Blackjack: the math behind the cards

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May 13, 2019

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Abstract

In this paper the reader will learn about the math behind the cards in the game of Blackjack. Blackjack or "21" has been played around the world with various rules and regulations in both professional and informal environments. The ultimate objective of the game is to receive a total card value of 21, or as close to 21 as possible without exceeding it, from the cards in a player's hand in order to beat the dealer's total. The goal of this project is to calculate the probabilities of various hands to determine the best strategies to win 21. The probabilities of receiving each combination of a two-card pair, as well as the probabilities of a player's best and worst case of receiving a third card to remain in the game, are calculated. These calculations are done for scenarios involving three players and a dealer using two to six card decks per scenario. The results indicate consistent probabilities of receiving the same two-card pair regardless of the number of decks used, but decreasing probabilities of receiving a third card to remain in the game. Thus, casinos are more likely to administer games with more decks in play. The goal is to determine whether to "hit" or "stand" when dealt certain a pair.

Keywords: 21, Blackjack, cards, combinatorial, combinatorics, probability

1 Introduction and Motivation

Blackjack is a card game typically played at casinos consisting of two to seven players (see [4] for more rules at casinos) and one dealer using two to eight card decks (see [1]). A standard card deck consists of 52 cards with fours suits: spades, hearts, diamonds, and clubs; giving each suit thirteen cards per deck. Each card is deemed a point value with number cards (2-10) worth their own numbers, face cards (Jack, Queen, and King) worth 10 points, and Ace cards worth 1 or 11 points used at the player's discretion. The objective of the game is receiving cards with a total point value of 21, hence giving Blackjack its nickname "21". Each player and dealer is guaranteed two cards at the beginning of each round. To remain in the game, a player has multiple options, two of which are to either "Stand" or "Hit". If a player chooses to stand, then he or she is content with their point total and no longer wishes to receive any cards. If a player chooses to hit, he or she receives another card to try to reach 21 or as close to it. According to [8], casinos have a five-card rule meaning the maximum amount of cards a player can have is five, thus a player can choose to hit only a maximum of three times. Once a player collects a total of 21 or the highest total among the other players and dealer without exceeding 21, the player wins. However, if the player collects a third, fourth, or fifth card to push them over the goal of 21, then he or she "busts", which means he or she is out of the game. The purpose of this work is to analyze the likelihood of receiving pairs with totals of 2 to 21, then calculate the success rate of receiving a third card to total the cards to 21 or less.

Assumptions for this project were created in order to simplify the calculations produced for Blackjack. We began by understanding the probabilities of the game was analyzing 1 deck of cards [10]. Since casinos play with more than one deck, we have investigated scenarios using two through six decks of cards and have considered three players and a dealer as the number of participants in the game. With Player 1 being dealt to first, each card is dealt face down by the dealer, where each player and dealer has a minimum of two cards and a maximum of three cards per hand.

The initial interest for this research was understanding how to count cards from watching the movie "21" [3], which is a method of keeping a count of cards that are still in the deck while playing 21. From researching the mathematics the movie, which is based on a true story, uses behind counting cards, we realized we could take a step back and analyze how the game itself is played (see [5] for a description on the math used in the movie). After realizing there are numerous probabilities in receiving various card pairs, it was decided to analyze the cards of Blackjack as well as creating the "Best Case" and the "Worst Case" scenarios for the pairs. While calculating the probabilities of receiving card pairs, the interest of gaining a third card was introduced, creating these Best and Worst Case scenarios. Each of these scenarios were formulated depending on the specific card pair a player has in his or her hand. For instance, if a player has a value of 14 points within the pair, what are the chances the third card received is a 7 or less in order to remain in the game. Both the total number of points and the individual card points are crucial to the probability of receiving a desired third card. An example of the calculations for card pairs valuing at 14 can be found in "V. Data and Models" as well as the full table of computed values for card pairs valuing at 2 - 21 can be found in "IX. Appendices". For more information on basic probabilities, [9] is available.

2 Background Literature and Related Studies

The card game, Blackjack, is widely played around the world for both enjoyment of the game and the winnings that go along with it. As the most popular card game, Blackjack can be found in nearly every casino. In order to "win" the players compete against the dealer to earn money by collecting 21 or close enough to 21 without busting or going over. Each player and dealer is dealt two cards each from the two to eight standard card decks in play. When Blackjack was first introduced, a typical game was played with one deck, however this made it easy for players to count cards (or keep track of which cards remain in the deck) to help them decide whether to hit or stand in order to win [4]. Thus, the game has changed to using more decks to prevent this strategy. The standard deck of cards include the numerical cards corresponding to their individual numerical value, "face cards" (Jack, Queen, and King) with values of 10 each, and of course the Aces cards with a value of 1 or 11. While rules of casinos may vary, the objective of the game remains the same - collect 21. The main variations that casinos are likely to have include using different numbers of decks and players,leaving dealt cards face up or face down for the table to view or not, and restricting players to only receiving a maximum amount of cards per hand. Given the background of Blackjack, assumptions have been put in place for this research in order to create specific scenarios including three players, one dealer, two to six decks per game, and a maximum hand of three cards.

3 Methods

3.1 The Initial Pairs

The methodology for this research began by collecting all of the possible two-card pairs that can be created from the initial deal of two through six decks. The next step was to calculate the probabilities of receiving each pair. We used a combinatorial approach to calculate these probabilities and relied heavily on "choose" functions. For more information on choose functions, we direct the reader to [2]. These functions were used because they create ratios of the possible desired pairs to the total pairs available in the deck. The combinatorial functions yield rates of success for the player to obtain the pairs. These calculations varied from deck to deck because of the different number of decks used per scenario. For example, if a player wanted to know the likelihood of receiving a 21 with two cards, she would have to know only an Ace and a card valued at 10 would create a 21. From there, she would also have to account the number of decks used in the game. Hence, if two card decks were being used, then by using the function, she could determine that there are eight Aces and thirty-two 10 valued cards available to create a 21 as well as there being 104 total cards in the deck. With this information, the function is used by stating: there are 8 Aces with only one needing to be obtained multiplied by 32 10-valued cards with only one needing to be obtained all divided by 104 total cards with only two needing to be obtained to create the pair. The terminology is portrayed as "8 choose 1 multiplied by 32 choose 1, all divided by 104 choose 2." This specific calculation gives a value of a 4.78% chance the player receives 21 on the first two cards she receives. Now, by changing the number of decks to six, the calculation changes to "24 Aces choose 1 multiplied by 96 10-valued cards choose 1, all divided by 312 total cards choose 2", giving a 4.75% chance of receiving a 21 card pair.

Analysis of these values can be found in the "VI. Results" of this paper.

3.2 A Third Card

Another calculation was created when a third card was introduced to create the "Best Case Scenarios" and the "Worst Case Scenarios" of a player receiving a total three-card point value of 21 or less. The best and the worst case scenario calculations were devised with a thought process of "If a player wants to receive a third card, what is the interval of likelihood the player receives the third card and still remains under or receives 21?" Again, we used a combinatorial approach for the probability calculations. An example scenario to retrieve the interval would be: If Player 1 has a total of 20 points in her hand, she would need a Ace to remain in the game. The best case scenario is no one else playing at the table (Player 2, Player 3, and the Dealer) has any Ace cards, which means all of those cards are still in the deck for Player 1 to receive. The worst case scenario is everyone else playing has only Ace cards, leaving less desired cards in the deck and yielding a lower chance of remaining in the game. These calculations were simple: if the game was being played with two decks and the players and the dealer already have cards in their hands, that gives 104 possible cards in the deck to start the game subtracted by the 8 cards that are in the player's hands, leaves 96 remaining cards still in the deck. Since the player needs an Ace to stay in the game, the Best Case Scenario would be a ratio of 8 Aces to 96 total cards in the deck yielding an 8.33% chance of receiving an Ace. On the other hand, the Worst Case Scenario would be Player 2, Player 3, and the Dealer only have Aces in their hands, which takes out 6 Aces in the 96 card deck, leaving only 2 to be obtained. Thus, the ratio would be 2 Aces to 96 total cards, yielding a 2.08% chance to receive an Ace. These scenarios create an interval of success for Player 1. The likelihood of her receiving the desired card, regardless of how many desired cards are or are not in the deck, would be in this interval every time given the same two-card pair and the same number of decks used. These best and worst case scenarios were calculated for each two-card pair to receive the wanted card using two through six decks. Analysis of these values can also be found in "VI. Results" of this paper.

3.3 The Overall Intervals

Since receiving a third card is an independent event from receiving the initial pair, the probabilities of receiving the pairs can be multiplied by the intervals of the Worst Case to the Best Case. This creates an overall interval of the probability of receiving a player's three card hand. The initial pairs are independent from the third card because of the assumption that the player does not know the exact cards in the other players' hands. These initial pair calculations and Worst Case Scenario to Best Case Scenario Intervals are independent because after the initial pair was calculated, the cards were taken out of the remaining deck of cards. After taking out the initial pairs of all players, we now only have 96 options. Assuming the 8 cards that have been removed are not the cards we want, this creates the two exclusive sets - Set One: removing the 8 cards from the deck; Set Two: analyzing the remaining 96 cards in the deck. For further information on probability on independent events, refer to [6] or [7].

