 5.1.2: If $f \in L^2(\Omega)$, and $a, b \in H^1(\Omega)$, then there exists a unique weak solution of Equations 5.2-5.4.

Proof: See Theorem 2.1 in [15]. \qed

5.2 Numerical Method

A cell $K$ with corners $k_1, k_2, k_3$ belongs to one of three different sets

$\Lambda_1 = \{ \Delta_k \subset \Omega \mid k_1, k_2, k_3 \text{ are in the same domain among } \Omega_j, \ j = 1, 2, 3 \}$,

$\Lambda_2 = \{ \Delta_k \subset \Omega \mid k_1, k_2, k_3 \text{ are in two different domains among } \Omega_j, \ j = 1, 2, 3 \}$,

$\Lambda_3 = \{ \Delta_k \subset \Omega \mid k_1, k_2, k_3 \text{ are in three different domains among } \Omega_j, \ j = 1, 2, 3 \}$

If $K \in \Lambda_1$ or $K \in \Lambda_2$, it has the same definition as in Section 3.2, Chapter 3. If $K \in \Lambda_3$, Figure 5.2 shows the interfaces inside $K$.

Theorem 5.2.1: For all $u^h \in H^{1,h}, U^{h}(u^h)$ can be constructed uniquely, provided $T^h, \phi, a$ and $b$ are given.

Proof: See Theorem 3.2.1 in Chapter 3. \qed

Lemma 5.2.2: The coefficient matrix $A$ generated by the method above is independent of $a_j(x, y)$ and $b_j(x, y)$, $j = 1, 2, 3$.

Proof: See Lemma 3.2.3 in Chapter 3. \qed

Theorem 5.2.3: The coefficient matrix $A = (a_{ij})_{n \times n}$ generated by the method above is positive definite if $\beta_j$, $j = 1, 2, 3$ are positive definite.
Proof For any vector $c \in \mathbb{R}^n$, $c^T Ac > 0$ since
\[
c^T Ac = \sum_{i,j=1}^{n} a_{ij} c_i c_j = B \left[ \sum_{i=1}^{n} c_i u^i, \sum_{i=1}^{n} c_i \psi^i \right],
\]
where $u^i$ are basis functions for the solution and $\psi^i$ are the test functions. For the $i$-th grid point, $u^i$ and $\psi^i$ both have non-zero support only on the six triangles which have a vertex on the $i$-th grid point. $u^i$ can be decomposed into $u^i = \sum_{j=1}^{6} u_j^i$, where each $u_j^i$ has non-zero support only on the $j$-th triangle around the $i$-th grid point.

Let $m$ be the number of triangles on the whole domain $\Omega = \bigcup_{k=1}^{m} \Delta_k$. The summation of $u^i$ over all the triangles can be rewritten
\[
\sum_{i=1}^{n} c_i u^i = \sum_{i=1}^{n} \sum_{j=1}^{6} c_i u_j^i = \sum_{k=1}^{m} U_k,
\]
where $U_k$ is defined on $\Delta_k = \Delta_{k_1 k_2 k_3}$, and $U_k = c_{k_1} u_{k_1} + c_{k_2} u_{k_2} + c_{k_3} u_{k_3}$, $k_1, k_2, k_3$ are the three vertices of $\Delta_k$.

Similarly, the summation of $\psi^i$ over all the triangles can be rewritten
\[
\sum_{i=1}^{n} c_i \psi^i = \sum_{i=1}^{n} \sum_{j=1}^{6} c_i \psi_j^i = \sum_{k=1}^{m} \Psi_k.
\]
with

\[ \Psi_k = c_{k1}\psi_{k1} + c_{k2}\psi_{k2} + c_{k3}\psi_{k3} \]  \hspace{1cm} (5.13)

Consider the sets

\[ \Lambda_1 = \{\triangle_k \subset \Omega \mid k_1, k_2, k_3 \text{ are in the same domain among } \Omega_j, \ j = 1, 2, 3\}, \]
\[ \Lambda_2 = \{\triangle_k \subset \Omega \mid k_1, k_2, k_3 \text{ are in two different domains among } \Omega_j, \ j = 1, 2, 3\}, \]
\[ \Lambda_3 = \{\triangle_k \subset \Omega \mid k_1, k_2, k_3 \text{ are in three different domains among } \Omega_j, \ j = 1, 2, 3\} \]

Then

\[ \sum_{k=1}^{m} U_k = \sum_{\triangle_k \in \Lambda_1} U_k + \sum_{\triangle_k \in \Lambda_2} U_k + \sum_{\triangle_k \in \Lambda_3} U_k, \hspace{1cm} (5.14) \]

\[ \sum_{k=1}^{m} \Psi_k = \sum_{\triangle_k \in \Lambda_1} \Psi_k + \sum_{\triangle_k \in \Lambda_2} \Psi_k + \sum_{\triangle_k \in \Lambda_3} \Psi_k \hspace{1cm} (5.15) \]

The difference between \( U_k \) and \( \Psi_k \) is, \( U_k \) satisfies the jump conditions on the interface and \( \Psi_k \) is a simple linear function on \( \triangle_k \). So when \( \triangle_k \in \Lambda_1 \), there is no jump in \( \triangle_k \). Thus

\[ U_k(x,y) = \Psi_k(x,y), \ (x,y) \in \triangle_k, \ \triangle_k \in \Lambda_1 \]

When \( \triangle_k \in \Lambda_2 \), the proof of Theorem 3.2.4 in Chapter 3 shows that by adjusting the jump conditions \( a_j(x,y) \) and \( b_j(x,y) \), it can be obtained that

\[ U_k(x,y) = \Psi_k(x,y), \ (x,y) \in \triangle_k, \ \triangle_k \in \Lambda_2 \]

