Strategies and Algorithms of Sudoku

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May 13, 2020

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Abstract

This paper discusses different strategies for the game of Sudoku and how those strategies relate to other problem solving techniques while also attempting to use those other techniques in a way that improves the strategies for Sudoku. This includes a thorough analysis of the general algorithm and an algorithm that is formed by the Occupancy Theorem and Preemptive Sets. This paper also compares these algorithms that directly relate to Sudoku with algorithms to similar combinatorial problems such as the Traveling Salesman problem and more. With the study of game theory becoming more popular, these strategies have also been shown to help students in various ways in the classroom. To understand Sudoku on a deeper level, this paper demonstrates ways to model a puzzle by using permutation matrices and different symmetries.

Keywords: sudoku, strategies, algorithms, permutation matrix, symmetry, education

1 Introduction

						2		
	8				7		9	
6		2				5		
	7			6				
			9		1			
				2			4	
		5				6		3
	9		4				7	
		6						

(a) Sudoku Puzzle

9	5	7	6	1	3	2	8	4
4	8	3	2	5	7	1	9	6
6	1	2	8	4	9	5	3	7
1	7	8	3	6	4	9	5	2
5	2	4	9	7	1	3	6	8
3	6	9	5	2	8	7	4	1
8	4	5	7	9	2	6	1	3
2	9	1	4	3	6	8	7	5
7	3	6	1	8	5	4	2	9

(b) Solution

Sudoku is a puzzle in which players insert the numbers one through nine into a grid consisting of nine squares subdivided into a further nine smaller squares in such a way that every number appears once in each horizontal line, vertical line, and block [6]. The game has nothing to do with the actual use of the numbers so it is important to note that nine different symbols could be used instead of digits [9]. Sudoku was made possible by Euler's development of the Latin squares and magic squares in 1783 [6]. The conventional Sudoku is 9x9 while a 6x6 is a special puzzle called Rudoku; other sizes are still used in some areas of practice to increase difficulty, among other possibilities. Permutation matrices make it easier to see how a puzzle is formed while also giving a strategy for the unconventional sizes of a puzzle. There are many different strategies that are used to solve a puzzle, so every player has their own way of solving a puzzle. Depending on the puzzle difficulty, a player may choose to use a certain strategy or algorithm for the best outcome of a solved puzzle. The General Algorithm is based on two theorems such as the Occupancy Theorem and Preemptive Set Theorem. Sudoku is characterized as a combinatorial problem based on logic and number/symbol placement. Sudoku can also be related to many educational subjects and other classic combinatorial mathematics problems. To understand Sudoku puzzles, terminology is key. A square or cell is the smallest item in a puzzle; there are eighty-one squares. A block is a $3x3$ square in the puzzle. A conventional Sudoku puzzle is composed of nine blocks in a $3x3$ grid. A candidate is a possibility for a square. We say that a square has three virtual lines: the row, column, and block a square is in. We call the squares that are included in those virtual lines buddies.

2 Mathematical Models and Formulations

2.1 Permutation Matrices

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

A permutation matrix is a matrix obtained by permuting the rows of an nxn identity matrix according to some permutation of the numbers 1 to n. Every row and column therefore contains precisely a single one with zeros everywhere else, and every permutation corresponds to a unique permutation matrix. There are $n!$ permutation matrices of size n, where $n!$ is a factorial [3]. By using permutation matrices, you can see how the Sudoku puzzle is initially modeled by using the zeros and ones. The ones represent one of the digits while the zeros represent all of the other digits. Stars can be used to represent the forbidden positions so zeros would automatically be placed there. By using zeros and ones, the permutation matrix can help a person to visualize how the simple Sudoku puzzle is solvable by visualizing the given and forbidden positions. Every time you add a digit and take away zeros, the puzzle begins to form. The puzzle is complete when the last set of zeros turn into the nth digit. The digits would then be permuted, or mixed up, to determine the puzzle's level of difficulty using the digits 1 to n. By using permutation matrices, it can become a strategy or algorithm in itself on how to solve a Sudoku puzzle of different sizes because the permutation aspect lets you choose both n and N. The amount of rows or columns in the block is n. The number of blocks in a row or column of the puzzle is N.

2.2 Symmetries

Symmetries in Sudoku puzzles have a way of forming new Sudoku puzzles. Using Sudoku symmetries, a person can solve a puzzle by hand or input the symmetries into a computer for an easy solve. This paper strictly focuses on the human solving strategy of the symmetries. On a standard $9x9$ Sudoku puzzle, the three columns of $3x3$ blocks are called stacks, and the three rows of $3x3$ blocks are called bands. If two whole rows in a band or two whole columns in a stack are interchanged, it creates a symmetry. If two entire bands or two entire stacks are interchanged, this creates new symmetries as well. If a puzzle is reflected over any axis of symmetry, whether it's vertical, horizontal, or diagonal, it again creates new symmetries. All of these actions made on the Sudoku board generate the set of the complete group of Sudoku symmetries. Each time a symmetry is made, the puzzle changes but is "essentially the same" [1].

