Fall 11-11-2018

A Complete Characterization of Near Outer-Planar Graphs

Tanya Allen Lueder genannt Luehr

A COMPLETE CHARACTERIZATION OF NEAR OUTER-PLANAR GRAPHS

by

Tanya Allen Lueder genannt Luehr, BS, MS

A Dissertation Presented in Partial Fulfillment
of the Requirements for the Degree
Doctor of Philosophy of Computational Analysis & Modeling

COLLEGE OF ENGINEERING AND SCIENCE
LOUISIANA TECH UNIVERSITY

November 2018

Replace this page with the Signature Page.

ABSTRACT

A graph is *outer-planar* (OP) if it has a plane embedding in which all of the vertices lie on the boundary of the outer face. A graph is *near outer-planar* (NOP) if it is edgeless or has an edge whose deletion results in an outer-planar graph. An edge of a non outer-planar graph whose removal results in an outer-planar graph is a *vulnerable edge*. This dissertation focuses on near outer-planar (NOP) graphs. We describe the class of all such graphs in terms of a finite list of excluded graphs, in a manner similar to the well-known Kuratowski Theorem for planar graphs. The class of NOP graphs is not closed by the minor relation, and the list of minimal excluded NOP graphs is not finite by the topological minor relation. Instead, we use the domination relation to define minimal excluded near outer-planar graphs, or XNOP graphs. To complete the list of 58 XNOP graphs, we give a description of those members of this list that dominate W_3 or W_4, wheels with three and four spokes, respectively.

To do this, we introduce the concepts of skeletons, joints and limbs. We find an infinite list of possible skeletons of XNOP graphs, as well as a finite list of possible limbs. With the list of skeletons, we permute the edges of a skeleton with the finite list of limbs to find the complete list of XNOP graphs. In this process, we also develop algorithms in SageMath to prove the list of full-K^4 XNOP graphs and prove that the list of skeletons of XNOP graphs is finite.

Replace this page with the approval for scholarly dissemination form.

DEDICATION

It is dedicated to Markus Lueder, Hayden Lueder, Aaron Lueder, Dr. Charles Allen, and Susan Allen for their support and encouragement.

TABLE OF CONTENTS

LIST OF FIGURES

ACKNOWLEDGMENTS

This dissertation would not be possible without several contributions. It is a pleasure to thank Dr. Jinko Kanno and Dr. Bogdan Oporowski for their years of direction in my education as well as this project. Dr. Oporowski mentored me from start to finish and put up with many years of downtime. Dr. Kanno has encouraged me in directions that I needed and has helped me to grow in so many ways. I would not have been able to complete this task without their complete belief in me.

I also would like to thank Dr. Ben Choi, Dr. Pradeep Chowriappa, Dr. Richard Greechie, Dr. Songming Hou and Dr. Galen Turner for their time and knowledge in serving on my dissertation committee. Having many minds from different fields as made this dissertation into a better product.

A special thanks to Dr. Stacey McAdams for her support and friendship during my time at Louisiana Tech University. Also, I could not have finished this program without the understanding and help of my LSUA family, especially Dr. Nathan Ponder and Renee West.

There are not enough words to express the appreciation of my family. They supported my dream in whatever ways they could. My husband, as well as my parents and sister, shouldered many of my responsibilities while I worked on courses and research. I hope that one day I am able to help support our sons in a similar way.

CHAPTER 1

INTRODUCTION

1.1 Overview

In this dissertation, the graphs are undirected and finite. In a series of over twenty papers, spanning over twenty years, Robertson and Seymour developed the Graph Minor Theorem, a major result in the field of graph theory. Within this body of work, they proved the following theorem, known as Wagner's Conjecture.

Theorem 1.1. ([12]) *Every infinite set of finite graphs contains two graphs, such that one is a minor of the other.*

This theorem tells us that if a class of graphs is closed under the minor relation, then we can characterize the class by a finite number of excluded minor minimal graphs. We call this finite set of graphs an *excluded minor* list.

In this dissertation, our goal is to find such a characterization of near outer-planar (NOP) graphs, or graphs that contain an edge whose deletion results in an outer-planar graph. We want to characterize the class of NOP graphs by finding an excluded list of graphs that are not NOP. The following theorem and corollary motivate our quest.

Theorem 1.2. ([9], [13]) *A graph is planar if and only if it does not contain K^5 or $K_{3,3}$ as a topological minor.*

The following corollary can be easily derived from this theorem.

Corollary 1.3. ([4]) *A graph is outer-planar if and only if it does not contain* K^4 *or* $K_{2,3}$ *as a topological minor.*

This theorem and corollary are not just cornerstones in most graph theory texts, but the main examples given in most discussions of the Graph Minor Theorem. It is amazing that an entire class of graphs (planar or outer-planar) can be described by excluding exactly two graphs. This has inspired us to find a similar class of graphs in a minimal way. We use a modification of Theorem 1.2 and Corollary 1.3 to find a finite list of minimal excluded near outer-planar graphs. Our main theorem is the following:

Theorem 1.4. *A graph is near outer-planar if and only if it does not dominate one of the following 58 graphs:* $K_{3,3}$, D_1, D_2, D_3, CV_1, CV_2, CV_3, CV_4, CV_5, CV_6, $K_{2,4}$, S_1, S_2, S_3, S_4, S_5, S_6, DE_1, KF_{1A}, KF_{1B}, KF_{2A}, KF_{2B}, KF_{3A}, KF_{3B}, KF_{3C}, KF_{3D}, KF_{3E}, KF_{3F}, KF_{3G}, KF_{3H}, KF_{4A}, KF_{4B}, KF_{4C}, KF_{4D}, KF_{4E}, KF_{4F}, KF_{5A}, KF_{5B}, KF_{5C}, KF_{5D}, KF_{5E}, KF_{6A}, KF_{6B}, KF_{6C}, DE_2, WF_1, WF_2, WF_3, TP_1, TP_2, TP_3, TP_4, TP_5, TP_6, $CUBE$, DH_1, $K_5\backslash e$, $CUBE/e$.

1.2 Definitions

We base our terminology on [14]. A graph G is a triple $(V, E, \mathfrak{I})$ where V is a set whose elements are called *vertices*; E is a set disjoint from V whose elements are called *edges*; and $\mathfrak{I}$, called the *incidence relation*, is a subset of $V \times E$ in which each edge is in relation with exactly two distinct vertices, u and v, called its *endpoints*. Thus, we exclude loops in this dissertation. The edge e with endpoints u and v is sometimes written uv. If vertex u is an endpoint of edge e, then u and e are *incident*. Two vertices that are connected by an edge are *adjacent*. Likewise, two edges that are connected by a vertex are *adjacent*. The *degree* of a vertex is the number of edges incident to the vertex. The number of vertices of a graph G is the *order* of G and is indicated by $|V(G)|$ or $|G|$. If two edges are incident to the same pair of vertices, then we call them *parallel edges*. A graph without parallel edges is called a *simple graph*. The *simplification* of a graph G is the graph that results in deleting the least number of edges from G such that the resulting graph has no loops or parallel edges. A graph H is a *subgraph* of G if $V(H) \subseteq V(G)$, $E(H) \subseteq E(G)$, and $\mathcal{I}(H) \subseteq \mathcal{I}(G)$. A graph H is a *proper subgraph* of G if $H \subset G$ and $H \neq G$. If H is a proper subgraph of G, then G is a *supergraph* of H. If $H \subseteq G$ and H contains all the edges of G whose endpoints belong to $V(H)$, then H is an *induced subgraph* of G. A subgraph of given graph G is *maximal* for a particular property if it has that property but no other supergraph of it that is also a subgraph of G also has the same property.

A *trail* is a sequence $v_0, e_0, v_1, e_1, \ldots, e_n, v_n$ where each edge, e_i, is incident with vertices, v_i and v_{i+1}, and no edge is repeated. A *path* is a trail with no repeated vertices. The *length* of a path is the number of edges it contains. The first and the last vertices of a path are its *endpoints*. All other vertices of a path are its *internal vertices*. Two paths are *independent* if no vertex of one is an internal vertex of the other. An *isomorphism* between two graphs G and H is a pair of bijections, φ and ψ, such that $\varphi : V(G) \to V(H)$ and $\psi : E(G) \to E(H)$, where $(u, e) \in \mathcal{I}(G)$ if and only if $(\varphi(u), \psi(e)) \in \mathcal{I}(H)$.