Now let \( \triangle_k \in \Lambda_3 \) It has already been shown that \( U_k(k_j) = \Psi_k(k_j), \ j = 1, 2, 3 \) and it needs to be shown that

\[ U_k(x,y) = \Psi_k(x,y), \ \forall(x,y) \in \triangle_k \]
By the method used for computation, it is assumed that three interfaces $\Gamma_1$, $\Gamma_2$, and $\Gamma_3$ intersect at the point $p_0$ inside $\Delta_k$, and each $\Gamma_j$ intersects with one side of $\Delta_k$ at the point $p_j$ for $j = 1, 2, 3$, (see Figure 5.3)

![Figure 5.3 Interface triangle $\Delta_k$ belongs to \Lambda_3

Without loss of generality, it is assumed that $k_1 \in \Omega_1$, $k_2 \in \Omega_2$, and $k_3 \in \Omega_3$

First let

$$a_1(p_1) = 0, \quad a_2(p_2) = 0, \quad a_3(p_3) = 0,$$

and

$$a_1(p_0) = a_2(p_0) = a_3(p_0) = 0$$

Then $U_k(x, y)$ is piecewise linear on each sub-triangles $\Delta_{k_1p_0p_2}$, $\Delta_{k_1p_3p_0}$, $\Delta_{k_2p_0p_3}$, $\Delta_{k_2p_1p_0}$, $\Delta_{k_3p_0p_1}$, $\Delta_{k_3p_2p_0}$, and it can be determined by values at $p_0$, $p_1$, $p_2$, $p_3$ since $U_k(k_1)$, $U_k(k_2)$, $U_k(k_3)$ are given and fixed.

First fix $U_k(p_0)$ and consider $\Delta_{k_2p_1p_0}$ and $\Delta_{k_3p_0p_1}$. It can be easily confirmed that when ranging $U_k(p_3)$ from $-\infty$ to $\infty$, $b_1(p_{01})$ also ranges from $-\infty$ to $\infty$, and vice versa. Monotonicity implies $U_k(p_1)$ is uniquely determined by $b_1(p_{01})$. Similarly, $U_k(p_2)$ and $U_k(p_3)$ are uniquely determined by $b_2(p_{02})$ and $b_3(p_{03})$, respectively.
Therefore, after applying jump conditions $a_j$ and $b_j$ for $j = 1, 2, 3$, the $U_k(x, y)$ is uniquely determined inside $\Delta_k$ corresponding to the value of $U_k(p_0)$.

Then it is shown that $U_k(p_0)$ is unique after applying the conditions that $U_k(p_0)$, $U_k(p_1)$, $U_k(p_2)$ and $U_k(k_3)$ are in the same plane. Suppose $U_k(x, y)$ and $V_k(x, y)$ are two piecewise linear functions which satisfy the same jump conditions $a_j$ and $b_j$ and value at $p_0$, $p_1$, $p_2$, and $k_3$ are in the same plane.

If

$$U_k(p_0) = V_k(p_0),$$

then

$$U_k(x, y) = V_k(x, y), \ \forall (x, y) \in \Delta_k$$

If

$$U_k(p_0) \neq V_k(p_0),$$

and it is assumed

$$U_k(p_0) < V_k(p_0),$$

and since $U_k$ and $V_k$ both satisfy jump condition $b_1$ at $p_{01}$, it can be obtained that

$$U_k(p_1) > V_k(p_1)$$

Similarly, the result is

$$U_k(p_2) > V_k(p_2),$$

by applying jump condition $b_2$ at $p_{02}$

$U_k(p_0)$ and $V_k(p_0)$ can be also gotten by

$$\{U_k(p_1), U_k(p_1), U_k(k_3)\},$$
and
\[ \{V_k(p_1), V_k(p_2), V_k(k_3)\}, \]
respectively, since \( U_k \) and \( V_k \) are both linear functions on points \( p_0, p_1, p_2, \) and \( k_3 \)

Since \( U_k(k_3) = V_k(k_3), U_k(p_1) > V_k(p_1), \) and \( U_k(p_2) > V_k(p_2), \) it can be concluded
that \( U_k(p_0) > V_k(p_0) \) which contradicts the assumption that \( U_k(p_0) < V_k(p_0) \)

Therefore \( U_k \) is unique under these nine jump condition values \( a_1(p_0), a_1(p_1), \)
\( a_2(p_0), a_2(p_2), a_3(p_0), a_3(p_3), b_1(p_{01}), b_2(p_{02}), \) and \( b_3(p_{03}) \) If those jump condition
values are chosen under the function \( \Psi_k, \) then \( U_k = \Psi_k \) in \( \Delta_k \)

Therefore
\[
\sum_{\Delta_k \in \Lambda_3} U_k = \sum_{\Delta_k \in \Lambda_3} \Psi_k, 
\]
and the results are combined in \( \Lambda_j, \quad j = 1, 2, 3 \) to get
\[
\sum_{i=1}^{n} c_i u^i = \sum_{i=1}^{n} c_i \psi^i 
\]

It now follows from the positive definiteness of \( \beta \) that
\[
c^T A c = B \left[ \sum_{i=1}^{n} c_i u^i, \sum_{i=1}^{n} c_i \psi^i \right] > 0 
\]

Therefore, \( A \) is positive definite \( \square \)

From Remark 2 in Chapter 3, it is known that a positive definite matrix has
positive determinant, and is therefore invertible The linear system \( A x = b \) can be
solved efficiently

5.3 Numerical Experiments

In all examples of this section, the \( \phi_j, \beta_j \) and \( u_j \) are given for \( j = 1, 2, 3 \) Hence \( f_j, \)
\( a_j, b_j \) can be calculated on \( \Omega \) \( g_j \) is obtained from the solutions as a proper Dirichlet
boundary condition. All errors in solutions are measured in the $L^\infty$ norm in the whole domain $\Omega$.

Four numerical examples are presented in this chapter to demonstrate the effectiveness of this method.