4 Data and Models

4.1 The Initial Pairs

Suppose two card decks are in play, then by using the combinatorial function, the player could determine that there are eight Aces and thirty-two 10 valued cards available to create a 21 as well as there being 104 total cards in the deck. With this information, the function is used by stating: there are 8 Aces with only one needing to be obtained multiplied by 32 10-valued cards with only one needing to be obtained all divided by 104 total cards with only two needing to be obtained to create the pair. The terminology is portrayed as "8 choose 1 multiplied by 32 choose 1, all divided by 104 choose 2." This specific calculation gives a value of a 4.78% chance the player receives 21 on the first two cards she receives. That is, if we let J, Q, and K denote face cards Jack, Queen, and King, we have :

In the Case of 2 Decks:

$$P(\{\text{A, 10}\} \text{ or } \{\text{A, J}\} \text{ or } \{\text{A, Q}\} \text{ or } \{\text{A, K}\}) = \frac{\binom{8}{1}\binom{32}{1}}{\binom{104}{2}} = 4.78\% \tag{1}$$

In the Case of 3 Decks:

$$P(\{\text{A, 10}\} \text{ or } \{\text{A, J}\} \text{ or } \{\text{A, Q}\} \text{ or } \{\text{A, K}\}) = \frac{\binom{12}{1}\binom{48}{1}}{\binom{156}{2}} = 4.76\% \tag{2}$$

In the Case of 4 Decks:

$$P(\{\text{A, 10}\} \text{ or } \{\text{A, J}\} \text{ or } \{\text{A, Q}\} \text{ or } \{\text{A, K}\}) = \frac{\binom{16}{1}\binom{64}{1}}{\binom{208}{2}} = 4.76\% \tag{3}$$

In the Case of 5 Decks:

$$P(\{\text{A, 10}\} \text{ or } \{\text{A, J}\} \text{ or } \{\text{A, Q}\} \text{ or } \{\text{A, K}\}) = \frac{\binom{20}{1}\binom{80}{1}}{\binom{260}{2}} = 4.75\% \tag{4}$$

In the Case of 6 Decks:

$$P(\{\text{A, 10}\} \text{ or } \{\text{A, J}\} \text{ or } \{\text{A, Q}\} \text{ or } \{\text{A, K}\}) = \frac{\binom{24}{1}\binom{96}{1}}{\binom{312}{2}} = 4.75\% \tag{5}$$

The following table shows the probability of receiving specific two-card pair values for two through six decks of cards for points totaling 12-21. The reason for examining these point values more closely is because there is a chance of exceeding 21 on a third card dealt. The full table can be found as Figure 13 in "IX. Appendices".

Value of Two Cards	Possible Pairs	Probability of Receiving Pair				
		2 Decks	3 Decks	4 Decks	5 Decks	6 Decks
12	{A, A}	0.52%	0.55%	0.56%	0.56%	0.57%
	{2, (10, J, Q, or K)}	4.78%	4.764%	4.757%	4.75%	4.75%
	{3, 9}	1.19%	1.19%	1.19%	1.19%	1.19%
	{4, 8}	1.19%	1.19%	1.19%	1.19%	1.19%
	{5, 7}	1.19%	1.19%	1.19%	1.19%	1.19%
	{6, 6}	0.52%	0.55%	0.56%	0.56%	0.57%
13	{2, A}	1.19%	1.19%	1.19%	1.19%	1.19%
	{3, (10, J, Q, or K)}	4.78%	4.76%	4.76%	4.75%	4.75%
	{4, 9}	1.19%	1.19%	1.19%	1.19%	1.19%
	{5, 8}	1.19%	1.19%	1.19%	1.19%	1.19%
	{6, 7}	1.19%	1.19%	1.19%	1.19%	1.19%
14	{3, A}	1.19%	1.19%	1.19%	1.19%	1.19%
	{4, (10, J, Q, or K)}	4.78%	4.76%	4.76%	4.75%	4.75%
	{5, 9}	1.19%	1.19%	1.19%	1.19%	1.19%
	{6, 8}	1.19%	1.19%	1.19%	1.19%	1.19%
	{7, 7}	0.52%	0.55%	0.56%	0.56%	0.57%
15	{4, A}	1.19%	1.19%	1.19%	1.19%	1.19%
	{5, (10, J, Q, or K)}	4.78%	4.76%	4.76%	4.75%	4.75%
	{6, 9}	1.19%	1.19%	1.19%	1.19%	1.19%
	{7, 8}	1.19%	1.19%	1.19%	1.19%	1.19%
16	{5, A}	1.19%	1.19%	1.19%	1.19%	1.19%
	{6, (10, J, Q, or K)}	4.78%	4.76%	4.76%	4.75%	4.75%
	{7, 9}	1.19%	1.19%	1.19%	1.19%	1.19%
	{8, 8}	0.52%	0.55%	0.56%	0.56%	0.57%
17	{6, A}	1.19%	1.19%	1.19%	1.19%	1.19%
	{7, (10, J, Q, or K)}	4.78%	4.76%	4.76%	4.75%	4.75%
	{8, 9}	1.19%	1.19%	1.19%	1.19%	1.19%
18	{7, A}	1.19%	1.19%	1.19%	1.19%	1.19%
	{8, (10, J, Q, or K)}	4.78%	4.76%	4.76%	4.75%	4.75%
	{9, 9}	0.52%	0.55%	0.56%	0.56%	0.57%
19	{8, A}	1.19%	1.19%	1.19%	1.19%	1.19%
	{9, (10, J, Q, or K)}	4.78%	4.76%	4.76%	4.75%	4.75%
20	{9, A}	1.19%	1.19%	1.19%	1.19%	1.19%
	{(10, J, Q, or K), (10, J, Q, or K)}	9.26%	9.33%	9.36%	9.39%	9.40%
21	{(10, J, Q, or K), A}	4.78%	4.76%	4.76%	4.75%	4.75%

Figure 1: Point Value of Pairs Equaling 12 - 21

4.2 A Third Card

Suppose two card decks are still in play, then by using ratios, the player could determine whether to hit or stand based on her best and worst chances of receiving a third card without exceeding 21. For example, consider the player has a pair of two 10-valued cards (e.g. {10, 10}, {10, J}, {Q, K}, etc.) and would need only one of the possible eight Aces to not exceed 21. The scenario would proceed as follows: after each player and the dealer has their pair of cards, the deck has 96 cards remaining. The "Best Case Scenario" is all 8 Aces are still in the remaining deck of cards because no other player has any Aces among the eight cards in their hands combined. Thus, the ratio of 8 Aces to 96 cards shows my chances of receiving an Ace. For the "Worst Case Scenario", all other players have all of the Aces in their hands already, leaving two Aces in the deck. Only two are left because the player is aware of the eight possible Aces in the deck with the possibility of six being taken because she knows she does not have Aces in her own hand. This gives a ratio of 2 Aces to 96 cards chance of not exceeding 21. These scenarios create an interval of success for the player to remain in the game. In this case the player has a 2.08% - 8.33% chance of success in receiving an Ace. The Worst and Best Case scenarios are taking the two extremes of trying to not exceed 21, thus giving an interval for all other possible situations (e.g. only 3 Ace through 7 Aces in the remaining deck). Calculations are given for card pairs totaling at 20 with each card equaling 10 ($\{10, 10\}, \{10, \mathrm{J}\}, \{10, \mathrm{Q}\}, \{10, \mathrm{K}\}, \{\mathrm{J}, \mathrm{Q}\}, \{\mathrm{J}, \mathrm{K}\}, \{\mathrm{Q}, \mathrm{K}\}$) for decks two through six:

In the Case of 2 Decks:
Best Case: All 8 Aces Are in the Deck of 96 Cards

$$\text{8 Aces : 96 Cards} = \frac{8}{96} = 8.33\% \tag{6}$$

Worst Case: Only 2 Aces Are in the Deck of 96 Cards

$$\text{2 Aces : 96 Cards} = \frac{2}{96} = 2.08\% \tag{7}$$

Interval: $2.08\% - 8.33\%$

In the Case of 3 Decks:
Best Case: All 12 Aces Are in the Deck of 148 Cards

$$\text{12 Aces : 148 Cards} = \frac{12}{148} = 8.11\% \tag{8}$$

Worst Case: Only 6 Aces Are in the Deck of 148 Cards

$$\text{6 Aces : 148 Cards} = \frac{6}{148} = 4.05\% \tag{9}$$

Interval: $4.05\% - 8.11\%$

In the Case of 4 Decks:
Best Case: All 16 Aces Are in the Deck of 200 Cards

$$16 \text{ Aces} : 200 \text{ Cards} = \frac{16}{200} = 8.00\% \tag{10}$$

Worst Case: Only 10 Aces Are in the Deck of 200 Cards

$$10 \text{ Aces} : 200 \text{ Cards} = \frac{10}{200} = 5.00\% \tag{11}$$

Interval: 5.00% – 8.00%

In the Case of 5 Decks:
Best Case: All 20 Aces Are in the Deck of 252 Cards

$$20 \text{ Aces} : 252 \text{ Cards} = \frac{20}{252} = 7.94\% \tag{12}$$

Worst Case: Only 14 Aces Are in the Deck of 252 Cards

$$14 \text{ Aces} : 252 \text{ Cards} = \frac{14}{252} = 5.56\% \tag{13}$$

Interval: 5.56% – 7.94%

In the Case of 6 Decks:
Best Case: All 24 Aces Are in the Deck of 304 Cards

$$24 \text{ Aces} : 304 \text{ Cards} = \frac{24}{304} = 7.89\% \tag{14}$$

Worst Case: Only 18 Aces Are in the Deck of 304 Cards

$$18 \text{ Aces} : 304 \text{ Cards} = \frac{18}{304} = 5.92\% \tag{15}$$

Interval: 5.92% – 7.89%

While card pairs valuing from 2-20 all have a Worst to Best Case interval, 21 does not; the reason being the player already has the point value to win and would not want a third card. For card pairs with values of 2 - 11, their Worst Case to Best Case Interval is 100.00% - 100.00% because regardless of the value of the third card, the player cannot exceed 21 since 10 is the largest value she can receive. If she has a card pair total of 11 and receives an Ace, which can be used as an 11 at her discretion, she would not use the third card as an 11. She would instead use the Ace as a 1 because if not, then the total would equal 22 causing her to bust. However, if she has a pair totaling to 12 - 21, she does risk busting after receiving a third card. Hence, these intervals of success where created to examine the likelihood of adding a third card while still remaining in the game. Shown in the table below, labeled as Figure 2, are the possible pairs totaling to 12 - 21 along with their Worst Case to Best Case Interval for two to six card decks. The remaining table can be seen as Figure 14 in "IX. Appendices".

Possible Pairs	Intervals of Worst Case to Best Case				
	2 Decks	3 Decks	4 Decks	5 Decks	6 Decks
{A, A}	66.67% - 72.92%	67.57% - 71.62%	68.00% - 71.00%	68.25% - 70.63%	68.42% - 70.39%
{2, (10, J, Q, or K)}	67.71% - 73.96%	68.24% - 72.30%	68.50% - 71.50%	68.65% - 71.03%	68.75% - 70.72%
{3, 9}	66.67% - 72.92%	67.57% - 71.62%	68.00% - 71.00%	68.25% - 70.63%	68.42% - 70.39%
{4, 8}	66.67% - 72.92%	67.57% - 71.62%	68.00% - 71.00%	68.25% - 70.63%	68.42% - 70.39%
{5, 7}	66.67% - 72.92%	67.57% - 71.62%	68.00% - 71.00%	68.25% - 70.63%	68.42% - 70.39%
{6, 6}	66.67% - 72.92%	67.57% - 71.62%	68.00% - 71.00%	68.25% - 70.63%	68.42% - 70.39%
{2, A}	58.33% - 64.58%	59.46% - 63.51%	60.00% - 63.00%	60.32% - 62.70%	60.53% - 62.50%
{3, (10, J, Q, or K)}	59.38% - 65.63%	60.14% - 64.19%	60.50% - 63.50%	60.71% - 63.10%	60.86% - 62.83%
{4, 9}	59.38% - 65.63%	60.14% - 64.19%	60.50% - 63.50%	60.71% - 63.10%	60.86% - 62.83%
{5, 8}	58.33% - 64.58%	59.46% - 63.51%	60.00% - 63.00%	60.32% - 62.70%	60.53% - 62.50%
{6, 7}	58.33% - 64.58%	59.46% - 63.51%	60.00% - 63.00%	60.32% - 62.70%	60.53% - 62.50%
{3, A}	50.00% - 56.25%	51.35% - 55.41%	52.00% - 55.00%	52.38% - 54.76%	52.63% - 54.61%
{4, (10, J, Q, or K)}	51.04% - 57.29%	52.03% - 56.08%	52.50% - 55.50%	52.78% - 55.16%	52.96% - 54.93%
{5, 9}	51.04% - 57.29%	52.03% - 56.08%	52.50% - 55.50%	52.78% - 55.16%	52.96% - 54.93%
{6, 8}	51.04% - 57.29%	52.03% - 56.08%	52.50% - 55.50%	52.78% - 55.16%	52.96% - 54.93%
{7, 7}	50.00% - 56.25%	51.35% - 55.41%	52.00% - 55.00%	52.38% - 54.76%	52.63% - 54.61%
{4, A}	41.67% - 47.92%	43.24% - 47.30%	44.00% - 47.00%	44.44% - 46.83%	44.08% - 46.05%
{5, (10, J, Q, or K)}	42.71% - 48.96%	43.92% - 47.97%	44.50% - 47.50%	44.84% - 47.22%	44.41% - 46.38%
{6, 9}	42.71% - 48.96%	43.92% - 47.97%	44.50% - 47.50%	44.84% - 47.22%	44.41% - 46.38%
{7, 8}	43.75% - 50.00%	44.59% - 48.65%	45.00% - 48.00%	45.24% - 47.62%	44.74% - 46.71%
{5, A}	33.33% - 39.58%	35.14% - 39.19%	36.00% - 39.00%	36.51% - 38.89%	36.84% - 38.82%
{6, (10, J, Q, or K)}	35.42% - 41.67%	36.49% - 40.54%	37.00% - 40.00%	37.30% - 39.68%	37.50% - 39.47%
{7, 9}	35.42% - 41.67%	36.49% - 40.54%	37.00% - 40.00%	37.30% - 39.68%	37.50% - 39.47%
{8, 8}	35.42% - 41.67%	36.49% - 40.54%	37.00% - 40.00%	37.30% - 39.68%	37.50% - 39.47%
{6, A}	26.04% - 32.29%	27.70% - 31.76%	28.50% - 31.50%	28.97% - 31.35%	29.28% - 31.25%
{7, (10, J, Q, or K)}	27.08% - 33.33%	28.38% - 32.43%	29.00% - 32.00%	29.37% - 31.75%	29.61% - 31.58%
{8, 9}	27.08% - 33.33%	28.38% - 32.43%	29.00% - 32.00%	29.37% - 31.75%	29.61% - 31.58%
{7, A}	17.71% - 23.96%	19.59% - 23.65%	20.50% - 23.50%	21.03% - 23.41%	21.38% - 23.36%
{8, (10, J, Q, or K)}	18.75% - 25.00%	20.27% - 24.32%	21.00% - 24.00%	21.43% - 23.81%	21.71% - 23.68%
{9, 9}	18.75% - 25.00%	20.27% - 24.32%	21.00% - 24.00%	21.43% - 23.81%	21.71% - 23.68%
{8, A}	9.38% - 15.63%	11.49% - 15.54%	12.50% - 15.50%	13.10% - 15.48%	13.49% - 15.46%
{9, (10, J, Q, or K)}	10.42% - 16.67%	12.16% - 16.22%	13.00% - 16.00%	13.49% - 15.87%	13.82% - 15.79%
{9, A}	1.04% - 7.29%	3.38% - 7.43%	4.50% - 7.50%	5.16% - 7.54%	5.59% - 7.57%
{(10, J, Q, or K), (10, J, Q, or K)}	2.08% - 8.33%	4.05% - 8.11%	5.00% - 8.00%	5.56% - 7.94%	5.92% - 7.89%
{(10, J, Q, or K), A}	0.00% - 0.00%	0.00% - 0.00%	0.00% - 0.00%	0.00% - 0.00%	0.00% - 0.00%

Figure 2: Intervals of the Worst Case Scenarios to the Best Case Scenarios for Pairs Equaling 12 - 21 when a Third Card Is Introduced

4.3 The Overall Intervals

Since the probability of receiving the initial pairs and the Worst Case to Best Case Scenario Intervals are independent events, they can be multiplied to create an overall total interval. This interval is the Worst Case to Best Case Scenario of receiving the initial two pair along with receiving a third card, while still remaining at 21 or less. The calculations of the pairs {10, 10}, {10, J}, {10, Q}, {10, K}, {J, Q}, {J, K}, or {Q, K} are shown below - since the probabilities for each pair valuing at 20 are the same, {10, 10} is shown for conciseness:

In the Case of 2 Decks:

$$P(\{10,10\}) \cdot \text{WC to BC Interval} = 9.26\% \cdot 2.08\% \text{ to } 9.26\% \cdot 8.33\% = 0.19\% - 0.77\% \quad (16)$$

In the Case of 3 Decks:

$$P(\{10,10\}) \cdot \text{WC to BC Interval} = 9.33\% \cdot 4.05\% \text{ to } 8.11\% \cdot 8.33\% = 0.38\% - 0.75\% \quad (17)$$

In the Case of 4 Decks:

$$P(\{10,10\}) \cdot \text{WC to BC Interval} = 9.36\% \cdot 5.00\% \text{ to } 8.00\% \cdot 8.33\% = 0.46\% - 0.74\% \quad (18)$$

In the Case of 5 Decks:

$$P(\{10,10\}) \cdot \text{WC to BC Interval} = 9.39\% \cdot 5.56\% \text{ to } 7.94\% \cdot 8.33\% = 0.51\% - 0.73\% \quad (19)$$

In the Case of 6 Decks:

$$P(\{10,10\}) \cdot \text{WC to BC Interval} = 9.40\% \cdot 5.92\% \text{ to } 7.89\% \cdot 8.33\% = 0.55\% - 0.73\% \quad (20)$$

The overall total intervals are shown below for pairs equaling 12 - 21 as Figure 3. The full table, including pairs 2 - 11, can be found in "IX. Appendices" as Figure 15.