We call a graph *connected* if every pair of its vertices is connected by a path, and *disconnected*, otherwise. The maximal connected subgraphs of a graph are its *components*. A *cut vertex* in a graph is a vertex whose removal results in an increase in the number of components. The *connectivity* of a graph G is (1) zero if a graph is disconnected; (2) $|G| - 1$ if G is connected, but has no pair of distinct non-adjacent vertices; or (3) the size of the smallest set of vertices that disconnects G if G is connected, and has a pair of non-adjacent vertices. See Figure 1.1 for examples.

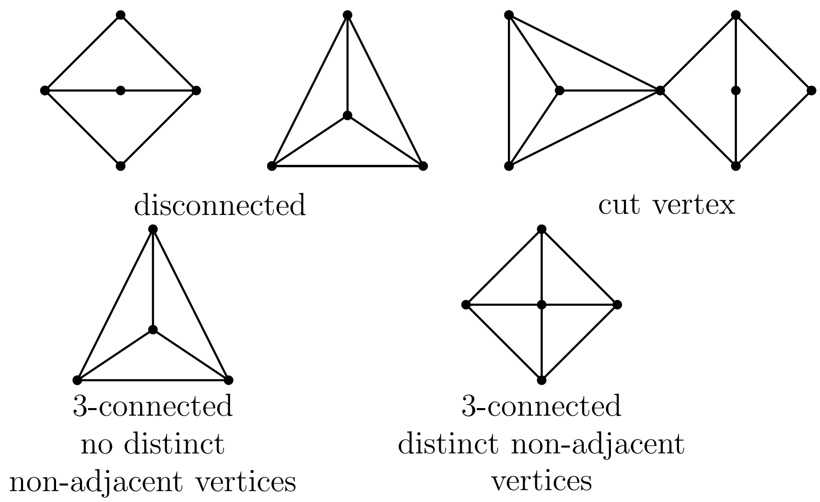


Figure 1.1: Examples of connectivity.

There are several classes of graphs that we will use in this dissertation. The examples of graphs listed in Figure 1.2 will play important roles in this paper. A *complete graph* is a simple graph in which every pair of vertices is connected by an edge. We denote complete graphs by K^n, where n is the number of vertices. A *bipartite graph* is composed of two disjoint sets of vertices such that each edge is incident to one vertex in each set. A *complete bipartite graph* is a simple bipartite graph in which each vertex is adjacent to every vertex in the other set. We denote a complete bipartite graph by $K_{r,s}$ where r and s denote the number of vertices in the disjoint sets. A *cycle* on n vertices, denoted C_n, is a trail of n vertices in which no vertices are repeated except the first equals the last.

Definition 1.5. A *wheel*, W_n, is obtained from a cycle, C_n, by adding a new vertex, called the *hub* and joining every vertex of C_n to the new vertex. The cycle, C_n, viewed as a subgraph of W_n is called the *rim*. The edges of the rim are the *rim edges*. The edges that connect the hub to the rim vertices are called *spokes*.

The wheel, W_3 is also known as K^4.

Definition 1.6. The *n-prism* is a polyhedral graph that is the Cartesian product $C_n \times K^2$. In other words, it is obtained from two cycles of C_n, say C_n^1 and C_n^2, with vertices v_i^1 and v_i^2, respectively, for $1 \leq i \leq n$, by adding an edge from v_i^1 of C_n^1 to v_i^2 of C_n^2 for $1 \leq i \leq n$. The edges of the cycles are called *cycle edges*. Each connecting edge is called a *spur*. See Figure 1.2 for an illustration of a 3-prism.The 3-prism and 4-prism are also known as the triangular prism and the cube, respectively.

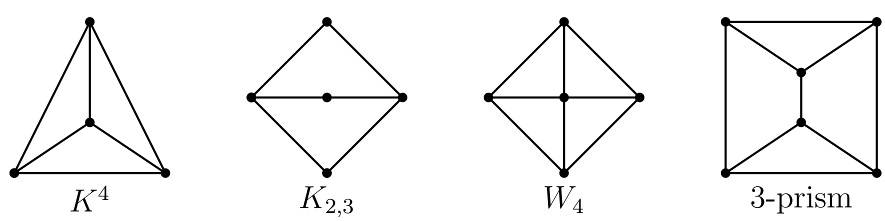


Figure 1.2: Examples of a complete graph (K^4), complete bipartite graph ($K_{2,3}$), wheel (W_4), and n-prism (3-prism).

A *planar embedding* of a graph G is a drawing of G in the plane where the vertices are represented by points, the edges by simple curves joining the endpoints, and the edges intersect only at their endpoints. A graph is *planar* if it has a planar embedding. In other words, a graph is planar if it can be drawn on the plane so that its edges only intersect at common vertices. A graph is *nonplanar* if it is not planar. The embedding a planar graph in the plane divides the plane into regions called *faces*. One face is unbounded; we call this the *outer face*. A graph is called *outer-planar* (OP) if it has a plane embedding in which all of the vertices lie on the boundary of the outer face. The focus of this dissertation is to describe graphs that are one edge away from being outer-planar in a finite manner.

To do this, we need the following relations on graphs. *Edge contraction* is an operation where an edge e, and all edges parallel to it, are removed from a graph and the two endpoints are identified to form a new vertex v. Any edges not parallel to e, but adjacent to e before the contraction, are incident to v after the contraction. We denote an edge contraction of G by G/e. *Edge deletion* is an operation in which an edge is removed from a graph and *vertex deletion* is an operation in which a vertex and its incident edges are removed, denoted by $G\backslash e$ and $G-v$, respectively. A graph H is a *minor* of G if a graph isomorphic to H can be obtained from G by a sequence of operations (possibly null), each of which is one of the following three operations: contracting an edge, deleting an edge, or deleting a vertex. We denote that a graph H is a minor of G by $G \geqslant_{\mathrm{m}} H$ or $H \leqslant_{\mathrm{m}} G$. Similarly, a *topological minor* is obtained by a sequence of operations (possibly null), each of which is one of the following: contracting an edge incident to a vertex of degree two, deleting an edge, or deleting a vertex. An edge, uv, is *subdivided* if it is replaced with a path, uwv of length two through a new vertex, w. A graph G is a *subdivision* of another graph H, if a graph isomorphic to G can be obtained by a sequence of subdivisions (possibly zero) of edges of H. An alternate way to say that G contains a subdivision of H as a subgraph is to describe H as a topological minor of G. A *co-simplification* of a graph G is the graph G' that results in a simplification of G, along with the contraction of the minimal number of edges incident to a vertex of degree two such that no vertex of G' has degree two.

One of the main ideas of this dissertation is the following definition.

Definition 1.7. A graph is *near outer-planar*, or NOP, if it is edgeless or has an edge whose deletion results in an outer-planar (OP) graph. A graph that is OP is also NOP.

The graph G, shown in Figure 1.3, is NOP, but $G\backslash e$ is OP. Theorem 1.2 and Corollary 1.3 motivate our quest for a finite list of NOP graphs. At the forefront of this are the two graphs that make up the excluded outer-planar graphs - $K_{2,3}$ and K^4.

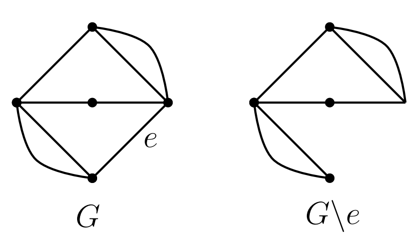


Figure 1.3: A graph G is NOP.

Throughout this dissertation, we will look at subdivisions of $K_{2,3}$ and K^4. The following definition will be useful in our proofs.