**Example 12** This example has smooth interfaces which are two circles with the same center $\phi_j(x, y), \beta_j(x, y)$ and $u_j(x, y)$ for $j = 1, 2, 3$, are given as

\begin{align*}
\phi_1(x, y) &= x^2 + y^2 - 0.25^2, \quad (5 \ 16) \\
\phi_2(x, y) &= -(x^2 + y^2 - 0.5^2), \quad (5 \ 17) \\
\phi_3(x, y) &= x^2 + y^2 - 0.8^2, \quad (5 \ 18) \\
\beta_1^+(x, y) &= \begin{pmatrix} x^2 + y^2 + 1 & x^2 + y^2 + 2 \\ x^2 + y^2 + 2 & x^2 + y^2 + 5 \end{pmatrix}, \quad (5 \ 19) \\
\beta_2^+(x, y) &= \begin{pmatrix} x^2 - y^2 + 3 & x^2 - y^2 + 1 \\ r^2 - y^2 + 1 & r^2 - y^2 + 4 \end{pmatrix}, \quad (5 \ 20) \\
\beta_3^+(x, y) &= \begin{pmatrix} ry + 2 & ry + 1 \\ xy + 1 & xy + 3 \end{pmatrix}, \quad (5 \ 21) \\
u_1(x, y) &= x^2 + y^3 - 1, \quad (5 \ 22) \\
u_2(x, y) &= \cos(\pi x) + \cos(\pi y) + 2, \quad (5 \ 23) \\
u_3(x, y) &= 10x^2 + \sin(x + y) + 5 \quad (5 \ 24)
\end{align*}

The computed solution with the current method using a $40 \times 40$ grid is shown in Figure 5.4. Table 5.1 shows the error on different grids. The numerical result shows close to second-order accuracy in the $L^\infty$ norm for the solution.
Example 13 This example has two triple junction points $\phi_j(x,y)$, $\beta_j(x,y)$ and $u_j(x,y)$ for $j = 1, 2, 3$, are given as

$$
\phi_1(x,y) = -((x + 0.17)^2 + y^2 - 0.317^2), \\
\phi_2(x,y) = (x - 0.153)^2 + y^2 - 0.41^2, \\
\phi_3(x,y) = (x + 0.17)^2 + y^2 - 0.317^2,
$$

Figure 5.4 Interface with the shape of two circles

Table 5.1 Interface with the shape of two circles

<table>
<thead>
<tr>
<th>$n_x \times n_y$</th>
<th>Error in $u$</th>
<th>Order</th>
</tr>
</thead>
<tbody>
<tr>
<td>$20 \times 20$</td>
<td>9.7176e-003</td>
<td></td>
</tr>
<tr>
<td>$40 \times 40$</td>
<td>2.7138e-003</td>
<td>1.84</td>
</tr>
<tr>
<td>$80 \times 80$</td>
<td>9.2766e-004</td>
<td>1.55</td>
</tr>
<tr>
<td>$160 \times 160$</td>
<td>2.3779e-004</td>
<td>1.96</td>
</tr>
</tbody>
</table>
\[ \beta_1^+(x,y) = \begin{pmatrix} x^2 + y^2 + 1 & x^2 + y^2 + 2 \\ x^2 + y^2 + 2 & x^2 + y^2 + 5 \end{pmatrix}, \quad (5.28) \]

\[ \beta_2^+(x,y) = \begin{pmatrix} x^4 + y^4 + 1 & x^4 + y^4 + 2 \\ x^4 + y^4 + 2 & x^4 + y^4 + 5 \end{pmatrix}, \quad (5.29) \]

\[ \beta_3^+(x,y) = \begin{pmatrix} x^2 + y^4 + 1 & x^2 + y^4 + 2 \\ x^2 + y^4 + 2 & x^2 + y^4 + 5 \end{pmatrix}, \quad (5.30) \]

\[ u_1(x,y) = x + e^y + 1, \quad (5.31) \]

\[ u_2(x,y) = \sin(2\pi x)\sin(2\pi y) + 6, \quad (5.32) \]

\[ u_3(x,y) = x^2 + y^3 + \sin(x + y) \quad (5.33) \]

The computed solution with the current method using a 40 \times 40 grid is shown in Figure 5.5. Table 5.2 shows the error on different grids. The numerical result shows close to second-order accuracy in the \( L^\infty \) norm for the solution.

Figure 5.5 Interface with the shape of an eclipse
Example 14 This example is two circles touching each other \( \phi_j(x, y), \beta_j(x, y) \) and \( u_j(x, y) \) for \( j = 1, 2, 3 \), are given as

Table 5.2 Interface with the shape of an eclipse

<table>
<thead>
<tr>
<th>( n_x \times n_y )</th>
<th>Error in ( U )</th>
<th>Order</th>
</tr>
</thead>
<tbody>
<tr>
<td>20 ( \times ) 20</td>
<td>1 5022e-001</td>
<td></td>
</tr>
<tr>
<td>40 ( \times ) 40</td>
<td>5 4492e-002</td>
<td>1.46</td>
</tr>
<tr>
<td>80 ( \times ) 80</td>
<td>1 6279e-002</td>
<td>1.74</td>
</tr>
<tr>
<td>160 ( \times ) 160</td>
<td>4 3505e-003</td>
<td>1.90</td>
</tr>
</tbody>
</table>

\[
\phi_1(x, y) = -(x + 0.35)^2 + y^2 - 0.35^2, \quad (5.34)
\]

\[
\phi_2(x, y) = (x - 0.35)^2 + y^2 - 0.35^2, \quad (5.35)
\]

\[
\phi_3(r, y) = r, \quad (5.36)
\]

\[
\beta_1^+(x, y) = \begin{pmatrix} x^2 + y^2 + 1 & x^2 + y^2 + 2 \\ x^2 + y^2 + 2 & x^2 + y^2 + 5 \end{pmatrix}, \quad (5.37)
\]

\[
\beta_2^+(x, y) = \begin{pmatrix} x^4 + y^4 + 1 & x^4 + y^4 + 2 \\ x^4 + y^4 + 2 & x^4 + y^4 + 5 \end{pmatrix}, \quad (5.38)
\]

\[
\beta_3^+(x, y) = \begin{pmatrix} x^2 + y^4 + 1 & x^2 + y^4 + 2 \\ x^2 + y^4 + 2 & x^2 + y^4 + 5 \end{pmatrix}, \quad (5.39)
\]

\[
u_1(x, y) = 5x + 6y + 1, \quad (5.40)
\]

\[
u_2(x, y) = -5x + 6y + 1, \quad (5.41)
\]

\[
u_3(x, y) = 2y^2 + \sin(2\pi x) - 2 \quad (5.42)
\]
The computed solution with the current method using a $40 \times 40$ grid is shown in Figure 5.6. Table 5.3 shows the error on different grids. The numerical result shows close to second-order accuracy in the $L^\infty$ norm for the solution.