3 Strategies, Algorithms, and Theorems

Tom Davis explains many strategies that can be used to solve a Sudoku puzzle [4]. However the basic strategies discussed below can be used on their own and in any chosen order, or they can be used in repetitive order to create its own algorithm if the puzzle is simple enough. Other algorithms can use these strategies as well but maybe not explicitly.

3.1 Unique missing candidates

The first strategy used in solving a Sudoku puzzle is looking for unique missing candidates. A unique missing candidate occurs when eight out of nine digits occur in a virtual line or block. This strategy is the most used strategy in solving Sudoku puzzles. Whether the difficulty allows for them to be apparent or not, a solver will always check for unique missing candidates first.

3.2 Naked Singles

By using naked singles, this strategy includes marking up a puzzle for the possible candidates. This markup includes listing the possible candidates for every open square in the respective square. Once a puzzle mark up is complete, a player should look to see if a square has only one possible candidate. If the square has only one candidate, then that square is determined as a naked single. After filling in the naked single with the correct digit, the player can update the puzzle markup by removing that digit from the remaining squares. By updating the puzzle markup, this could possibly uncover a hidden single.

3.3 Hidden Singles

Hidden singles occur after a mark up on the puzzle, and a player finds that there is only one candidate left for the different virtual lines after a few squares are completed. If there is only one candidate left in the square, that hidden single then becomes a unique missing candidate. The hidden candidates can also be pairs, triplets, quads, etc. That note is important for later if the need for a guess arises after all other strategies have been exhausted. If the need arises, see the Preemptive Set Theorem below in how to use the hidden sets to force a candidate.

3.4 Locked candidates

Locked candidates occur after you begin filling in the puzzle. They are also referred to as forced possibilities. Once a candidate is locked, a player moves on to solve the rest of the unknown possibilities in the square. With this strategy, if a player notices that two squares only have the same two possible candidates, then the player can eliminate those two possible candidates from the rest of the square's buddies, possibly exposing more hidden singles.

3.5 Theorems

The theorems listed below are used as the basis of the General Algorithm to solve Sudoku puzzles as Crook states in his article [2].

Theorem (Occupancy Theorem)**.** *Let X be a preemptive set in a Sudoku puzzle markup. Then every number in X that appears in the markup of cells not in X over the range of X cannot be a part of the puzzle solution.*

To explain how this theorem is used in solving Sudoku puzzles, one must understand how this theorem works. If a number appears in a virtual line, then the candidates for that square decrease. When a square becomes occupied, the set of possibilities for the other squares decreases by one each time. This strategy continues until the entire puzzle is occupied or solved.

Theorem (Preemptive Set Theorem)**.** *There is always a preemptive set that can be invoked to unhide a hidden set, which then changes the hidden set into a preemptive set except in the case of a singleton.*

These preemptive sets can be found in application to any virtual line. As mentioned earlier hidden multiples are "quite useful because they are often easier to spot than the accompanying preemptive set" [2]. Preemptive sets can sometimes be broken down into "the union of the two smaller preemptive sets" [2]. They can also be intersected to show an empty set. The exception of a singleton just points out that there will never be a set of one digit because that one digit would be a locked candidate for some square, therefore not a possibility for an unknown square.

3.6 General Algorithm

Crook states that the General Algorithm uses the Occupancy and Preemptive Set Theorem to set up the following steps to follow in order to solve a Sudoku puzzle [2].

1. Find forced numbers in the puzzle.
2. Mark up the puzzle.
3. Search iteratively for preemptive sets in all rows, columns, and boxes–taking cross out action as sets are discovered until...
   a. a solution is found, then end.
   b. a random choice must be made for continuation, then go back to step 3 and repeat.

4 Sudoku in Education

By incorporating Sudoku into K-12 education, it gives students the opportunity to learn essential skills without just practicing those skills in a regular classroom environment or out of a textbook. Since Sudoku is a game, students are more eager to learn. To motivate students in a mathematics classroom has become increasingly difficult because of "society's generally accepted poor attitudes towards mathematical success" [5]. Jerry Martin is an retired teacher and avid Sudoku puzzle solver who started an afternoon Sudoku Club for elementary school students. In his experience with the students, Sudoku "teaches the need for accuracy, persistence in problem solving, the difference between relevant and irrelevant information, cause and effect, pattern recognition, and an opportunity to develop longer attention spans" [8]. With young learners, teachers can also encourage collaboration between students to solve a puzzle together. When a square has been filled with a candidate, the student must prove why it is correct. By using this technique, it "reduced errors and improved communication and cooperation skills, important components of interpersonal intelligence" [8].