Definition 1.8. Let K be a graph with a subdivision of $K_{2,3}$ or K^4. Observe that $K_{2,3}$ and K^4 both have vertices of degree three. Also observe that by subdividing a $K_{2,3}$ or a K^4, the vertices of degree three do not change degree. We refer to these vertices in $K_{2,3}$ and K^4 and the corresponding vertices in K as *branch vertices.* In $K_{2,3}$, the *branch vertices* are the endpoints of three distinct paths, which we call the *legs* of $K_{2,3}$ or of K. A vertex on a *leg* of K that is not a branch vertex is called an *internal vertex.*

The following lemma is easy to verify.

Lemma 1.9. *A subdivision of $K_{2,3}$ or K^4 has three pairwise independent paths between two branch vertices.*

1.3 Finite Excluded List

In [10], we found that the class of NOP graphs is not closed under the taking of minors. Otherwise, we could use Theorem 1.1.

The‘list of minimal graphs that are not NOP under topological minors is not finite as shown in [10] due to the existence of *Robertson chains*, which we now define.

Definition 1.10. A *Robertson chain of length k* can be obtained from doubling the edges of a path of length k, such that the ends of the Robertson chain are the ends of the path.

Definition 1.11. A *quasi-ordering* is a binary relation that is reflexive and transitive. A quasi-order is *well-quasi-ordered* by $\leq$ if given a countable sequence, x_k of X, there are two elements x_i and x_j with $i < j$ and $x_i \leq x_j$.

Another important theorem that affirms the results of this dissertation is the following.

Theorem 1.12. ([11]) *For every positive integer k, the topological minor relation well-quasi-orders the graphs that do not contain a topological minor isomorphic to the Robertson chain of length k.*

To describe the class of NOP graphs by a finite list of minimal non-NOP graphs, we use the following operation and relation.

Definition 1.13. Suppose v is a vertex of G with exactly two distinct neighbors u and w, which may or may not be adjacent to each other. Let n denote the minimum of the number of uv edges and the number of vw edges in G. *Suppressing* the vertex v in G is the operation of replacing v and all its incident edges with n new uw edges. An example is given in Figure 1.4.

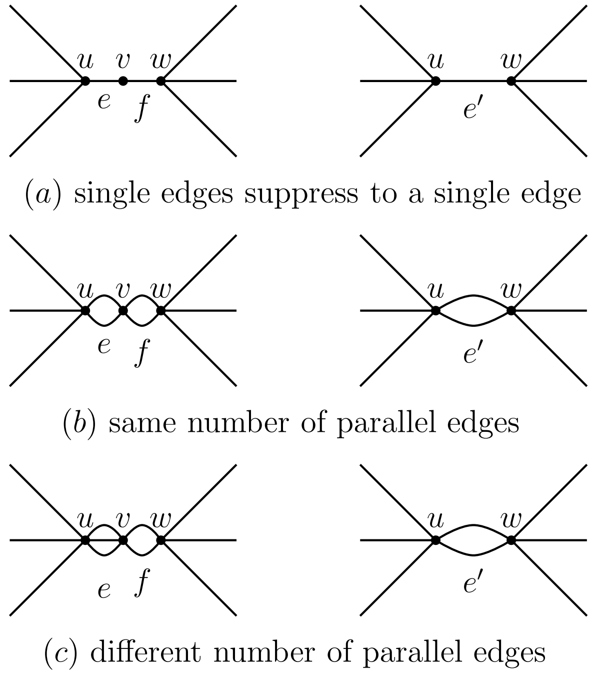


Figure 1.4: Suppression of v.

Definition 1.14. A graph H *dominates* a graph G, written $G \preceq H$, if G can be obtained from H by a sequence of operations each of which is one of the following:

- deleting an edge,
- deleting a vertex and all its incident edges, and
- suppressing a vertex with exactly two neighbors.

If H dominates G and is not isomorphic to G, then we say that it *properly dominates* G and write $G \prec H$. Note that if G is a topological minor of H, then $G \preceq H$.

In [10], we proved the following proposition and used the subsequent definition.

Proposition 1.15. *The class of NOP graphs is closed under domination.*

Definition 1.16. A graph G is *excluded near outer-planar* abbreviated *XNOP* if it is not NOP, but every graph properly dominated by G is NOP.

1.4 Strategy to Complete the XNOP list

In [10], we proved that the following sets are finite and provided a complete list of their members: nonplanar; disconnected; graphs with a cut vertex; and 2-connected graphs that do not dominate W_3. Our goal in this dissertation is to complete the list. To do this, we shall look at 2-connected graphs that dominate W_3 or W_4. The following concepts are key to our new findings and will be used frequently in the subsequent chapters.

Definition 1.17. Let G be a 2-connected graph. A *frame* F of G is a simple subgraph of G that is a subdivision of a 3-connected graph. The graph S is called the *skeleton* of G if S is a maximal co-simplification of F. A graph that has a skeleton S is a *full-*S. The vertices of G corresponding to the vertices of S are called the *joints* of G. Any vertex of G that is not a joint is an *internal vertex.*

For example, G in Figure 1.5 is a full-K^4.

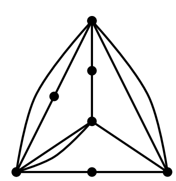


Figure 1.5: A full-K^4.

Definition 1.18. Let G be a 2-connected graph with a skeleton S, and let u and v be two adjacent joints of G. The graph $G' = G - \{u, v\}$ may have a single component or may have many components (see Figure 1.6 for an example). Let B'_0, B'_1, ..., B'_n

be the components of G', labeled such that the component of G' that contains the vertices of S is B'_0. Let $J = G[V(B'_0) \cup \{u, v\}]$. A *limb* L of G is the induced subgraph $G[V(G) - V(B'_0)]$. The *bridges* of J are the induced subgraphs $B_i = G[V(B'_i) \cup \{u, v\}]$ for $i \geq 1$. Note that: $\cup B_i = L$ and $\cap B_i = \{u, v\}$, as shown in Figure 1.6.

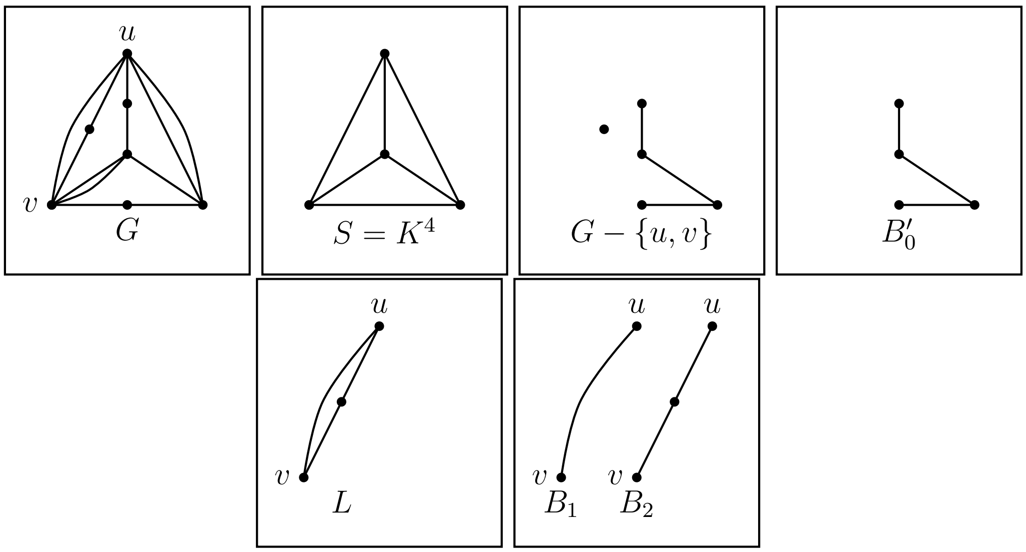


Figure 1.6: $L = G[V(B'_i) \cup \{u, v\}]$ for $i \geq 1$.

With this definition, we can deduce the following.

Remark 1.19. *A graph with a skeleton dominates* W_3.

In Chapter 2, we will list all of the different types of limbs and prove that the limb list is finite for XNOP graphs that dominate W_3.

Since the skeletons we seek are 3-connected, we must understand how to construct them from K^4. The following operations and theorem are necessary.