![Figure 5.6 Two circles touching](image)

**Table 5.3 Two circles touching**

<table>
<thead>
<tr>
<th>$n_x \times n_y$</th>
<th>Error in $U$</th>
<th>Order</th>
</tr>
</thead>
<tbody>
<tr>
<td>$20 \times 20$</td>
<td>3.0337e-002</td>
<td></td>
</tr>
<tr>
<td>$40 \times 40$</td>
<td>9.5274e-003</td>
<td>1.67</td>
</tr>
<tr>
<td>$80 \times 80$</td>
<td>2.6414e-003</td>
<td>1.85</td>
</tr>
<tr>
<td>$160 \times 160$</td>
<td>7.7858e-004</td>
<td>1.76</td>
</tr>
</tbody>
</table>

**Example 15** This example is a circle circumscribed on a star $\phi_j(x,y)$, $\beta_j(x,y)$ and $u_j(x,y)$ for $j = 1, 2, 3$, are given as

$$
\phi_1(r, \theta) = -\frac{R \sin(\theta_i/2)}{\sin(\theta_i/2 + \theta - \theta_r - 2\pi(i - 1)/5)} - r
$$
\[ \theta_r + \pi(2\ell - 2)/5 \leq \theta < \theta_r + \pi(2\ell - 1)/5, \quad (5.43) \]

\[ \phi_1(r, \theta) = -\left( \frac{R \sin(\theta_2/2)}{\sin(\theta_r/2 - \theta + \theta_r - 2\pi(\ell - 1)/5)} - r \right) \]

\[ \theta_r + \pi(2\ell - 3)/5 \leq \theta < \theta_r + \pi(2\ell - 2)/5, \quad (5.44) \]

with \( \theta_2 = \pi/5, \, \theta_r = \pi/7, \, R = 6/7 \) and \( \ell = 1, 2, 3, 4, 5, \)

\[ \phi_2(x, y) = x^2 + y^2 - (6/7)^2, \quad (5.45) \]

\[ \phi_3(x, y) = -(x^2 + y^2 - (6/7)^2), \quad (5.46) \]

\[ \begin{pmatrix} x^2 + y^2 + 1 & x^2 + y^2 + 2 \\ x^2 + y^2 + 2 & x^2 + y^2 + 5 \end{pmatrix}, \quad (5.47) \]

\[ \begin{pmatrix} x^2 - y^2 + 3 & x^2 - y^2 + 1 \\ x^2 - y^2 + 1 & x^2 - y^2 + 4 \end{pmatrix}, \quad (5.48) \]

\[ \begin{pmatrix} xy + 2 & xy + 1 \\ xy + 1 & xy + 3 \end{pmatrix}, \quad (5.49) \]

\[ u_1(x, y) = 2y + 1 + 0.1 \sin(2\pi(x^2 + y)), \quad (5.50) \]

\[ u_2(x, y) = 0, \quad (5.51) \]

\[ u_3(x, y) = y^3 + e^x + 1 \quad (5.52) \]

The computed solution with the current method using a 40 \times 40 grid is shown in Figure 5.7 Table 5.4 shows the error on different grids. The numerical result shows close to second-order accuracy in the \( L^\infty \) norm for the solution.
Figure 5.7  Interface with the shape of a star in a circle

Table 5.4  Interface with the shape of a star in a circle

<table>
<thead>
<tr>
<th>$n_x \times n_y$</th>
<th>Error in $U$</th>
<th>Order</th>
</tr>
</thead>
<tbody>
<tr>
<td>20 $\times$ 20</td>
<td>4.5391e-002</td>
<td></td>
</tr>
<tr>
<td>40 $\times$ 40</td>
<td>1.7135e-002</td>
<td>1.41</td>
</tr>
<tr>
<td>80 $\times$ 80</td>
<td>5.2382e-003</td>
<td>1.71</td>
</tr>
<tr>
<td>160 $\times$ 160</td>
<td>1.3995e-003</td>
<td>1.90</td>
</tr>
</tbody>
</table>
CHAPTER 6

3-D ELLIPTIC PROBLEM WITH TWO DOMAINS

In this chapter, a three-dimensional model is developed to solve the elliptic interface problem with two domains. The resulting linear system in three dimensions is also proved to be positive definite but not symmetric. Four examples are given, numerical results show that the three-dimensional model is second-order accurate. In all the examples, the interfaces contain sharp corners, which means that this method also works for the sharp interface problem.

6.1 Equations and Weak Formulations

The variable coefficient elliptic interface problem is given by

\[- \nabla \cdot (\beta(x) \nabla u(x)) = f(x), \quad x \in \Omega \setminus \Gamma, \quad (6.1)\]

where \(x = (x_1, \ldots, x_d)\) is the spatial variables, \(\beta(x)\) is a \(d \times d\) matrix that is uniformly elliptic on each disjoint subdomain, \(\Omega^-\) and \(\Omega^+\), \(f(x)\) is in \(L^2(\Omega)\).