Possible Pairs	Intervals of Worst Case to Best Case Multiplied by Probability of Receiving Pair				
	2 Decks	3 Decks	4 Decks	5 Decks	6 Decks
{A, A}	0.35% - 0.38%	0.35% - 0.37%	0.36% - 0.37%	0.36% - 0.37%	0.36% - 0.37%
{2, (10, J, Q, or K)}	3.24% - 3.53%	3.26% - 3.46%	3.27% - 3.42%	3.28% - 3.40%	3.29% - 3.38%
{3, 9}	0.80% - 0.87%	0.81% - 0.86%	0.81% - 0.85%	0.82% - 0.84%	0.82% - 0.84%
{4, 8}	0.80% - 0.87%	0.81% - 0.86%	0.81% - 0.85%	0.82% - 0.84%	0.82% - 0.84%
{5, 7}	0.80% - 0.87%	0.81% - 0.86%	0.81% - 0.85%	0.82% - 0.84%	0.82% - 0.84%
{6, 6}	0.35% - 0.38%	0.35% - 0.37%	0.36% - 0.37%	0.36% - 0.37%	0.36% - 0.37%
{2, A}	0.70% - 0.77%	0.71% - 0.76%	0.72% - 0.75%	0.72% - 0.75%	0.72% - 0.75%
{3, (10, J, Q, or K)}	2.84% - 3.14%	2.87% - 3.07%	2.89% - 3.04%	2.90% - 3.02%	2.91% - 3.00%
{4, 9}	0.71% - 0.78%	0.72% - 0.77%	0.72% - 0.76%	0.73% - 0.75%	0.73% - 0.75%
{5, 8}	0.70% - 0.77%	0.71% - 0.76%	0.72% - 0.75%	0.72% - 0.75%	0.72% - 0.75%
{6, 7}	0.70% - 0.77%	0.71% - 0.76%	0.72% - 0.75%	0.72% - 0.75%	0.72% - 0.75%
{3, A}	0.60% - 0.67%	0.61% - 0.66%	0.62% - 0.66%	0.63% - 0.65%	0.63% - 0.65%
{4, (10, J, Q, or K)}	2.44% - 2.74%	2.49% - 2.68%	2.51% - 2.65%	2.52% - 2.64%	2.53% - 2.63%
{5, 9}	0.61% - 0.68%	0.62% - 0.67%	0.63% - 0.66%	0.63% - 0.66%	0.63% - 0.66%
{6, 8}	0.61% - 0.68%	0.62% - 0.67%	0.63% - 0.66%	0.63% - 0.66%	0.63% - 0.66%
{7, 7}	0.26% - 0.29%	0.27% - 0.29%	0.27% - 0.29%	0.27% - 0.29%	0.28% - 0.29%
{4, A}	0.50% - 0.57%	0.52% - 0.57%	0.53% - 0.56%	0.53% - 0.56%	0.53% - 0.55%
{5, (10, J, Q, or K)}	2.04% - 2.34%	2.10% - 2.29%	2.13% - 2.27%	2.14% - 2.26%	2.12% - 2.22%
{6, 9}	0.51% - 0.59%	0.52% - 0.57%	0.53% - 0.57%	0.54% - 0.56%	0.53% - 0.55%
{7, 8}	0.52% - 0.60%	0.53% - 0.58%	0.54% - 0.57%	0.54% - 0.57%	0.53% - 0.56%
{5, A}	0.40% - 0.47%	0.42% - 0.47%	0.43% - 0.47%	0.44% - 0.46%	0.44% - 0.46%
{6, (10, J, Q, or K)}	1.69% - 1.99%	1.74% - 1.94%	1.77% - 1.91%	1.78% - 1.90%	1.79% - 1.89%
{7, 9}	0.42% - 0.50%	0.44% - 0.48%	0.44% - 0.48%	0.45% - 0.47%	0.45% - 0.47%
{8, 8}	0.19% - 0.22%	0.19% - 0.21%	0.19% - 0.21%	0.20% - 0.21%	0.20% - 0.21%
{6, A}	0.31% - 0.39%	0.33% - 0.38%	0.34% - 0.38%	0.35% - 0.37%	0.35% - 0.37%
{7, (10, J, Q, or K)}	1.29% - 1.59%	1.36% - 1.55%	1.39% - 1.53%	1.40% - 1.52%	1.42% - 1.51%
{8, 9}	0.32% - 0.40%	0.34% - 0.39%	0.35% - 0.38%	0.35% - 0.38%	0.35% - 0.38%
{7, A}	0.21% - 0.29%	0.23% - 0.28%	0.24% - 0.28%	0.25% - 0.28%	0.26% - 0.28%
{8, (10, J, Q, or K)}	0.90% - 1.19%	0.97% - 1.16%	1.00% - 1.15%	1.02% - 1.14%	1.04% - 1.13%
{9, 9}	0.10% - 0.13%	0.11% - 0.13%	0.11% - 0.13%	0.11% - 0.12%	0.11% - 0.12%
{8, A}	0.11% - 0.19%	0.14% - 0.19%	0.15% - 0.19%	0.16% - 0.18%	0.16% - 0.18%
{9, (10, J, Q, or K)}	0.50% - 0.80%	0.58% - 0.78%	0.62% - 0.76%	0.64% - 0.76%	0.66% - 0.75%
{9, A}	0.01% - 0.09%	0.04% - 0.09%	0.05% - 0.09%	0.06% - 0.09%	0.07% - 0.09%
{(10, J, Q, or K), (10, J, Q, or K)}	0.19% - 0.77%	0.38% - 0.75%	0.46% - 0.74%	0.51% - 0.73%	0.55% - 0.73%
{(10, J, Q, or K), A}	0.00% - 0.00%	0.00% - 0.00%	0.00% - 0.00%	0.00% - 0.00%	0.00% - 0.00%

Figure 3: Intervals of the Worst Case Scenarios to the Best Case Scenarios Multiplied by the Initial Probability of Receiving Pairs Equaling 12 - 21

5 Results

5.1 The Initial Pairs

Figures 4-7 shown below are the four categories that card pairs are divided by, labeled as Case 1, Case 2, Case 3, and Cases 4. All card pairs from the initial deal fall into one of these four cases using combinatorial math.

Case 1 contains the pairs of the same card with point values not equal to 10 (e.g. {A, A}, {2, 2}, {3, 3}, {4, 4}, {5, 5}, {6, 6}, {7, 7}, {8, 8}, and {9, 9}). As the card decks increase, the probability of receiving the card pairs increase slightly. Although there is an increase, the probabilities from the second to sixth deck only differ by 0.05% or 0.0005. This does not allow the player to realistically benefit from an increased number of card decks, but it is an increase, nonetheless.

Number of Decks	Calculation	Probability
2	$\frac{\binom{8}{2}}{\binom{104}{2}}$	0.52%
3	$\frac{\binom{12}{2}}{\binom{156}{2}}$	0.55%
4	$\frac{\binom{16}{2}}{\binom{208}{2}}$	0.56%
5	$\frac{\binom{20}{2}}{\binom{260}{2}}$	0.56%
6	$\frac{\binom{24}{2}}{\binom{312}{2}}$	0.57%

Figure 4: Case 1 - Pairs of the same card with point values not equal to 10

Case 2 contains the pairs where exactly one card is valued at 10 (e.g. {A, 10 or J or Q or K}, {2, 10 or J or Q or K}, {3, 10 or J or Q or K}, {4, 10 or J or Q or K}, {5, 10 or J or Q or K}, {6, 10 or J or Q or K}, {7, 10 or J or Q or K}, {8, 10 or J or Q or K}, {9, 10 or J or Q or K}.
As the card decks increase, the probability of receiving the card pairs decrease slightly. Although there is an decrease, the probabilities from the second to sixth deck again only differ by 0.05% or 0.0005. This does not allow the casino to realistically benefit from an increased number of card decks, but it is an beneficial decrease for the player.

Number of Decks	Calculation	Probability
2	$\frac{\binom{8}{1}\binom{32}{1}}{\binom{104}{2}}$	4.78%
3	$\frac{\binom{12}{1}\binom{48}{1}}{\binom{156}{2}}$	4.76%
4	$\frac{\binom{16}{1}\binom{64}{1}}{\binom{208}{2}}$	4.76%
5	$\frac{\binom{20}{1}\binom{80}{1}}{\binom{260}{2}}$	4.75%
6	$\frac{\binom{24}{1}\binom{96}{1}}{\binom{312}{2}}$	4.75%

Figure 5: Case 2 - Pairs where exactly one card is valued at 10

Case 3 contains the pairs where both cards value at 10 (e.g. {10, 10}, {10, J}, {10, Q}, {10, K}, {J, Q}, {J, K}, and {Q, K}). As the card decks increase, the probability of receiving the card pairs have the greater increase between itself and Case 1. The probabilities from the second to sixth deck differ by 0.14% or 0.0014, which is almost triple of the increase of Case 1. Although the increase is greater, it still yields too small of a change for the player to take advantage of the increased card deck.