Definition 1.20. The following operations are *Barnette-Grünbaum Operations* (*BG-operations*)*:*

1. adding an edge uv, possibly a parallel edge, for $u \neq v$ for u, $v \in V(G)$,
2. subdividing an edge uv, by adding a vertex w, and adding an edge wx for u, v, w, $x \in V(G)$ and $uv \in E(G)$,
3. subdividing two distinct, non-parallel (possibly adjacent) edges uv and wx by adding vertices y and z, respectively, and adding the edge yz.

We say that a BG-operation is *basic* if its application creates neither a loop nor a parallel edge. The following theorem and corollary are useful in our understanding of skeletons.

Theorem 1.21. ([2]) *A simple graph G is 3-connected if and only if G can be constructed from K^4 using basic BG-operations.*

Corollary 1.22. *Every skeleton has K^4 as a topological minor.*

A skeleton S of an XNOP graph G is a simple, 3-connected graph, as defined in Definition 1.17. By Theorem 1.21, they can be constructed from K^4 by basic BG-operations. We need not look at nonplanar graphs or graphs that dominate W_5 since in [10], we found all nonplanar XNOP graphs and proved that no XNOP graph dominates W_5. Hence, the skeleton of an XNOP graph that we seek is planar and does not dominate W_5.

Not every graph constructed from K^4 by BG-operations will help us to complete the list of XNOP graphs. The following lemma details which skeletons are necessary and which are not. This lemma will be used in subsequent chapters on skeletons.

Lemma 1.23. *A skeleton of an XNOP graph is NOP or XNOP, but it is not OP or does not properly dominate an XNOP graph.*

Proof. Since S is constructed from K^4, then S has K^4 as a minor and cannot be OP. Also, if a skeleton S properly dominates an XNOP graph, then a nontrivial, full-S also properly dominates an XNOP graph. So, S is NOP or XNOP, but not OP or does not properly dominate an XNOP graph. □

The graphs listed in Figure 1.7 are graphs that will be used in the proofs in this dissertation. The graphs are listed in the order that they will be used and separated by row to indicate the sections to which they belong. Since all possible skeletons are constructed from K^4, it is in the top row. The second row contains all possible skeletons that are obtained from one BG-operation on K^4. The details are in Section 3.1. The third row contains all possible skeletons that are obtained from one BG-operation on W_4. The details are in Section 4.1. The fourth row contains all possible skeletons that are obtained from one BG-operation on 3-prism and that are not already listed in the third row. The details are in Section 4.2. The last row contains all possible skeletons that are obtained from one BG-operation on the double hub. The details are in Section 5.1.

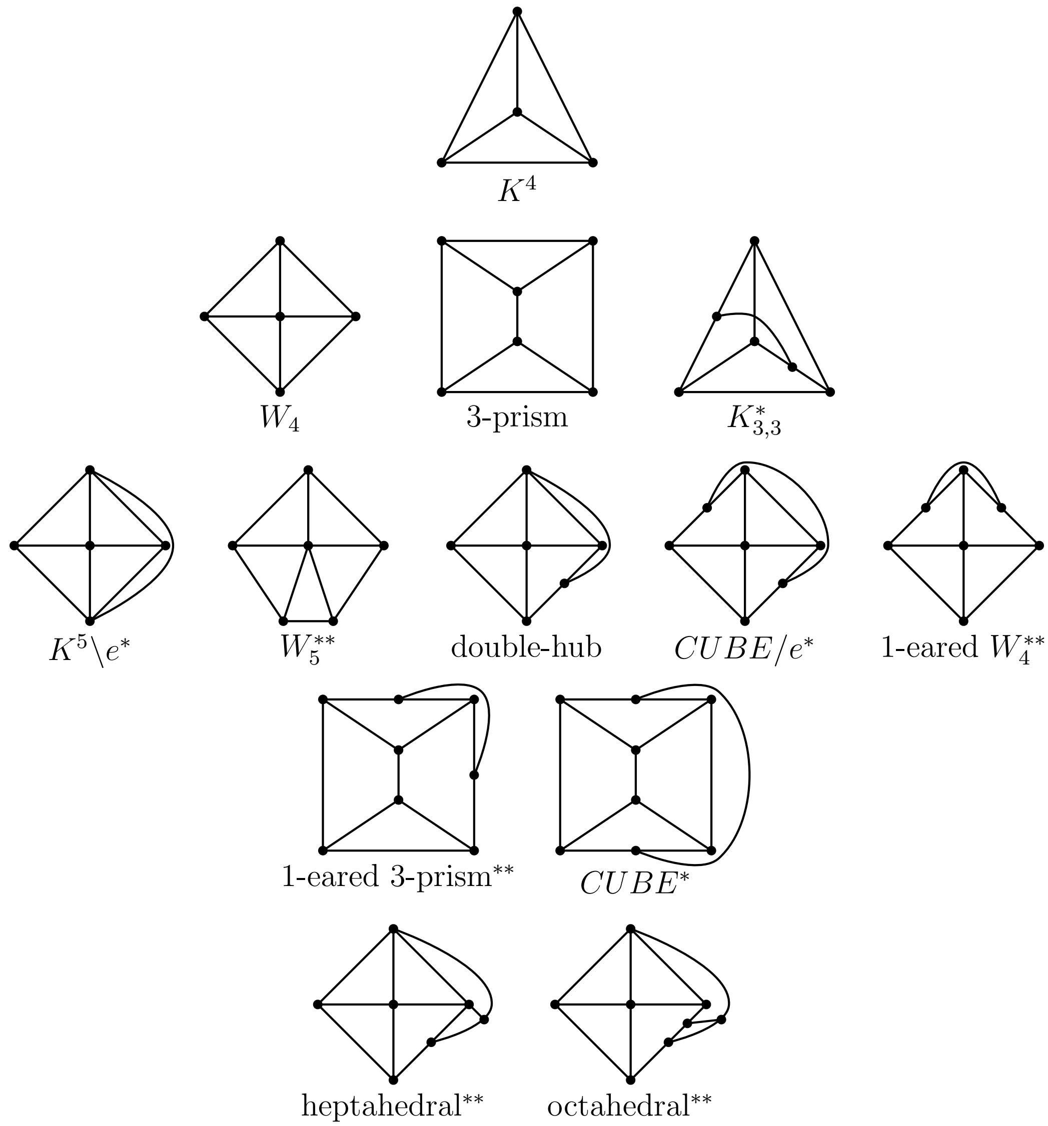


Figure 1.7: Skeletons of XNOP graphs and other relevant 3-connected graphs.

From the possible skeletons listed in Figure 1.7, we examined the list to identify which graphs can be skeletons of full-S XNOP graphs, which graphs are XNOP (designated as $*$), and which graphs cannot be skeletons of XNOP graphs (designated as $**$).

In Appendix A, we list the 58 graphs that constitute the XNOP graphs. Verifying that each of these is XNOP is tedious, so for the sake of brevity, we refer the reader to Appendix B of [10]. We have verified the other 57 graphs, but do not present the details here.

The set of all XNOP graphs, listed in Appendix A, may be divided into these sets (with the known graphs listed in parentheses):

I. Nonplanar graphs ($K_{3,3}$)

II. Disconnected graphs (D_1, D_2, D_3)

III. Graphs with a cut vertex (CV_1, CV_2, CV_3, CV_4, CV_5, CV_6)

IV. 2-connected graphs that do not dominate W_3 (DE_1, $K_{2,4}$, S_1, S_2, S_3, S_4, S_5, S_6)

V. 2-connected graphs that dominate W_3, but not W_4

   (a) and have K^4 as a skeleton (KF_{1A}, KF_{1B}, KF_{2A}, KF_{2B}, KF_{3A}, KF_{3B}, KF_{3C}, KF_{3D}, KF_{3E}, KF_{3F}, KF_{3G}, KF_{3H}, KF_{4A}, KF_{4B}, KF_{4C}, KF_{4D}, KF_{4E}, KF_{4F}, KF_{5A}, KF_{5B}, KF_{5C}, KF_{5D}, KF_{5E}, KF_{6A}, KF_{6B}, KF_{6C}, DE_2)

   (b) and have 3-prism as a skeleton (TP_1, TP_2, TP_3, TP_4, TP_5, TP_6)

   (c) and have *cube* as a skeleton ($CUBE$)

VI. 2-connected graphs that dominate W_4.

   (a) and have W_4 as a skeleton (WF_1, WF_2, WF_3)

   (b) and have $double-hub$ as a skeleton (DH_1)

   (c) and have $K_5\backslash e$ as a skeleton ($K_5\backslash e$)

   (d) and have $CUBE/e$ as a skeleton ($CUBE/e$)

VII. 2-connected graphs that dominate W_5 (none)

In [10], we devoted a chapter to each of I–IV to show that each set is finite and to present the complete list of its elements. In this dissertation, we prove that the sets V and VI are finite.