The jump conditions are prescribed as

\[
\begin{align*}
\{u\}_\Gamma(x) & \equiv u^+(x) - u^-(x) = a(x), \\
\{n(\beta \nabla u)\}_\Gamma(x) & \equiv n(\beta^+(x) \nabla u^+(x)) - n(\beta^-(x) \nabla u^-(x)) = b(x),
\end{align*}
\]

(6.2)

\(a\) and \(b\) are given functions along \(\Gamma\), "\pm" denote limits taken within \(\Omega^\pm\).
Function $g$ is given on $\partial \Omega$, the boundary condition is prescribed as

$$u(x) = g(x), \ x \in \partial \Omega$$  \hspace{1cm} (6.3)

The setup of the problem is illustrated in Figure 6.1

![Figure 6.1 Setup of the problem](image)

The weak formulation is generalized in [15, 16] for the elliptic equation with matrix coefficients. The usual Sobolev space $H^1(\Omega)$ is used. For $H^1_0(\Omega)$, an inner product is chosen as

$$B[u,v] = \int_{\Omega^+} \beta \nabla u \nabla v + \int_{\Omega^-} \beta \nabla u \nabla v$$  \hspace{1cm} (6.4)

**Definition 6.1.1** $u \in H(a,c)$ is called a weak solution of Equations 6.1-6.3, if $u$ satisfies, for all $\psi \in H^1_0(\Omega)$,

$$\int_{\Omega^+} \beta \nabla u \nabla \psi + \int_{\Omega^-} \beta \nabla u \nabla \psi = \int_{\Omega} f \psi + \int_{\Gamma} b \psi$$  \hspace{1cm} (6.5)
Theorem 6.1.2 If \( f \in L^2(\Omega) \), \( a, b \) and \( c \in H^1(\Omega) \), then there exists a unique weak solution of Equations 6.1-6.3 in \( H(a, c) \).

Proof See Theorem 2.1 in [15].

6.2 Numerical Method

For simplicity, the setup is restricted to a cube cell domain \( \Omega = (x_{\text{min}}, x_{\text{max}}) \times (y_{\text{min}}, y_{\text{max}}) \times (z_{\text{min}}, z_{\text{max}}) \) in three-dimensional space, and \( \beta \) is a \( 3 \times 3 \) matrix that is uniformly elliptic in each subdomain. Given positive integers \( I, J \) and \( K \), set 
\[
\Delta x = (x_{\text{max}} - x_{\text{min}}) / I, \quad \Delta y = (y_{\text{max}} - y_{\text{min}}) / J \quad \text{and} \quad \Delta z = (z_{\text{max}} - z_{\text{min}}) / K
\]
A uniform Cartesian grid is defined as 
\[
(x_i, y_j, z_k) = (x_{\text{min}} + i \Delta x, y_{\text{min}} + j \Delta y, z_{\text{min}} + k \Delta z)
\]
for \( i = 0, \ldots, I \), \( j = 0, \ldots, J \) and \( k = 0, \ldots, K \). Each \((x_i, y_j, z_k)\) is called a grid point. \( h = \max(\Delta x, \Delta y, \Delta z) > 0 \) is the grid size.

Two grid functions sets will be used

\[
H^{1,h} = \{ \omega^h = (\omega_{i,j,k}) \mid 0 \leq i \leq I, 0 \leq j \leq J, 0 \leq k \leq K \}
\]

and

\[
H_0^{1,h} = \{ \omega^h = (\omega_{i,j,k}) \in H^{1,h} \mid \omega_{i,j,k} = 0 \text{ if } i = 0, I \text{ or } j = 0, J \text{ or } k = 0, K \}
\]

Every cube cell region \([x_i, x_{i+1}] \times [y_j, y_{j+1}] \times [z_k, z_{k+1}]\) is cut into six tetrahedron regions. The tetrahedron regions are collected, and a uniform tetrahedralization \( T^h \cup_{L \in T^h} L \) is obtained. (See Figure 6.2 and Figure 6.3)

If \( \phi(x_i, y_j, z_k) \leq 0 \), the grid point \((x_i, y_j, z_k)\) is counted as in \( \Omega^- \), otherwise it is counted as in \( \Omega^+ \).
A cell $\Delta_L$ with corners $L_1, L_2, L_3, L_4$ belongs to one of two different sets

$$\Lambda_1 = \{\Delta_L \subset \Omega \mid L_1, L_2, L_3, L_4 \text{ are in the same domain among } \Omega^\pm\},$$

$$\Lambda_2 = \{\Delta_L \subset \Omega \mid L_1, L_2, L_3, L_4 \text{ are in two different domains among } \Omega^\pm\}.$$
If a cell belongs to $\Lambda_1$, it is a regular cell, otherwise it is an interface cell, written as $L = L^+ \cup L^-$ $L^+$ and $L^-$ are separated by a plane segment, denoted by $\Gamma^h_L$. There are two kinds of plane segments, see Figure 6.4 and Figure 6.5.

**Figure 6.4** Case 1 The interface segment is a triangle

**Figure 6.5** Case 2 The interface segment is a polygon
Since the solution bases and test function bases are different, the matrix $A$ for the linear system generated by the current method is not symmetric in the presence of an interface. However, it can be proved that it is positive definite.

**Theorem 6.2.1** If $\beta$ is positive definite, then the $n \times n$ matrix $A$ for the linear system generated by the current method is positive definite.