4.1 Basic Educational Skills

Critical Thinking/Logic: Critical thinking is the use of rational problem solving. Students can learn to solve problems by focusing on what information in the problem is relevant. Students also learn that decisions are consequential, but they also learn how to make those decisions. Logic is defined as learning how to think rather than what to think. Logic can be seen as a mix of deliberation and debate. This explains why Sudoku is referred to as mental yoga.

Creative Thinking: When solving a Sudoku puzzle, guessing is sometimes necessary. This would encourage students to think creatively in order to find a solution. Students must think of new tactics.

Flexibility: When solving Sudoku puzzles, a player should always solve them with a pencil because mistakes will be made. Sudoku puzzles can be challenging at times, and when mistakes are made, it shows that failure is okay. Learning comes from failure. Decisions in Sudoku need to be flexible because different guesses can lead to success or failure many times before the correct decision is made. Solving Sudoku puzzles also teaches discipline and perseverance. Just because a mistake is made does not mean that puzzle is unsolvable.

4.2 Relevancy to Other Mathematical Subjects

Combinatorial problems

Traveling Salesman Problem

$$\sum d(A_{\mathrm{pi(ł)}}, A_{\mathrm{pi(ł+1)}}) + d(A_{\mathrm{pi(m)}}, A_{\mathrm{pi(1)}})$$

This problem is a classic combinatorial optimization problem that tries to find the shortest and most efficient route between all of the cities the salesman must travel to. The shortest amount of time relates to the shortest amount of distance the salesman travels. The salesman plans the itinerary around the cities as fast as possible to be efficient. This problem uses finite sets and distance to determine the itinerary. This problem relates to Sudoku because a player wants to find as many digits for the squares as fast as possible. The equation above uses the sum of all of distances between each city on the traveler's itinerary to find the most efficient route.

Set Theory

Sudoku can help students learn elementary set theory. Teachers can use an Excel workbook to show basic strategy and use it as a "sudoku helper" [10]. By solving Sudoku puzzles, students would come across basic definitions in set theory and see how they are used. A complement of a set could be shown as the possible numbers of a square depending on if it is in relation to the union of the row, column, or block. The collection of possibles in a block could be shown as a partition of the set of all possibles. Students could look at the different sets of possibles and consider if they are disjoint or non-disjoint. A more difficult Sudoku puzzle could need the use of a binary search with backtracking which could introduce a discussion of a binary tree. This way of solving a Sudoku puzzle includes entering the information into Excel which leads to computer formatting skills for students as well.

Computer Science

In computer science, codes and algorithms are very important. With computer science skills, a player can encode a Sudoku puzzle as a CNF formula. CNF stands for Conjunctive Normal Form. A study has shown there exists an "optimized CNF encoding in order to deal with large instances of Sudoku puzzles" [7]. Clauses are very important to CNF coding, and Sudoku puzzles are great to teach students about how to set up certain clauses. To set up clauses for Sudoku, one must define the qualities for the clauses such as definedness, uniqueness, definedness constraints, and uniqueness constraints. The clauses also use modulo operation within the cells, rows, columns, and blocks. A challenge activity for students could be to use a set of clauses to formulate Sudoku. If and only if the the puzzle has a solution, the puzzle can be encoded as various sets of clauses. [7]

5 Conclusion and Further Research

5.1 Conclusions

To understand how to solve a Sudoku puzzle, you must learn how a Sudoku puzzle is formed or modeled. Permutation matrices are the basics of each Sudoku puzzle while symmetries create "essentially the same" puzzle [1]. The basic strategies can be used as an algorithm to solve Sudoku puzzles most efficiently. The General Algorithm can be used with a higher difficulty to force possibilities to show themselves. Through studying how Sudoku puzzles are formed and solved, ways to use Sudoku in an educational environment became apparent. Sudoku can be used to encourage critical, creative, and flexible thinking in all areas of learning but especially subjects such as mathematics and computer science.

5.2 Further Research

This paper concentrated on the strategies and algorithms that can be done with ease on paper. However, depending on the difficulty, more involved methods must be used to solve the puzzle. Sudoku puzzles of large size also require other algorithms that are more involved on different qualities of a Sudoku puzzle. Permutation matrices can be used to solve those larger puzzles with the help of computer formatting. Excel spreadsheets and CNF coding have certain algorithms that can be used to solve Sudoku puzzles. By studying these more extensive strategies and algorithms, we could find more resources and ways to use Sudoku puzzles.

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