We begin our proof of V and VI by looking at the smallest skeleton, K^4, and its possible limbs. We then look at the skeletons obtained by adding edges to K^4 using one BG-operation, then two BG-operations, and so forth.

CHAPTER 2

FULL-K^4 XNOP GRAPHS

This chapter is divided into three sections. In the first two sections, we look at what types of limbs are possible or impossible for an XNOP graph that is a full-K^4. In the last section, we share the algorithm and results. We use symmetry to specify which edges of K^4 will be permuted with the types of limbs found in Section 2.1 to reduce the programming load.

We will use the following illustration and definitions to prove the results about XNOP graphs that have K^4 as a skeleton.

Let G be an XNOP graph that is a full-K^4 (See Definition 1.17). Throughout this chapter, we will examine a particular limb L. Sometimes we will also focus on the limbs that are adjacent to L. Other times, we will focus on the singular limb M, that is non-adjacent to L, as shown in Figure 2.1.

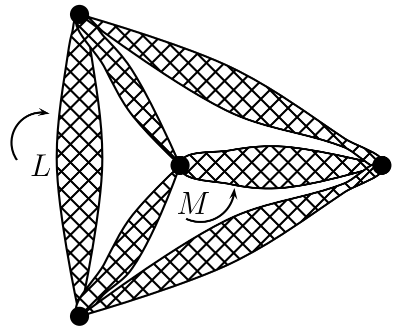


Figure 2.1: A representation of a full-K^4, with a limb L, and its non-adjacent limb M.

The following definition and remarks form the basis for many of the proofs that follow.

Definition 2.1. Let u and v be the joints of the limb L of an XNOP graph G. We say that L is 1-*edge-separable from its joint* u, or *edge-separable*, if there exists an edge $e \in E(L)$ such that $L\backslash e$ has no path from u to v in $L\backslash e$. The edge e is called a *separating edge* . We say that $G\backslash e$ *edge-separates* or *separates* a limb L from G if $L\backslash e$ is *edge-separable* from its joint.

Remark 2.2. *It is helpful to note the following points about an XNOP graph G that is a full-K^4:*

- *$G\backslash e$ is NOP for all $e \in E(G)$ (see Definition 1.7)*
- *there exists f for $f \in E(G\backslash e)$ such that $G\backslash e\backslash f$ is OP (see Definition 1.7)*
- *$G\backslash e\backslash f$ does not dominate $K_{2,3}$ or K^4 (see Theorem 1.2)*
- *$G\backslash e$ or $G\backslash e\backslash f$ separates at least one limb of G, otherwise $G\backslash e\backslash f \succ K^4$ (see Theorem 1.2).*

Remark 2.3. *A graph that properly dominates an XNOP graph by a single edge is not XNOP.*

2.1 Overview of All Possible Limbs of Full-K^4 XNOP Graphs

To prove the main theorem in Section 2.3, we must first identify and prove that the limbs in Figure 2.2 are the only possible limbs of a full-K^4. Then, we will use the algorithms in Section 2.3 to prove that the list of full-K^4 graphs that are XNOP is complete.

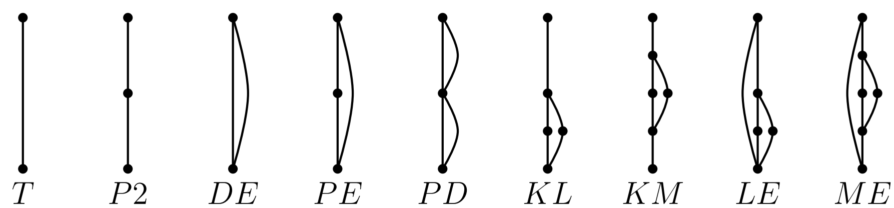


Figure 2.2: Limbs of a full-K^4.

The proof of all possible limbs is long and in many cases, very similar. For brevity, in this section, we outline the proof of all possible limbs with a listing of the lemmas, corollaries, and proposition, with summary figures, but without proofs of the lemmas, etc. We save the proofs of those lemmas, etc. for Section 2.2.

The first lemma gives us an upper bound on the structure of the limb. From this lemma, we learn that, although a graph must dominate K^4 or $K_{2,3}$ to be XNOP, a graph whose limb dominates K^4 or $K_{2,3}$ is not an XNOP graph, but properly dominates one.

Theorem 2.4. *If a full-K^4 graph G has a limb that dominates $K_{2,3}$ or K^4, then G is non-planar or G properly dominates at least one of the following graphs: $K_{2,4}$, S_3, S_4, S_5, S_6, KF_{1A}, KF_{1B}, KF_{2A}, KF_{2B}.*

The following definition and theorem are helpful in understanding the construction of a limb that does not dominate K^4. Specifically, a limb of an XNOP graph that dominates K^4 must be series parallel and therefore constructed by subdividing an edge or adding an edge in parallel to an existing edge.

Definition 2.5. ([7]) A graph *is series parallel* if it can be made from a loop, then applying a sequence of one of the following operations:

- replace an edge by two edges in series or
- replace an edge by two edges in parallel.

Theorem 2.6. ([6], [7]) *Excluding K^4 yields a series parallel graph.*

Now that we have the upper bound on limbs of a full-K^4, we can look at minimizing the types of bridges in a lemma and two corollaries. These can be proved by observing that a contradiction to the lemma and corollaries results in a graph that dominates $K_{2,4}$. See Figures 2.3, 2.4, and 2.5, respectively.

Lemma 2.7. *If a limb of a full-K^4 graph G has two or more bridges with internal vertices, then G dominates $K_{2,4}$.*

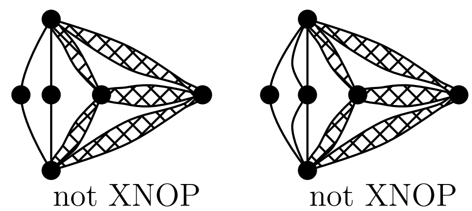


Figure 2.3: Lemma 2.7 - a limb with two or more bridges with internal vertices dominates $K_{2,4}$.

The two following corollaries will be useful in later proofs and can be easily derived.

Corollary 2.8. *If a limb of a full-K^4 graph G has more than one edge-disjoint x-y path and an edge or path connects two x-y paths, then $G \succ K_{2,4}$.*

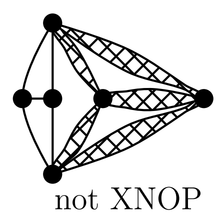


Figure 2.4: Corollary 2.8 - no edge or path connects two edge-disjoint x-y paths.

Corollary 2.9. *If a limb L of an XNOP graph G has a bridge with an internal vertex, then if L has another bridge, it is a single edge.*

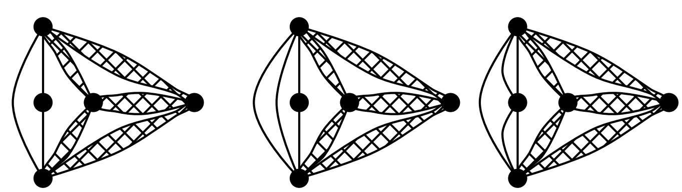


Figure 2.5: Corollary 2.9 - if G has a limb with a bridge with an internal vertex, then all other bridges are edges.

The following proposition is important in that it gives us an upper bound on the number of bridges of a limb. The proof, along with the other proofs from this section, is in Section 2.2 for brevity.