*Proof* For any vector $c \in \mathbb{R}^n$,

$$c^T A c = \sum_{i,j=1}^{n} a_{ij} c_i c_j = B \left[ \sum_{i=1}^{n} c_i u^i, \sum_{i=1}^{n} c_i \psi^i \right],$$

where $u^i$ and $\psi^i$ are basis functions for the solution and the test function, respectively. Note that they have compact support and have nonzero values inside the 24 tetrahedra around the $i$th grid point. For ease of discussion, each of $u^i$ and $\psi^i$ is decomposed into 24 parts, so that each part has nonzero values only inside one tetrahedra. Now the summation over $i$ is equivalent to a summation over all the tetrahedra, and there are four terms, $c_1 u_1 + c_2 u_2 + c_3 u_3 + c_4 u_4$, $c_1 \psi_1 + c_2 \psi_2 + c_3 \psi_3 + c_4 \psi_4$ for each tetrahedron, where $u_1, u_2, u_3, u_4, \psi_1, \psi_2, \psi_3, \psi_4$ equals 1 on one vertex of a tetrahedron and zero on three other vertices. The difference between $u_i$ and $\psi_i$ is, $u_i$ depends on the location of the interface and $\psi_i$ does not. $c_1 u_1 + c_2 u_2 + c_3 u_3 + c_4 u_4$ is a piecewise linear function satisfying the jump conditions and $c_1 \psi_1 + c_2 \psi_2 + c_3 \psi_3 + c_4 \psi_4$ is a linear function. At the four vertices, the two functions coincide. Now the jump conditions can be set as $a = 0$ and $b$ can be set to have the value in the tetrahedron such that $c_1 u_1 + c_2 u_2 + c_3 u_3 + c_4 u_4 = c_1 \psi_1 + c_2 \psi_2 + c_3 \psi_3 + c_4 \psi_4$ everywhere. In other words, the jump in $\beta$ is compensated by using $b$ to make sure the gradients on both sides of the interface coincide. Since Lemma 3.2.2 and Lemma 3.2.3 in Chapter 3 imply the
matrix \( A \) is independent of \( a, b \), choosing the above \( a, b \) would not change the matrix \( A \) and would only change the constant term, i.e., the right hand side of the linear system. When the tetrahedra are summed all over, the result is

\[
\sum_{i=1}^{n} c_i u^i = \sum_{i=1}^{n} c_i \psi^i
\]

It now follows from the positive definiteness of \( \beta \) that

\[
c^T A c = B \left[ \sum_{i=1}^{n} c_i u^i, \sum_{i=1}^{n} c_i u^i \right] > 0
\]

Therefore \( A \) is positive definite.

From Remark 2 in Chapter 3, it is known that a positive definite matrix has a positive determinant, and is therefore invertible. The linear system \( Ax = b \) can be solved efficiently.

\section*{6.3 Numerical Experiments}

Consider the problem

\[
-\nabla \, (\beta \nabla u) + p \, \nabla u + qu = f, \text{ in } \Omega^\pm, \tag{6.6}
\]

\[
[u] = a, \text{ on } \Gamma, \tag{6.7}
\]

\[
[(\beta \nabla u) \, n] = b, \text{ on } \Gamma, \tag{6.8}
\]

\[
u = g, \text{ on } \partial \Omega, \tag{6.9}
\]

on the domain \( \Omega = (x_{\min}, x_{\max}) \times (y_{\min}, y_{\max}) \times (z_{\min}, z_{\max}) \). \( \Gamma \) is an interface prescribed by the level-set function \( \phi(x, y, z) \). \( n = \frac{\nabla \phi}{|\nabla \phi|} \) is the unit normal vector of \( \Gamma \) pointing from \( \Omega^- \) to \( \Omega^+ \).
In all examples of this section, given \( \phi(x, y, z) \), \( \beta^\pm(x, y, z) \) and
\[
\begin{align*}
  u &= u^+(x, y, z), \text{ in } \Omega^+, & (6.10) \\
  u &= u^-(x, y, z), \text{ in } \Omega^-, & (6.11)
\end{align*}
\]
such that, on \( \Omega \)
\[
\begin{align*}
  f &= -\nabla (\beta \nabla u), & (6.12) \\
  a &= u^+ - u^-, & (6.13) \\
  b &= (\beta^+ \nabla u^+) \cdot n - (\beta^- \nabla u^-) \cdot n & (6.14)
\end{align*}
\]
\( q \) is obtained from the solutions as a proper Dirichlet boundary condition

All errors in solutions are measured in the \( L^\infty \) norm in the whole domain \( \Omega \)

**Example 16** The interface of this example is an intersection of a few balls \( \beta^\pm \) and \( u^\pm \) are
\[
\begin{align*}
  \beta^+(x, y, z) &= \begin{pmatrix} 4 \sin(x)^2 + 6 & \sin(y + x)z & yx \\ 
  \sin(y + x)z & 2z^2 + \cos(x^2)^2 + 3 & 0.5 \sin(xy) \\ 
  yx & 0.5 \sin(xy) & \cos(xy + z)^2 + 5 \end{pmatrix}, & (6.15) \\
  \beta^-(x, y, z) &= \begin{pmatrix} xz + \cos(x + y) + 3 & x & 0.2 \sin(y - x) \\ 
  x & z^2 + 5 & yz \\ 
  0.2 \sin(y - x) & yz & \sin(z)^2 + 2 \end{pmatrix}, & (6.16) \\
  u^+(x, y, z) &= 10 - x^3 + 2y^2 - 2z + \sin(x + y + z) + \sin(x) + z, & (6.17) \\
  u^-(x, y, z) &= z^3 + y^2 - 2x & (6.18)
\end{align*}
\]

When the level-set function \( \phi \) is given as
\[
\phi(x, y, z) = \min((x - 0.2)^2 + y^2 + z^2 - 0.25, (x + 0.2)^2 + y^2 + z^2 - 0.25), & (6.19)
\]
Figure 6.6 shows the computed error on the interface with the current method using 24 grid points in $x$, $y$ and $z$ directions, different colors denote different values of the error. Table 6.1 shows the error on different grids.