Number of Decks	Calculation	Probability
2	$\frac{\binom{32}{2}}{\binom{104}{2}}$	9.26%
3	$\frac{\binom{48}{2}}{\binom{156}{2}}$	9.33%
4	$\frac{\binom{64}{2}}{\binom{208}{2}}$	9.36%
5	$\frac{\binom{80}{2}}{\binom{260}{2}}$	9.39%
6	$\frac{\binom{96}{2}}{\binom{312}{2}}$	9.40%

Figure 6: Case 3 - Pairs where both cards value at 10

Case 4 contains all of the other pairs possible in the deck (e.g. {A, 2 or 3 or 4 or 5 or 6 or 7 or 8 or 9}, {2, 3 or 4 or 5 or 6 or 7 or 8 or 9}, {3, 4 or 5 or 6 or 7 or 8 or 9}, {4, 5 or 6 or 7 or 8 or 9}, {5, 6 or 7 or 8 or 9}, {6, 7 or 8 or 9}, {7, 8 or 9}, and {8, 9}). As the card decks increase, the probability of receiving the card pairs remains constant. Without any change, neither the casino, nor the player have an upper hand advantage to win.

Number of Decks	Calculation	Probability
2	$\frac{\binom{8}{1}\binom{8}{1}}{\binom{104}{2}}$	1.19%
3	$\frac{\binom{12}{1}\binom{12}{1}}{\binom{156}{2}}$	1.19%
4	$\frac{\binom{16}{1}\binom{16}{1}}{\binom{208}{2}}$	1.19%
5	$\frac{\binom{20}{1}\binom{20}{1}}{\binom{260}{2}}$	1.19%
6	$\frac{\binom{24}{1}\binom{24}{1}}{\binom{312}{2}}$	1.19%

Figure 7: Case 4 - All other pairs

Although Case 1 and 3 increase, Case 2 decreases, and Case 4 remains constant while more decks are added to the game, neither the casino nor the player will gain an excess advantage over the other. This poses the question: “Why would casinos use a higher number of decks when their chances of winning do not vary much between decks?” The answer is in the Worst Case to Best Case Scenario Intervals.

5.2 A Third Card

The following figures are for card pairs equaling 14 points. The tables include the probability of receiving the two cards as well as the interval of success of the best and the worst case scenarios when a third card is introduced for two through six decks. The overall probability of receiving the pair multiplied by the interval is also included.

Hand	Probability of Hand	Interval of Worst Case to Best Case	Probability of Receiving Pair Multipied by Interval
{3, A}	1.19%	50.00% - 56.25%	0.60% - 0.67%
{4, (10, J, Q, or K)}	4.78%	51.04% - 57.29%	2.44% - 2.74%
{5, 9}	1.19%	51.04% - 57.29%	0.61% - 0.68%
{6, 8}	1.19%	51.04% - 57.29%	0.61% - 0.68%
{7, 7}	0.52%	50.00% - 56.25%	0.26% - 0.29%

Figure 8: 2 Decks

Hand	Probability of Hand	Interval of Worst Case to Best Case	Probability of Receiving Pair Multipied by Interval
{3, A}	1.19%	51.35% - 55.41%	0.61% - 0.66%
{4, (10, J, Q, or K)}	4.76%	52.03% - 56.08%	2.48% - 2.67%
{5, 9}	1.19%	52.03% - 56.08%	0.62% - 0.67%
{6, 8}	1.19%	52.03% - 56.08%	0.62% - 0.67%
{7, 7}	0.55%	51.35% - 55.41%	0.28% - 0.30%

Figure 9: 3 Decks

Hand	Probability of Hand	Interval of Worst Case to Best Case	Probability of Receiving Pair Multipied by Interval
{3, A}	1.19%	52.00% - 55.00%	0.62% - 0.65%
{4, (10, J, Q, or K)}	4.76%	52.50% - 55.50%	2.50% - 2.64%
{5, 9}	1.19%	52.50% - 55.50%	0.62% - 0.66%
{6, 8}	1.19%	52.50% - 55.50%	0.62% - 0.66%
{7, 7}	0.56%	52.00% - 55.00%	0.29% - 0.31%

Figure 10: 4 Decks

Hand	Probability of Hand	Interval of Worst Case to Best Case	Probability of Receiving Pair Multipied by Interval
{3, A}	1.19%	52.38% - 54.76%	0.62% - 0.65%
{4, (10, J, Q, or K)}	4.75%	52.78% - 55.16%	2.51% - 2.62%
{5, 9}	1.19%	52.78% - 55.16%	0.63% - 0.66%
{6, 8}	1.19%	52.78% - 55.16%	0.63% - 0.66%
{7, 7}	0.56%	52.38% - 54.76%	0.30% - 0.31%

Figure 11: 5 Decks

Hand	Probability of Hand	Interval of Worst Case to Best Case	Probability of Receiving Pair Multipied by Interval
{3, A}	1.19%	52.63% - 54.61%	0.62% - 0.65%
{4, (10, J, Q, or K)}	4.75%	52.96% - 54.93%	2.52% - 2.61%
{5, 9}	1.19%	52.96% - 54.93%	0.63% - 0.65%
{6, 8}	1.19%	52.96% - 54.93%	0.63% - 0.65%
{7, 7}	0.57%	52.63% - 54.61%	0.30% - 0.31%

Figure 12: 6 Decks

Note: The pair {3, A} can be used as a 14 (Ace as an 11) as well as a 4 (Ace as a 1). Since using the pair as a 4 is further from 21 than using the pair as a 14, players will most likely count a 14. However, in the events of the Worst Case to Best Case, there is a 100% chance the player would not go over 21 while counting the pair as a 4. The chance is flawless because the greatest number card a player can receive is a 10 valued card, which would put the player at a total of 14. Although a player could receive an Ace and value it as an 11 at her discretion, it guaranteed the highest value of cards in the remaining deck is a 10. This same reasoning goes for all cards 2 through 9 paired with an Ace.

5.3 The Overall Intervals

As shown, not only do the values of the worst case and best case get smaller, but the intervals themselves decrease in size also. This decreases the likelihood for a player to receive her desired third card to receive or remain below 21. Again, since the probability of receiving the pair and the probability of receiving the third card are two independent events, they can be multiplied to create an overall interval off success for the player. These overall intervals display the decreased chance per deck with all three cards taken into account. Theses decreases benefit the casinos because player's are less likely to win and lose money, which in turn generates for revenue for the business. The reader can refer to the full table in Figures 14 and 15 in "IX. Appendices".

6 Conclusion

Although the probabilities of receiving the initial pair either increase or decrease slightly or even remain the same, the Worst to Best Case Interval does decrease both in length and value as the decks increase. Because it is typical for players to receive a third card to each 21, casinos are more inclined to use more decks to cause a greater likelihood for players to lose. When players lose a game, consequently they lose their money, which they have bet against the casino, to the dealer. This gives the casinos more money, which keeps them in business.

For future studies, it would be interesting to explore more scenarios. These include exploring seven and eight decks to determine if the probabilities continue along the same trend as decks two though six. Going further, analyzing nine or even ten decks would be interesting to investigate the casinos' purpose in playing with the maximum of eight card decks. With these future studies, it is captivating to understand not only the probabilities of these results, but also identifying casinos' business strategies in order to earn more money. However, with the calculations that have already been investigated, we can strategize both the player's and casino's best chances of winning. As far as deciding on "which side" the readers would want to help win more money, depends on one variable - whichever side is paying them.

7 References

[1] "4-Deck to 8-Deck Blackjack Strategy - Wizard of Odds". Wizard of Odds Consulting, Inc. Retrieved 30 April 2014.

[2] "Chapter 1: 'Let's Count!'." Discrete Mathematics: Elementary and Beyond, by Lovasz L. et al., Springer, 2003, pp. 15–21.

[3] Luketic, Robert, director. 21. Columbia Pictures, 2008.

[4] McLeod, John. "Blackjack." Card Games, 2018, www.pagat.com/banking/blackjack.html.

[5] Moehlis, Jeff. "The Mathematics of the Movie '21.'" The Mathematics of the Movie "21", sites.me.ucsb.edu/ moehlis/21.html.

[6] Probability and Statistics for Engineering and the Sciences, by Jay L. Devore, Cengage Learning, 2016, pp. 1–20.

[7] "Section 2.7 Multiplicative Rules." Probability and Statistics for Engineers and Scientists, by Ronald E. Walpole and Raymond H Myers, Prentice Hall, 1993, pp. 37–37.

[8] Stantic, Zoran. "Probability and Strategy in a Variation of a Blackjack Game." Arxiv, Faculty of Mathematics, University of Belgrade, Apr. 2013, arxiv.org/pdf/1304.5455.pdf.

[9] "Summary of Basic Probability." Sources of Error, www2.southeastern.edu/Academics/Faculty/dgurney/Math241/StatTopics/Bas%20Prob.htm.

[10] "The Wizard of Odds". Fine points of basic strategy in single-deck blackjack. Retrieved December 8, 2006.