Proposition 2.10. *A full-K^4 graph that has a limb with more than two bridges is not XNOP.*

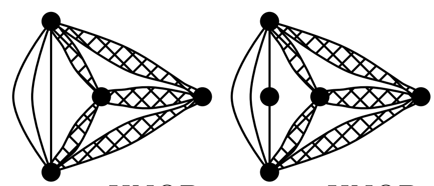


Figure 2.6: Proposition 2.10 - full-K^4 with three bridges.

With the above general ideas on bridges, we can begin to investigate the bridges in a more specific manner. Let G be a full-K^4 graph with limb L. The limb L can be edge-separable or not. First, we will explore when L is edge-separable. During this discussion, we may discover some ideas about limbs that are not edge-separable, but after Lemma 2.23, we will look seriously at limbs that are not edge-separable.

Suppose that $G \backslash e$ separates L from one of its joints as in Figure 2.7. Without loss of generality, an edge-separation from a joint that results in a cut vertex at a joint does not affect outer-planar properties and will not be depicted in further figures in this dissertation. Since $G \backslash e$ is NOP, then it must dominate $K_{2,3}$. By Lemma 2.4, we assume that a single limb does not dominate $K_{2,3}$. Then, a limb can dominate two legs of $K_{2,3}$, one leg of $K_{2,3}$, or one edge of $K_{2,3}$. We explore each of these scenarios, as well as whether each of these can be edge-separable or not, after this paragraph, after Lemma 2.15, and after Lemma 2.21, respectively.

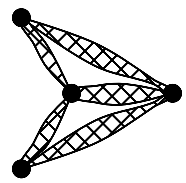


Figure 2.7: An illustration of $G \backslash e$ that separates L from one or more of its joints.

Suppose that two of the legs, a $K_{2,2}$, are in one limb. The pair of legs are in a limb adjacent to L or they are not, as in Figure 2.8.

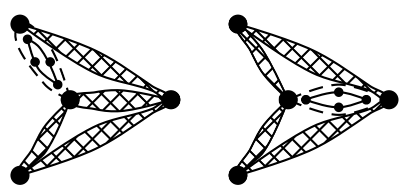


Figure 2.8: Illustrations of a pair of $K_{2,2}$ legs in a limb of $G \backslash e$.

The branch vertices (see Definition 1.8) of the subdivision of $K_{2,2}$ are either joints of their respective limbs or they are not. If both are joints, then $G \succ K_{2,4}$.

Hence, at least one branch vertex is not a joint. By Lemma 2.7 and Proposition 2.10, if the limb has a bridge which dominates $K_{2,2}$, then there are no other bridges of the limb, or there is one other bridge, which consists of a single edge from u to v. See Figure 2.9 for examples of limbs that are eliminated.

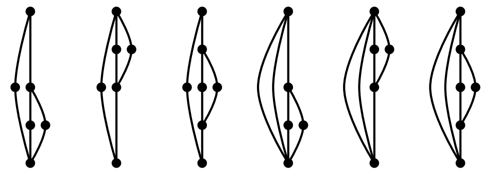


Figure 2.9: Illustration of limbs that dominate $K_{2,2}$ and are contradictions to Lemma 2.7 and Proposition 2.10.

To recap, we have $G\backslash e$ which separates L from one of its joints; some limb M, of $G\backslash e$ dominates $K_{2,2}$; at least one branch vertex of the $K_{2,2}$ is not a joint of G; M has no more than two bridges; and if M does have two bridges, one is a single edge. The next proposition and three lemmas eliminate some graphs with limbs that dominate $K_{2,2}$.

The following proposition is important to the remaining proofs and to reducing the programming load. It can be proved by realizing that a graph that properly dominates an XNOP graph is not XNOP. Note that KF_{1A} and KF_{1B} have exactly one nontrivial limb (type LE or ME, respectively). If a limb of a full-K^4 graph G dominates a limb of type LE or ME, then G is not XNOP.

Proposition 2.11. *If a full-K^4 graph G has a limb which dominates a limb of type LE or ME, then G dominates KF_{1A} or KF_{1B}.*

The following lemma is easy to verify. See Figure 2.10.

Lemma 2.12. *If a full-K^4 graph G has two limbs each of which dominates a limb of type KL or KM, then G properly dominates S_4, S_5 or S_6.*

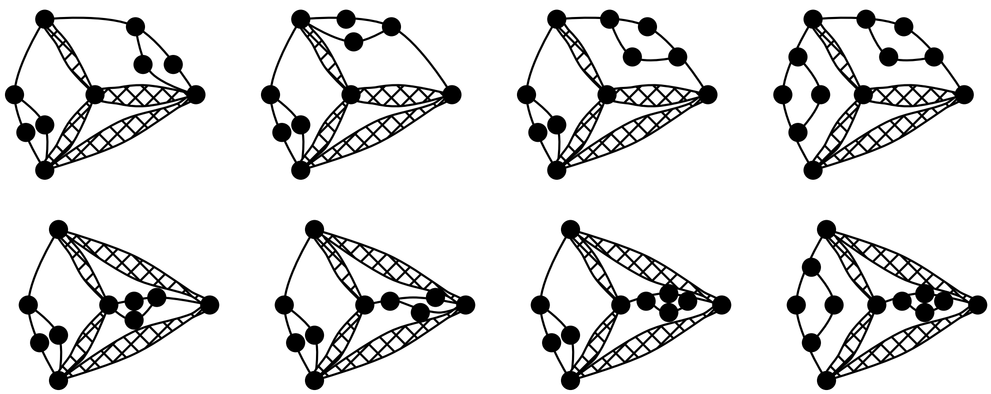


Figure 2.10: Lemma 2.12 - two limbs of KL or KM result in a graph that dominates S_4, S_5, or S_6.

The next two lemmas minimize a limb that dominates $K_{2,2}$.

Lemma 2.13. *If a full-K^4 XNOP graph has a limb that dominates $K_{2,2}$ but is not of type LE or ME, then the limb must be edge-separable from a joint that is not a branch vertex of the $K_{2,2}$ subdivision.*

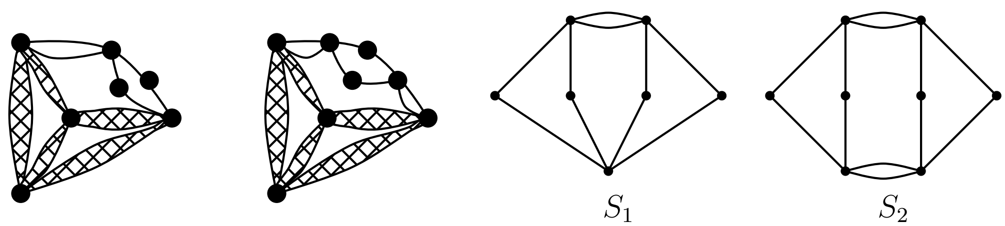


Figure 2.11: Lemma 2.13 - graphs with a limb that is not edge-separable, but dominates $K_{2,2}$ properly, dominate S_1 or S_2.

Lemma 2.14. *If an XNOP graph has an edge-separable limb L, that dominates $K_{2,2}$, then L is either type KL or KM, and the non-adjacent limb to L must be type $P2$.*

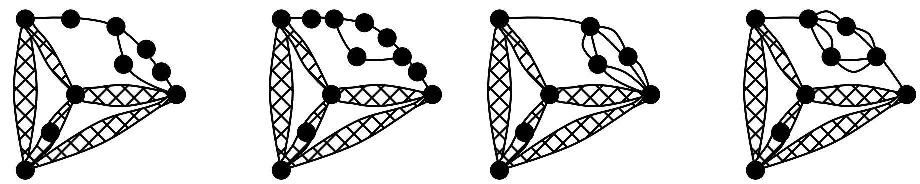


Figure 2.12: Lemma 2.14 - graphs that properly dominate KF_{2A} and KF_{2B}.

With the above lemmas and the existence of graphs that dominate these types of limbs in KF_{1A}, KF_{1B}, KF_{2A}, and KF_{2B}, we have proved that limbs of type KL, KM, LE, or ME are possible and that an XNOP graph that has a limb that dominates $K_{2,2}$ must be type KL, KM, LE, or ME.

Now that we have all of the limbs that dominate $K_{2,2}$, we can prove the following lemma that will be useful in the remaining lemmas.