![Intersection of two balls](image)

**Figure 6.6** Intersection of two balls

**Table 6.1** Intersection of two balls

<table>
<thead>
<tr>
<th>$n_x \times n_y \times n_z$</th>
<th>Error in $U$</th>
<th>Order</th>
</tr>
</thead>
<tbody>
<tr>
<td>$6 \times 6 \times 6$</td>
<td>0.02400</td>
<td></td>
</tr>
<tr>
<td>$12 \times 12 \times 12$</td>
<td>0.00742</td>
<td>1.6944</td>
</tr>
<tr>
<td>$24 \times 24 \times 24$</td>
<td>0.00220</td>
<td>1.7557</td>
</tr>
<tr>
<td>$48 \times 48 \times 48$</td>
<td>0.00060</td>
<td>1.8746</td>
</tr>
<tr>
<td>$96 \times 96 \times 96$</td>
<td>0.00015</td>
<td>1.9909</td>
</tr>
</tbody>
</table>
When the level-set function $\phi$ is given as

$$
\phi(x, y, z) = \min(\min((x - 0.4)^2 + y^2 + z^2 - 0.25, (x + 0.3)^2 + y^2 + z^2 - 0.25)) - 0.25, x^2 + (y + 0.5)^2 + z^2 - 0.25), \quad (6.20)
$$

Figure 6.7 shows the computed error on the interface with the current method using 24 grid points in $x$, $y$ and $z$ directions, different colors denote different values of the error. Table 6.2 shows the error on different grids.

![Figure 6.7 Intersection of three balls](image)

When the level-set function $\phi$ is given as

$$
\phi(x, y, z) = \min(x^2 + y^2 + (z + 0.5)^2 - 0.25, \min((x - 0.4)^2 + y^2 + z^2 - 0.25, (x + 0.3)^2 + y^2 + z^2 - 0.25)) - 0.25, x^2 + (y + 0.5)^2 + z^2 - 0.25), \quad (6.21)
$$

Figure 6.8 shows the computed error on the interface with the current method using 24 grid points in $x$, $y$ and $z$ directions, different colors denote different values of the error. Table 6.3 shows the error on different grids.
Table 6.2 Intersection of three balls

<table>
<thead>
<tr>
<th>$n_x \times n_y \times n_z$</th>
<th>Error in $U$</th>
<th>Order</th>
</tr>
</thead>
<tbody>
<tr>
<td>$6 \times 6 \times 6$</td>
<td>0.04143</td>
<td></td>
</tr>
<tr>
<td>$12 \times 12 \times 12$</td>
<td>0.01427</td>
<td>1.5374</td>
</tr>
<tr>
<td>$24 \times 24 \times 24$</td>
<td>0.00370</td>
<td>1.9479</td>
</tr>
<tr>
<td>$48 \times 48 \times 48$</td>
<td>0.00100</td>
<td>1.8938</td>
</tr>
<tr>
<td>$96 \times 96 \times 96$</td>
<td>0.00025</td>
<td>2.0011</td>
</tr>
</tbody>
</table>

Figure 6.8 Intersection of four balls

Example 17 The interface of this example is an intersection of two balls $\phi$, $u^\pm \beta^\pm$ are

$$\phi(x, y, z) = \min((x - 0.2)^2 + y^2 + z^2 - 0.25, (x + 0.2)^2 + y^2 + z^2 - 0.25),$$
Table 6.3 Intersection of four balls

<table>
<thead>
<tr>
<th>$n_x \times n_y \times n_z$</th>
<th>Error in $U$</th>
<th>Order</th>
</tr>
</thead>
<tbody>
<tr>
<td>$6 \times 6 \times 6$</td>
<td>0.04193</td>
<td></td>
</tr>
<tr>
<td>$12 \times 12 \times 12$</td>
<td>0.01426</td>
<td>1.5556</td>
</tr>
<tr>
<td>$24 \times 24 \times 24$</td>
<td>0.00370</td>
<td>1.9467</td>
</tr>
<tr>
<td>$48 \times 48 \times 48$</td>
<td>0.00100</td>
<td>1.8939</td>
</tr>
<tr>
<td>$96 \times 96 \times 96$</td>
<td>0.00025</td>
<td>2.0010</td>
</tr>
</tbody>
</table>

\[
\beta^+(x, y, z) = \begin{pmatrix}
4x^2 + 6 & \sin(y + x) & yx \\
\sin(y + x) & 2z^2 + 3 & 0.5\sin(x) \\
yx & 0.5\sin(x) & \cos(xy + z)^2 + 5
\end{pmatrix}, \quad (6.22)
\]

\[
\beta^-(x, y, z) = \begin{pmatrix}
\cos(x + y)^2 + 3 & z & 0.2\sin(z - x) \\
z & z^2 + 5 & y \\
0.2\sin(z - x) & y & \sin(z)^2 + 2
\end{pmatrix}, \quad (6.23)
\]

\[
u^+(x, y, z) = 10 - 2x^3 + 3y^2 + \sin(z - y), \quad (6.24)
\]

\[
u^-(x, y, z) = -6\sin(x) + 3y + 5z^3 \quad (6.25)
\]

Figure 6.9 shows the computed error on the interface with the current method using 24 grid points in $x$, $y$ and $z$ directions, different colors denote different values of the error. Table 6.4 shows the error on different grids.
Figure 6.9 Example of three-dimensional problems: Two balls 1

Table 6.4 Example of three-dimensional problems: Two balls 1

<table>
<thead>
<tr>
<th>( n_x \times n_y \times n_z )</th>
<th>Error in ( U )</th>
<th>Order</th>
</tr>
</thead>
<tbody>
<tr>
<td>6 \times 6 \times 6</td>
<td>0.05242</td>
<td></td>
</tr>
<tr>
<td>12 \times 12 \times 12</td>
<td>0.01400</td>
<td>1.9043</td>
</tr>
<tr>
<td>24 \times 24 \times 24</td>
<td>0.00370</td>
<td>1.9204</td>
</tr>
<tr>
<td>48 \times 48 \times 48</td>
<td>0.00099</td>
<td>1.9036</td>
</tr>
<tr>
<td>96 \times 96 \times 96</td>
<td>0.00024</td>
<td>2.0141</td>
</tr>
</tbody>
</table>