8 Appendices

8.1 Appendix A: The Initial Pairs

Value of Two Cards	Possible Pairs	Probability of Receiving Pair				
		2 Decks	3 Decks	4 Decks	5 Decks	6 Decks
2	{A, A}	0.52%	0.55%	0.56%	0.56%	0.57%
3	{A, 2}	1.19%	1.19%	1.19%	1.19%	1.19%
4	{A, 3}	1.19%	1.19%	1.19%	1.19%	1.19%
	{2, 2}	0.52%	0.55%	0.56%	0.56%	0.57%
5	{A, 4}	1.19%	1.19%	1.19%	1.19%	1.19%
	{2, 3}	1.19%	1.19%	1.19%	1.19%	1.19%
6	{A, 5}	1.19%	1.19%	1.19%	1.19%	1.19%
	{2, 4}	1.19%	1.19%	1.19%	1.19%	1.19%
	{3, 3}	0.52%	0.55%	0.56%	0.56%	0.57%
7	{A, 6}	1.19%	1.19%	1.19%	1.19%	1.19%
	{2, 5}	1.19%	1.19%	1.19%	1.19%	1.19%
	{3, 4}	1.19%	1.19%	1.19%	1.19%	1.19%
8	{A, 7}	1.19%	1.19%	1.19%	1.19%	1.19%
	{2, 6}	1.19%	1.19%	1.19%	1.19%	1.19%
	{3, 5}	1.19%	1.19%	1.19%	1.19%	1.19%
	{4, 4}	0.52%	0.55%	0.56%	0.56%	0.57%
9	{A, 8}	1.19%	1.19%	1.19%	1.19%	1.19%
	{2, 7}	1.19%	1.19%	1.19%	1.19%	1.19%
	{3, 6}	1.19%	1.19%	1.19%	1.19%	1.19%
	{4, 5}	1.19%	1.19%	1.19%	1.19%	1.19%
10	{A, 9}	1.19%	1.19%	1.19%	1.19%	1.19%
	{2, 8}	1.19%	1.19%	1.19%	1.19%	1.19%
	{3, 7}	1.19%	1.19%	1.19%	1.19%	1.19%
	{4, 6}	1.19%	1.19%	1.19%	1.19%	1.19%
	{5, 5}	0.52%	0.55%	0.56%	0.56%	0.57%
11	{A, (10, J, Q, or K)}	----------	----------	----------	----------	----------
	{9, 2}	1.19%	1.19%	1.19%	1.19%	1.19%
	{8, 3}	1.19%	1.19%	1.19%	1.19%	1.19%
	{7, 4}	1.19%	1.19%	1.19%	1.19%	1.19%
	{6, 5}	1.19%	1.19%	1.19%	1.19%	1.19%
12	{A, A}	0.52%	0.55%	0.56%	0.56%	0.57%
	{2, (10, J, Q, or K)}	4.78%	4.764%	4.757%	4.75%	4.75%
	{3, 9}	1.19%	1.19%	1.19%	1.19%	1.19%
	{4, 8}	1.19%	1.19%	1.19%	1.19%	1.19%
	{5, 7}	1.19%	1.19%	1.19%	1.19%	1.19%
	{6, 6}	0.52%	0.55%	0.56%	0.56%	0.57%
13	{2, A}	1.19%	1.19%	1.19%	1.19%	1.19%
	{3, (10, J, Q, or K)}	4.78%	4.76%	4.76%	4.75%	4.75%
	{4, 9}	1.19%	1.19%	1.19%	1.19%	1.19%
	{5, 8}	1.19%	1.19%	1.19%	1.19%	1.19%
	{6, 7}	1.19%	1.19%	1.19%	1.19%	1.19%

Value of Two Cards	Possible Pairs	Probability of Receiving Pair				
		2 Decks	3 Decks	4 Decks	5 Decks	6 Decks
14	{3, A}	1.19%	1.19%	1.19%	1.19%	1.19%
	{4, (10, J, Q, or K)}	4.78%	4.76%	4.76%	4.75%	4.75%
	{5, 9}	1.19%	1.19%	1.19%	1.19%	1.19%
	{6, 8}	1.19%	1.19%	1.19%	1.19%	1.19%
	{7, 7}	0.52%	0.55%	0.56%	0.56%	0.57%
15	{4, A}	1.19%	1.19%	1.19%	1.19%	1.19%
	{5, (10, J, Q, or K)}	4.78%	4.76%	4.76%	4.75%	4.75%
	{6, 9}	1.19%	1.19%	1.19%	1.19%	1.19%
	{7, 8}	1.19%	1.19%	1.19%	1.19%	1.19%
16	{5, A}	1.19%	1.19%	1.19%	1.19%	1.19%
	{6, (10, J, Q, or K)}	4.78%	4.76%	4.76%	4.75%	4.75%
	{7, 9}	1.19%	1.19%	1.19%	1.19%	1.19%
	{8, 8}	0.52%	0.55%	0.56%	0.56%	0.57%
17	{6, A}	1.19%	1.19%	1.19%	1.19%	1.19%
	{7, (10, J, Q, or K)}	4.78%	4.76%	4.76%	4.75%	4.75%
	{8, 9}	1.19%	1.19%	1.19%	1.19%	1.19%
18	{7, A}	1.19%	1.19%	1.19%	1.19%	1.19%
	{8, (10, J, Q, or K)}	4.78%	4.76%	4.76%	4.75%	4.75%
	{9, 9}	0.52%	0.55%	0.56%	0.56%	0.57%
19	{8, A}	1.19%	1.19%	1.19%	1.19%	1.19%
	{9, (10, J, Q, or K)}	4.78%	4.76%	4.76%	4.75%	4.75%
20	{9, A}	1.19%	1.19%	1.19%	1.19%	1.19%
	{(10, J, Q, or K), (10, J, Q, or K)}	9.26%	9.33%	9.36%	9.39%	9.40%
21	{(10, J, Q, or K), A}	4.78%	4.76%	4.76%	4.75%	4.75%

Figure 13: Probability of Receiving Pairs with Values of 2 - 21

Note 1: The pair {A, 10 or J or Q or K} would not be used as 11 points; it would be used as 21 to win - thus, the dashed lines for receiving the pair. Probabilities of this pair can be seen under the point value ”21”.

Note 2: Adding all of the probabilities of receiving the possible pairs would give a total that exceeds 100%. Pairs involving Aces were counted twice because they can be used as either a 1 or 11. If the duplicate pairs are removed from the sum of possible pair probabilities, the total would equal 100%.