Lemma 2.15. *If an XNOP graph has a limb L of type DE, then the non-adjacent limb to L is also type DE.*

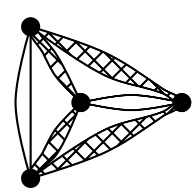


Figure 2.13: Lemma 2.15 - a limb of type DE has a non-adjacent limb of type DE.

We are still looking at an XNOP graph G such that $G \backslash e$ separates L from one of its joints and $G \backslash e \succ K_{2,3}$. By Lemma 2.4, we can assume that the three legs of the $K_{2,3}$ are not in one limb, and we have found all possible limbs that have two of the legs of the $K_{2,3}$ in one limb. Now, we will explore the possible limbs that have one leg of the $K_{2,3}$. Since $G\backslash e \succ K_{2,3}$, the limb M, that is non-adjacent to L, must have an internal vertex as in Figure 2.14.

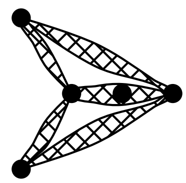


Figure 2.14: $G\backslash e \succ K_{2,3}$ and the M has an internal vertex.

The limb M can be edge-separable or not. If it is edge-separable, then at least one edge of M is not part of a multiple edge. At a minimum, the limb could be a path of length two, and at a maximum, the limb does not dominate $K_{2,2}$. See Figure 2.15 for some potential limbs.

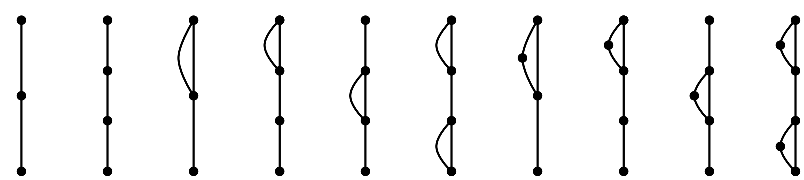


Figure 2.15: Potential limbs of M that have one leg of $K_{2,3}$.

The following lemmas about limbs of type P2 are useful in improving the list of possible limbs of a full-K^4 XNOP graph.

Lemma 2.16. *If a limb of an XNOP graph is edge-separable, has an internal vertex, and does not dominate $K_{2,2}$, then it is type P2.*

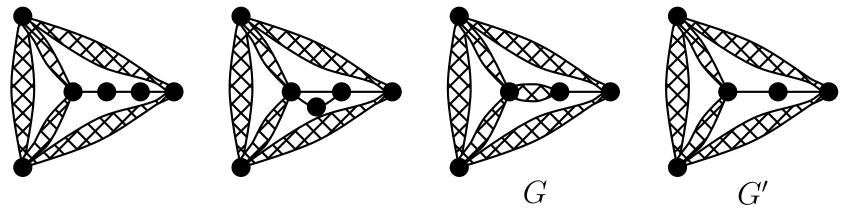


Figure 2.16: Lemma 2.16 - graphs with an edge-separable limb, which has internal vertices and which does not dominate $K_{2,2}$.

Lemma 2.17. *If a limb of a full-K^4 graph is type $P2$ and its non-adjacent limb is type T, then G is not XNOP.*

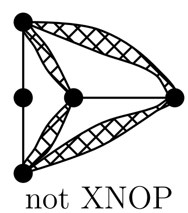


Figure 2.17: Lemma 2.17 - L is type T and M is type P2.

With the previous lemmas, we have proved (1) that limb-type P2 can be a limb of a full-K^4, (2) that other edge-separable limbs that do not dominate $K_{2,2}$ and that have internal vertices are not minimal, and (3) limbs of type T and type P2 are not be in non-adjacent limbs of a full-K^4 XNOP graph. There are no other possible edge-separable limbs that can be a leg of $K_{2,3}$.

Now we explore the limb M, that is not edge-separable and dominates one leg of $K_{2,3}$. We assume from Corollary 2.9 that M does not have more than two bridges. We can also assume from Proposition 2.10, that if it has two bridges, then one is a single edge. The nontrivial bridge can be edge-separable or not. We use the next three lemmas to establish some ideas about limbs that are not edge-separable. The limbs in Figure 2.18 are a guide to the next lemmas.

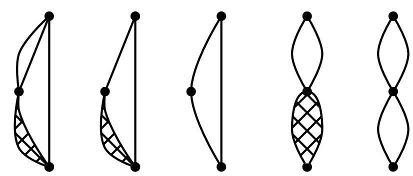


Figure 2.18: Limbs that are not edge-separable and that dominate one leg of $K_{2,3}$.

Lemma 2.18. *If a limb of a full-K^4 graph G consists of two bridges, one of which has an internal vertex and is not edge-separable, then G is not XNOP.*

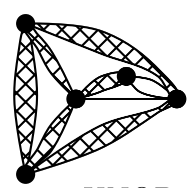


Figure 2.19: Lemma 2.18 - G has two bridges, one that is not edge-separable.

Hence, if a limb has two bridges, both must be edge-separable. We can also prove the following about its non-adjacent limb in a similar proof to the previous lemma.

Lemma 2.19. *If a limb of an XNOP graph consists of two bridges, one of which has an internal vertex, then the limb is type PE and the non-adjacent limb is trivial.*

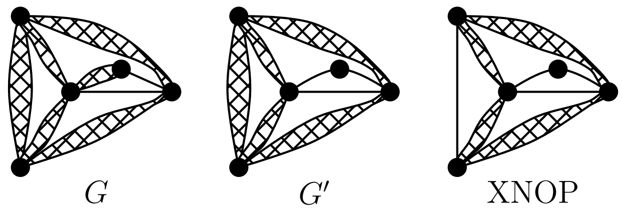


Figure 2.20: Lemma 2.19 - M has two bridges, one that is edge-separable and the other trivial.

The following lemma is easy to verify by using the previous lemma and replacing limb M with a limb of type PD, and the edge e with an edge of M.

Lemma 2.20. *If a limb of an XNOP graph consists of a single, not edge-separable bridge with an internal vertex, then the limb is type PD and the non-adjacent limb is trivial.*

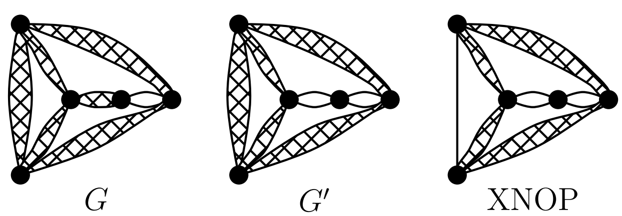


Figure 2.21: Lemma 2.20 - M has one bridge that is not edge-separable.

With the proofs of the previous lemmas, we can prove another lemma about a limb of type P2 that will be used later.

Lemma 2.21. *A limb of a full-K^4 XNOP graph G is type P2, if and only if its non-adjacent limb is either type P2, KL, or KM.*

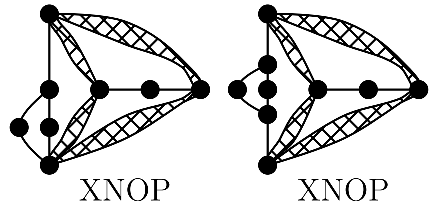


Figure 2.22: Lemma 2.16 - both L and M are type P2.

This corollary follows.

Corollary 2.22. *If a limb of a full-K^4 XNOP graph G is KL or KM, then G is KF_{2A} or KF_{2B}.*

With the conclusion of the previous lemma, we have found every possibility of a limb that dominates one leg of $K_{2,3}$. If a limb does not dominate a single leg of $K_{2,3}$, then the limb must dominate a single edge of the leg of $K_{2,3}$. This can be edge-separable or not. If it is not edge-separable, then it must be a limb of type DE, which was addressed in Lemma 2.15. If it is edge-separable, then it must be type T. The following lemma proves that a limb can be type T.

Lemma 2.23. *If a limb of an XNOP graph is type T, then the non-adjacent limb has an internal vertex and is not edge-separable.*

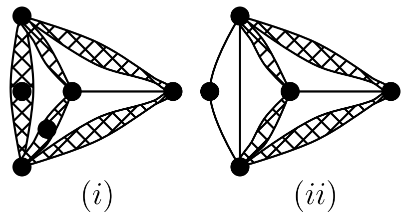


Figure 2.23: Lemma 2.23 - G for Figure 2.60.