**Example 18** The interface of this example is also an intersection of two balls \( \phi \), \( u^\pm \) and \( \beta^\pm \) are

\[
\phi(x, y, z) = \min((x - 0.2)^2 + y^2 + z^2 - 0.25, (x + 0.2)^2 + y^2 + z^2 - 0.25),
\]
\[
\beta^+(x, y, z) = \begin{pmatrix}
4x^2 + 6 & \sin(y + x) & yx \\
\sin(y + x) & 2z^2 + 3 & 0.5\sin(x) \\
yx & 0.5\sin(x) & \cos(xy + z)^2 + 5
\end{pmatrix}, \quad (6.26)
\]

\[
\beta^-(r, y, z) = \begin{pmatrix}
\cos(x + y)^2 + 3 & z & 0.2\sin(z - x) \\
z & z^2 + 5 & y \\
0.2\sin(z - r) & y & \sin(z)^2 + 2
\end{pmatrix}, \quad (6.27)
\]

\[
u^+(x, y, z) = 10\cos(x)\cos(y)\cos(z) + 20, \quad (6.28)
\]

\[
u^-(r, y, z) = \exp\left(-\frac{r^2 + y^2 + z^2}{20}\right) \quad (6.29)
\]

Figure 6.10 shows the computed error on the interface with the current method using 24 grid points in \( r, y \) and \( z \) directions, different colors denote different values of the error. Table 6.5 shows the error on different grids.
Table 6.5  Example of three-dimensional problems  Two balls 2

<table>
<thead>
<tr>
<th>$n_x \times n_y \times n_z$</th>
<th>Error in $U$</th>
<th>Order</th>
</tr>
</thead>
<tbody>
<tr>
<td>$6 \times 6 \times 6$</td>
<td>0.10308</td>
<td></td>
</tr>
<tr>
<td>$12 \times 12 \times 12$</td>
<td>0.02780</td>
<td>1.8909</td>
</tr>
<tr>
<td>$24 \times 24 \times 24$</td>
<td>0.00764</td>
<td>1.8628</td>
</tr>
<tr>
<td>$48 \times 48 \times 48$</td>
<td>0.00201</td>
<td>1.9254</td>
</tr>
<tr>
<td>$96 \times 96 \times 96$</td>
<td>0.00052</td>
<td>1.9441</td>
</tr>
</tbody>
</table>

Example 19  This example has a singular point on the interface  $\phi, u^\pm$ and $\beta^\pm$ are

$$\phi(r, y, z) = (r - 0.4)^2 + y^2 + z^2 - 0.16,$$  \hspace{1cm} (6.30)

$$\beta^+(x, y, z) = \begin{pmatrix} 4x^2 + 6 & \sin(y + x) & yx \\ \sin(y + x) & 2z^2 + 3 & 0.5 \sin(x) \\ yx & 0.5 \sin(x) & \cos(xy + z)^2 + 5 \end{pmatrix},$$  \hspace{1cm} (6.31)

$$\beta^-(x, y, z) = \begin{pmatrix} \cos(x + y)^2 + 3 & z & 0.2 \sin(z - x) \\ z & z^2 + 5 & y \\ 0.2 \sin(z - x) & y & \sin(z)^2 + 2 \end{pmatrix},$$  \hspace{1cm} (6.32)

$$u^+(x, y, z) = (x^2 + y^2 + z^2)^{5/6},$$  \hspace{1cm} (6.33)

$$u^-(x, y, z) = \sin(x + y).$$  \hspace{1cm} (6.34)

Figure 6.11 shows the computed error on the interface with the current method using 24 grid points in $x$, $y$ and $z$ directions, different colors denote different values of the error  Table 6.6 shows the error on different grids.
Figure 6.11  Singular point on the interface in three dimensions

Table 6.6  Singular point on the interface in three dimensions

<table>
<thead>
<tr>
<th>$n_x \times n_y \times n_z$</th>
<th>Error in $U$</th>
<th>Order</th>
</tr>
</thead>
<tbody>
<tr>
<td>$6 \times 6 \times 6$</td>
<td>0.02227</td>
<td></td>
</tr>
<tr>
<td>$12 \times 12 \times 12$</td>
<td>0.00722</td>
<td>1.6262</td>
</tr>
<tr>
<td>$24 \times 24 \times 24$</td>
<td>0.00225</td>
<td>1.6816</td>
</tr>
<tr>
<td>$48 \times 48 \times 48$</td>
<td>0.00069</td>
<td>1.6951</td>
</tr>
<tr>
<td>$96 \times 96 \times 96$</td>
<td>0.00021</td>
<td>1.7208</td>
</tr>
</tbody>
</table>
CHAPTER 7

CONCLUSIONS AND FUTURE WORK

This dissertation extends the idea presented in [15] for solving matrix coefficient second-order elliptic equations for interface problems with two domains in two dimensions. Parts of Chapter 3 have been published and can be found in [16].

This method is extended to solve second-order elasticity equations for interface problems with two domains in two dimensions, second-order elliptic equations for interface problems with three domains in two dimensions and second-order elliptic equations for interface problems with two domains in three dimensions. This dissertation generalized the theorems in [15] and proofs are provided. It is also proved that the matrix for the linear system generated by the current method is positive definite (but not symmetric). Through numerical experiments, this method achieved second-order accuracy in the $L^\infty$ norm, and can handle the difficulties of sharp-edged interfaces and oscillatory solutions. Compared with the previous work in [15], the order of accuracy for sharp-edged interfaces is improved from 0.8th to close to second order. Compared with the result in [39], the more oscillatory the solution is, the more advantageous the current method is.

The focus of the future work will be on the following topics:

(1) Since the numerical results for two-dimensional/three-dimensional elliptic/elasticity
interface problems with two/three domains have been obtained, proofs of the convergence of this method for all the four topics will be the next step of research

(2) Elasticity interface problem with three domains in two dimensions

(3) Elliptic interface problem with three domains in three dimensions

(4) Elasticity interface problem with two domains in three dimensions

(5) Elasticity interface problem with three domains in three dimensions is a further extension of the above topics, it will be under consideration for future research

(6) Moving interface problems are more practical but yet more complicated Elliptic and elasticity problems with moving interface is another challenging research topic

(7) Some applications on solving the elliptic and elasticity interface problems, such as in biomathematics, fluid dynamics, etc
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