8.2 Appendix B: A Third Card

Possible Pairs	Intervals of Worst Case to Best Case				
	2 Decks	3 Decks	4 Decks	5 Decks	6 Decks
{A, A}	100.00% - 100.00%	100.00% - 100.00%	100.00% - 100.00%	100.00% - 100.00%	100.00% - 100.00%
{A, 2}	100.00% - 100.00%	100.00% - 100.00%	100.00% - 100.00%	100.00% - 100.00%	100.00% - 100.00%
{A, 3}	100.00% - 100.00%	100.00% - 100.00%	100.00% - 100.00%	100.00% - 100.00%	100.00% - 100.00%
{2, 2}	100.00% - 100.00%	100.00% - 100.00%	100.00% - 100.00%	100.00% - 100.00%	100.00% - 100.00%
{A, 4}	100.00% - 100.00%	100.00% - 100.00%	100.00% - 100.00%	100.00% - 100.00%	100.00% - 100.00%
{2, 3}	100.00% - 100.00%	100.00% - 100.00%	100.00% - 100.00%	100.00% - 100.00%	100.00% - 100.00%
{A, 5}	100.00% - 100.00%	100.00% - 100.00%	100.00% - 100.00%	100.00% - 100.00%	100.00% - 100.00%
{2, 4}	100.00% - 100.00%	100.00% - 100.00%	100.00% - 100.00%	100.00% - 100.00%	100.00% - 100.00%
{3, 3}	100.00% - 100.00%	100.00% - 100.00%	100.00% - 100.00%	100.00% - 100.00%	100.00% - 100.00%
{A, 6}	100.00% - 100.00%	100.00% - 100.00%	100.00% - 100.00%	100.00% - 100.00%	100.00% - 100.00%
{2, 5}	100.00% - 100.00%	100.00% - 100.00%	100.00% - 100.00%	100.00% - 100.00%	100.00% - 100.00%
{3, 4}	100.00% - 100.00%	100.00% - 100.00%	100.00% - 100.00%	100.00% - 100.00%	100.00% - 100.00%
{A, 7}	100.00% - 100.00%	100.00% - 100.00%	100.00% - 100.00%	100.00% - 100.00%	100.00% - 100.00%
{2, 6}	100.00% - 100.00%	100.00% - 100.00%	100.00% - 100.00%	100.00% - 100.00%	100.00% - 100.00%
{3, 5}	100.00% - 100.00%	100.00% - 100.00%	100.00% - 100.00%	100.00% - 100.00%	100.00% - 100.00%
{4, 4}	100.00% - 100.00%	100.00% - 100.00%	100.00% - 100.00%	100.00% - 100.00%	100.00% - 100.00%
{A, 8}	100.00% - 100.00%	100.00% - 100.00%	100.00% - 100.00%	100.00% - 100.00%	100.00% - 100.00%
{2, 7}	100.00% - 100.00%	100.00% - 100.00%	100.00% - 100.00%	100.00% - 100.00%	100.00% - 100.00%
{3, 6}	100.00% - 100.00%	100.00% - 100.00%	100.00% - 100.00%	100.00% - 100.00%	100.00% - 100.00%
{4, 5}	100.00% - 100.00%	100.00% - 100.00%	100.00% - 100.00%	100.00% - 100.00%	100.00% - 100.00%
{A, 9}	100.00% - 100.00%	100.00% - 100.00%	100.00% - 100.00%	100.00% - 100.00%	100.00% - 100.00%
{2, 8}	100.00% - 100.00%	100.00% - 100.00%	100.00% - 100.00%	100.00% - 100.00%	100.00% - 100.00%
{3, 7}	100.00% - 100.00%	100.00% - 100.00%	100.00% - 100.00%	100.00% - 100.00%	100.00% - 100.00%
{4, 6}	100.00% - 100.00%	100.00% - 100.00%	100.00% - 100.00%	100.00% - 100.00%	100.00% - 100.00%
{5, 5}	100.00% - 100.00%	100.00% - 100.00%	100.00% - 100.00%	100.00% - 100.00%	100.00% - 100.00%
{A, (10, J, Q, or K)}	----------- -----------	----------- - -----------	----------- - -----------	----------- - -----------	----------- - -----------
{9, 2}	100.00% - 100.00%	100.00% - 100.00%	100.00% - 100.00%	100.00% - 100.00%	100.00% - 100.00%
{8, 3}	100.00% - 100.00%	100.00% - 100.00%	100.00% - 100.00%	100.00% - 100.00%	100.00% - 100.00%
{7, 4}	100.00% - 100.00%	100.00% - 100.00%	100.00% - 100.00%	100.00% - 100.00%	100.00% - 100.00%
{6, 5}	100.00% - 100.00%	100.00% - 100.00%	100.00% - 100.00%	100.00% - 100.00%	100.00% - 100.00%
{A, A}	66.67% - 72.92%	67.57% - 71.62%	68.00% - 71.00%	68.25% - 70.63%	68.42% - 70.39%
{2, (10, J, Q, or K)}	67.71% - 73.96%	68.24% - 72.30%	68.50% - 71.50%	68.65% - 71.03%	68.75% - 70.72%
{3, 9}	66.67% - 72.92%	67.57% - 71.62%	68.00% - 71.00%	68.25% - 70.63%	68.42% - 70.39%
{4, 8}	66.67% - 72.92%	67.57% - 71.62%	68.00% - 71.00%	68.25% - 70.63%	68.42% - 70.39%
{5, 7}	66.67% - 72.92%	67.57% - 71.62%	68.00% - 71.00%	68.25% - 70.63%	68.42% - 70.39%
{6, 6}	66.67% - 72.92%	67.57% - 71.62%	68.00% - 71.00%	68.25% - 70.63%	68.42% - 70.39%
{2, A}	58.33% - 64.58%	59.46% - 63.51%	60.00% - 63.00%	60.32% - 62.70%	60.53% - 62.50%
{3, (10, J, Q, or K)}	59.38% - 65.63%	60.14% - 64.19%	60.50% - 63.50%	60.71% - 63.10%	60.86% - 62.83%
{4, 9}	59.38% - 65.63%	60.14% - 64.19%	60.50% - 63.50%	60.71% - 63.10%	60.86% - 62.83%
{5, 8}	58.33% - 64.58%	59.46% - 63.51%	60.00% - 63.00%	60.32% - 62.70%	60.53% - 62.50%
{6, 7}	58.33% - 64.58%	59.46% - 63.51%	60.00% - 63.00%	60.32% - 62.70%	60.53% - 62.50%

Possible Pairs	Intervals of Worst Case to Best Case				
	2 Decks	3 Decks	4 Decks	5 Decks	6 Decks
{3, A}	50.00% - 56.25%	51.35% - 55.41%	52.00% - 55.00%	52.38% - 54.76%	52.63% - 54.61%
{4, (10, J, Q, or K)}	51.04% - 57.29%	52.03% - 56.08%	52.50% - 55.50%	52.78% - 55.16%	52.96% - 54.93%
{5, 9}	51.04% - 57.29%	52.03% - 56.08%	52.50% - 55.50%	52.78% - 55.16%	52.96% - 54.93%
{6, 8}	51.04% - 57.29%	52.03% - 56.08%	52.50% - 55.50%	52.78% - 55.16%	52.96% - 54.93%
{7, 7}	50.00% - 56.25%	51.35% - 55.41%	52.00% - 55.00%	52.38% - 54.76%	52.63% - 54.61%
{4, A}	41.67% - 47.92%	43.24% - 47.30%	44.00% - 47.00%	44.44% - 46.83%	44.08% - 46.05%
{5, (10, J, Q, or K)}	42.71% - 48.96%	43.92% - 47.97%	44.50% - 47.50%	44.84% - 47.22%	44.41% - 46.38%
{6, 9}	42.71% - 48.96%	43.92% - 47.97%	44.50% - 47.50%	44.84% - 47.22%	44.41% - 46.38%
{7, 8}	43.75% - 50.00%	44.59% - 48.65%	45.00% - 48.00%	45.24% - 47.62%	44.74% - 46.71%
{5, A}	33.33% - 39.58%	35.14% - 39.19%	36.00% - 39.00%	36.51% - 38.89%	36.84% - 38.82%
{6, (10, J, Q, or K)}	35.42% - 41.67%	36.49% - 40.54%	37.00% - 40.00%	37.30% - 39.68%	37.50% - 39.47%
{7, 9}	35.42% - 41.67%	36.49% - 40.54%	37.00% - 40.00%	37.30% - 39.68%	37.50% - 39.47%
{8, 8}	35.42% - 41.67%	36.49% - 40.54%	37.00% - 40.00%	37.30% - 39.68%	37.50% - 39.47%
{6, A}	26.04% - 32.29%	27.70% - 31.76%	28.50% - 31.50%	28.97% - 31.35%	29.28% - 31.25%
{7, (10, J, Q, or K)}	27.08% - 33.33%	28.38% - 32.43%	29.00% - 32.00%	29.37% - 31.75%	29.61% - 31.58%
{8, 9}	27.08% - 33.33%	28.38% - 32.43%	29.00% - 32.00%	29.37% - 31.75%	29.61% - 31.58%
{7, A}	17.71% - 23.96%	19.59% - 23.65%	20.50% - 23.50%	21.03% - 23.41%	21.38% - 23.36%
{8, (10, J, Q, or K)}	18.75% - 25.00%	20.27% - 24.32%	21.00% - 24.00%	21.43% - 23.81%	21.71% - 23.68%
{9, 9}	18.75% - 25.00%	20.27% - 24.32%	21.00% - 24.00%	21.43% - 23.81%	21.71% - 23.68%
{8, A}	9.38% - 15.63%	11.49% - 15.54%	12.50% - 15.50%	13.10% - 15.48%	13.49% - 15.46%
{9, (10, J, Q, or K)}	10.42% - 16.67%	12.16% - 16.22%	13.00% - 16.00%	13.49% - 15.87%	13.82% - 15.79%
{9, A}	1.04% - 7.29%	3.38% - 7.43%	4.50% - 7.50%	5.16% - 7.54%	5.59% - 7.57%
{(10, J, Q, or K), (10, J, Q, or K)}	2.08% - 8.33%	4.05% - 8.11%	5.00% - 8.00%	5.56% - 7.94%	5.92% - 7.89%
{(10, J, Q, or K), A}	0.00% - 0.00%	0.00% - 0.00%	0.00% - 0.00%	0.00% - 0.00%	0.00% - 0.00%

Figure 14: The Worst Case to the Best Case Scenario Intervals for Receiving a Desired Third Card to Not Exceed 21, Given the Initial Card Pairs

8.3 Appendix C: The Overall Intervals

Value of Two Cards	Possible Pairs	Intervals of Worst Case to Best Case Multiplied by Probability of Receiving Pair				
		2 Decks	3 Decks	4 Decks	5 Decks	6 Decks
2	{A, A}	0.52% - 0.52%	0.52% - 0.52%	0.52% - 0.52%	0.52% - 0.52%	0.52% - 0.52%
3	{A, 2}	1.19% - 1.19%	1.19% - 1.19%	1.19% - 1.19%	1.19% - 1.19%	1.19% - 1.19%
4	{A, 3}	1.19% - 1.19%	1.19% - 1.19%	1.19% - 1.19%	1.19% - 1.19%	1.19% - 1.19%
	{2, 2}	0.52% - 0.52%	0.52% - 0.52%	0.52% - 0.52%	0.52% - 0.52%	0.52% - 0.52%
5	{A, 4}	1.19% - 1.19%	1.19% - 1.19%	1.19% - 1.19%	1.19% - 1.19%	1.19% - 1.19%
	{2, 3}	1.19% - 1.19%	1.19% - 1.19%	1.19% - 1.19%	1.19% - 1.19%	1.19% - 1.19%
6	{A, 5}	1.19% - 1.19%	1.19% - 1.19%	1.19% - 1.19%	1.19% - 1.19%	1.19% - 1.19%
	{2, 4}	1.19% - 1.19%	1.19% - 1.19%	1.19% - 1.19%	1.19% - 1.19%	1.19% - 1.19%
	{3, 3}	0.52% - 0.52%	0.52% - 0.52%	0.52% - 0.52%	0.52% - 0.52%	0.52% - 0.52%
7	{A, 6}	1.19% - 1.19%	1.19% - 1.19%	1.19% - 1.19%	1.19% - 1.19%	1.19% - 1.19%
	{2, 5}	1.19% - 1.19%	1.19% - 1.19%	1.19% - 1.19%	1.19% - 1.19%	1.19% - 1.19%
	{3, 4}	1.19% - 1.19%	1.19% - 1.19%	1.19% - 1.19%	1.19% - 1.19%	1.19% - 1.19%
8	{A, 7}	1.19% - 1.19%	1.19% - 1.19%	1.19% - 1.19%	1.19% - 1.19%	1.19% - 1.19%
	{2, 6}	1.19% - 1.19%	1.19% - 1.19%	1.19% - 1.19%	1.19% - 1.19%	1.19% - 1.19%
	{3, 5}	1.19% - 1.19%	1.19% - 1.19%	1.19% - 1.19%	1.19% - 1.19%	1.19% - 1.19%
	{4, 4}	0.52% - 0.52%	0.52% - 0.52%	0.52% - 0.52%	0.52% - 0.52%	0.52% - 0.52%
9	{A, 8}	1.19% - 1.19%	1.19% - 1.19%	1.19% - 1.19%	1.19% - 1.19%	1.19% - 1.19%
	{2, 7}	1.19% - 1.19%	1.19% - 1.19%	1.19% - 1.19%	1.19% - 1.19%	1.19% - 1.19%
	{3, 6}	1.19% - 1.19%	1.19% - 1.19%	1.19% - 1.19%	1.19% - 1.19%	1.19% - 1.19%
	{4, 5}	1.19% - 1.19%	1.19% - 1.19%	1.19% - 1.19%	1.19% - 1.19%	1.19% - 1.19%
10	{A, 9}	1.19% - 1.19%	1.19% - 1.19%	1.19% - 1.19%	1.19% - 1.19%	1.19% - 1.19%
	{2, 8}	1.19% - 1.19%	1.19% - 1.19%	1.19% - 1.19%	1.19% - 1.19%	1.19% - 1.19%
	{3, 7}	1.19% - 1.19%	1.19% - 1.19%	1.19% - 1.19%	1.19% - 1.19%	1.19% - 1.19%
	{4, 6}	1.19% - 1.19%	1.19% - 1.19%	1.19% - 1.19%	1.19% - 1.19%	1.19% - 1.19%
	{5, 5}	0.52% - 0.52%	0.52% - 0.52%	0.52% - 0.52%	0.52% - 0.52%	0.52% - 0.52%
11	{A, (10, J, Q, or K)}	-------- - --------	-------- - --------	-------- - --------	-------- - --------	-------- - --------
	{9, 2}	1.19% - 1.19%	1.19% - 1.19%	1.19% - 1.19%	1.19% - 1.19%	1.19% - 1.19%
	{8, 3}	1.19% - 1.19%	1.19% - 1.19%	1.19% - 1.19%	1.19% - 1.19%	1.19% - 1.19%
	{7, 4}	1.19% - 1.19%	1.19% - 1.19%	1.19% - 1.19%	1.19% - 1.19%	1.19% - 1.19%
	{6, 5}	1.19% - 1.19%	1.19% - 1.19%	1.19% - 1.19%	1.19% - 1.19%	1.19% - 1.19%
12	{A, A}	0.35% - 0.38%	0.35% - 0.37%	0.36% - 0.37%	0.36% - 0.37%	0.36% - 0.37%
	{2, (10, J, Q, or K)}	3.24% - 3.53%	3.26% - 3.46%	3.27% - 3.42%	3.28% - 3.40%	3.29% - 3.38%
	{3, 9}	0.80% - 0.87%	0.81% - 0.86%	0.81% - 0.85%	0.82% - 0.84%	0.82% - 0.84%
	{4, 8}	0.80% - 0.87%	0.81% - 0.86%	0.81% - 0.85%	0.82% - 0.84%	0.82% - 0.84%
	{5, 7}	0.80% - 0.87%	0.81% - 0.86%	0.81% - 0.85%	0.82% - 0.84%	0.82% - 0.84%
	{6, 6}	0.35% - 0.38%	0.35% - 0.37%	0.36% - 0.37%	0.36% - 0.37%	0.36% - 0.37%
13	{2, A}	0.70% - 0.77%	0.71% - 0.76%	0.72% - 0.75%	0.72% - 0.75%	0.72% - 0.75%
	{3, (10, J, Q, or K)}	2.84% - 3.14%	2.87% - 3.07%	2.89% - 3.04%	2.90% - 3.02%	2.91% - 3.00%
	{4, 9}	0.71% - 0.78%	0.72% - 0.77%	0.72% - 0.76%	0.73% - 0.75%	0.73% - 0.75%
	{5, 8}	0.70% - 0.77%	0.71% - 0.76%	0.72% - 0.75%	0.72% - 0.75%	0.72% - 0.75%
	{6, 7}	0.70% - 0.77%	0.71% - 0.76%	0.72% - 0.75%	0.72% - 0.75%	0.72% - 0.75%

Value of Two Cards	Possible Pairs	Intervals of Worst Case to Best Case Multiplied by Probability of Receiving Pair				
		2 Decks	3 Decks	4 Decks	5 Decks	6 Decks
14	{3, A}	0.60% - 0.67%	0.61% - 0.66%	0.62% - 0.66%	0.63% - 0.65%	0.63% - 0.65%
	{4, (10, J, Q, or K)}	2.44% - 2.74%	2.49% - 2.68%	2.51% - 2.65%	2.52% - 2.64%	2.53% - 2.63%
	{5, 9}	0.61% - 0.68%	0.62% - 0.67%	0.63% - 0.66%	0.63% - 0.66%	0.63% - 0.66%
	{6, 8}	0.61% - 0.68%	0.62% - 0.67%	0.63% - 0.66%	0.63% - 0.66%	0.63% - 0.66%
	{7, 7}	0.26% - 0.29%	0.27% - 0.29%	0.27% - 0.29%	0.27% - 0.29%	0.28% - 0.29%
15	{4, A}	0.50% - 0.57%	0.52% - 0.57%	0.53% - 0.56%	0.53% - 0.56%	0.53% - 0.55%
	{5, (10, J, Q, or K)}	2.04% - 2.34%	2.10% - 2.29%	2.13% - 2.27%	2.14% - 2.26%	2.12% - 2.22%
	{6, 9}	0.51% - 0.59%	0.52% - 0.57%	0.53% - 0.57%	0.54% - 0.56%	0.53% - 0.55%
	{7, 8}	0.52% - 0.60%	0.53% - 0.58%	0.54% - 0.57%	0.54% - 0.57%	0.53% - 0.56%
16	{5, A}	0.40% - 0.47%	0.42% - 0.47%	0.43% - 0.47%	0.44% - 0.46%	0.44% - 0.46%
	{6, (10, J, Q, or K)}	1.69% - 1.99%	1.74% - 1.94%	1.77% - 1.91%	1.78% - 1.90%	1.79% - 1.89%
	{7, 9}	0.42% - 0.50%	0.44% - 0.48%	0.44% - 0.48%	0.45% - 0.47%	0.45% - 0.47%
	{8, 8}	0.19% - 0.22%	0.19% - 0.21%	0.19% - 0.21%	0.20% - 0.21%	0.20% - 0.21%
17	{6, A}	0.31% - 0.39%	0.33% - 0.38%	0.34% - 0.38%	0.35% - 0.37%	0.35% - 0.37%
	{7, (10, J, Q, or K)}	1.29% - 1.59%	1.36% - 1.55%	1.39% - 1.53%	1.40% - 1.52%	1.42% - 1.51%
	{8, 9}	0.32% - 0.40%	0.34% - 0.39%	0.35% - 0.38%	0.35% - 0.38%	0.35% - 0.38%
18	{7, A}	0.21% - 0.29%	0.23% - 0.28%	0.24% - 0.28%	0.25% - 0.28%	0.26% - 0.28%
	{8, (10, J, Q, or K)}	0.90% - 1.19%	0.97% - 1.16%	1.00% - 1.15%	1.02% - 1.14%	1.04% - 1.13%
	{9, 9}	0.10% - 0.13%	0.11% - 0.13%	0.11% - 0.13%	0.11% - 0.12%	0.11% - 0.12%
19	{8, A}	0.11% - 0.19%	0.14% - 0.19%	0.15% - 0.19%	0.16% - 0.18%	0.16% - 0.18%
	{9, (10, J, Q, or K)}	0.50% - 0.80%	0.58% - 0.78%	0.62% - 0.76%	0.64% - 0.76%	0.66% - 0.75%
20	{9, A}	0.01% - 0.09%	0.04% - 0.09%	0.05% - 0.09%	0.06% - 0.09%	0.07% - 0.09%
	{(10, J, Q, or K), (10, J, Q, or K)}	0.19% - 0.77%	0.38% - 0.75%	0.46% - 0.74%	0.51% - 0.73%	0.55% - 0.73%
21	{(10, J, Q, or K), A}	0.00% - 0.00%	0.00% - 0.00%	0.00% - 0.00%	0.00% - 0.00%	0.00% - 0.00%

Figure 15: The Worst Case to the Best Case Scenario Intervals Multiplied by the Probability of Receiving Pairs