With the ideas from these theorems and lemmas, we can further investigate the limbs of a full-K^4. From Remark 2.2, we know that either $G \backslash e$ or $G \backslash e \backslash f$ separates a limb of G.

Suppose that $G \backslash e$ does not separate a limb, but $G \backslash e \backslash f$ does. In this case, the limb L must have at least two bridges or two paths. We have already investigated limbs of this type with Lemma 2.15, Lemma 2.18, Lemma 2.19, and Lemma 2.20, but we must make sure that we have investigated all of them. If the limb has two bridges, then by Corollary 2.9, one bridge must be a single edge. The other limb must be edge-separable by Lemma 2.18. If it is edge-separable and has an internal vertex, then by Lemma 2.19, the limb of G must be type PE. If it is not type PE and it has two bridges, then it must be type DE as in Lemma 2.33. Type DE is the minimum limb of two bridges, so if the limb is not edge-separable, then the limb must have two paths. If the limb is not edge-separable and has two paths, then the limb is type PD by Lemma 2.20. There are no other possibilities of limbs of two bridges or two paths.

With this list of limbs, we can permute the limbs in Sage Math and verify that we have all of the XNOP graphs that have K^4 as a skeleton.

2.2 Proofs of All Possible Limbs of Full-K^4 XNOP Graphs

Lemma 2.4. *If a full-K^4 graph G has a limb that dominates $K_{2,3}$ or K^4, then G is non-planar or G properly dominates at least one of the following graphs: $K_{2,4}$, S_3, S_4, S_5, S_6, KF_{1A}, KF_{1B}, KF_{2A}, KF_{2B}.*

Proof. Suppose that L, with joints u and v, dominates $K_{2,3}$ or K^4, and G is not in the list. Let P be the subdivision of $K_{2,3}$ or K^4 in L. Let H be a subgraph of G such that $H \cup L = G$, $H \cap L = \{u, v\}$, and $H \succeq K^4 \backslash e$.

The joints u and v share vertices with P or they do not. This gives us three cases: P shares both joints, P shares exactly one joint, and P shares no joints as in Figure 2.24.

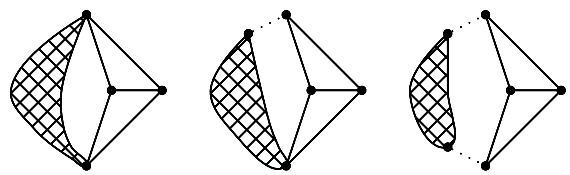


Figure 2.24: Cases of P in the graph $L \cup H$.

Case (i): Suppose both u and v share vertices with P. Then G is nonplanar or G dominates $K_{2,4}$, KF_{1A}, or KF_{1B}. (The figures with the graphs associated with the cases of this lemma are large. They are placed in Apendix B to save room and to allow for clarity of the rest of the chapter.) See Figure B.1.

Case (ii): Suppose exactly one of u or v share vertices with P. Then G is nonplanar or G dominates $K_{2,4}$, S_3, S_4, S_5, KF_{1A}, or KF_{2A}. See Figure B.2.

Case (iii): Suppose neither u or v share vertices with P. Then G is nonplanar or G dominates $K_{2,4}$, S_5, S_6, KF_{1B}, or KF_{2B}. See Figure B.3. □

Proposition 2.10. *A full-K^4 graph that has a limb with more than two bridges is not XNOP.*

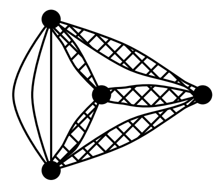


Figure 2.25: Full-K^4 with three bridges.

Proof. Suppose that L is a limb with three bridges. From Corollary 2.9, L can only have one bridge of length two or longer as in Figure 2.25.

Let e be an edge of G such that $e \in E(L)$. Since G is XNOP, then $G \backslash e$ is NOP. $G \backslash e$ dominates K^4 or $K_{2,3}$ and must be as in Figure 2.26. Without loss of generality, an edge-separation of a joint that results in a cut vertex does not affect outer-planar properties and will not be depicted in further figures of this proof.

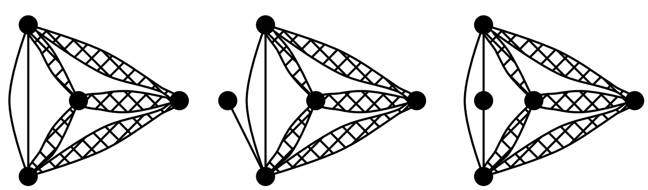


Figure 2.26: $G \backslash e$ for $e \in E(L)$ for Figure 2.25.

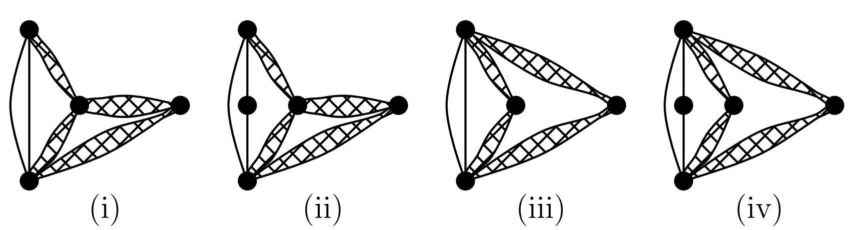


Figure 2.27: Four cases of $G \backslash e \backslash f$ for $f \notin E(L)$ for Figure 2.26.

Then, $G\backslash e\backslash f$ is OP for some edge $f \in E(G\backslash e)$. Since G is a full-K^4, then $G\backslash e\backslash f$ must separate at least one limb of G. Otherwise, it dominates K^4. There are at least two bridges in $L\backslash e$, so $f \notin E(L\backslash e)$. Without loss of generality, there are four possibilities of $G\backslash e\backslash f$ as depicted in Figure 2.27 (i)–(iv). The last possibility (iv) is a contradiction, since it dominates $K_{2,3}$ and is therefore not OP. We can also better describe (i) and (ii), as we may assume from Lemma 2.4 that a single limb does not dominate $K_{2,3}$. In these two cases, the limb of $G\backslash e\backslash f$ that is adjacent to all remaining limbs does not have an internal vertex. Hence, it can have infinitely many bridges of length one. In Figure 2.28 we show graph (i) with one, two, and three bridges of length one, but only show one case of graph (ii).

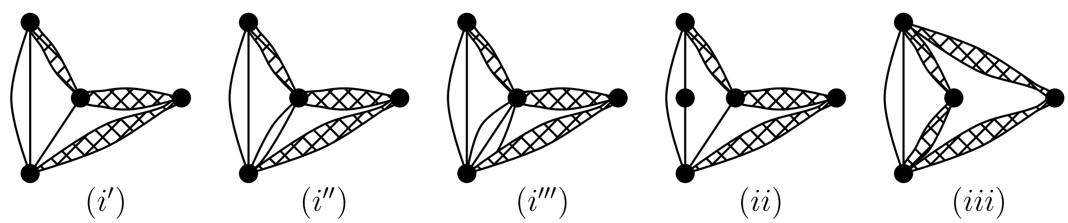


Figure 2.28: Refinement of cases of $G\backslash e\backslash f$ for Figure 2.27.

Since G is XNOP, $G\backslash f$ must not be OP. Then $G\backslash f$ dominates $K_{2,3}$ or K^4. See Figure 2.29. Since the removal of f separates a limb, then $G\backslash f$ should not dominate K^4. So, $G\backslash f \succ K_{2,3}$. By Lemma 2.4, no limb of G can dominate $K_{2,3}$ or K^4, so structurally, one of the limbs must dominate $K_{2,2}$. Then a case of $G\backslash e\backslash f$ must have the same limbs that dominate $K_{2,2}$. See three possibilities of Figure 2.29 (i') in Figure 2.30. However, if a single limb of $G\backslash e\backslash f$ has a limb that dominates $K_{2,2}$, then all cases of $G\backslash e\backslash f$ dominate $K_{2,3}$, a contradiction to it being OP. Hence, a limb of G does not have three or more bridges. □

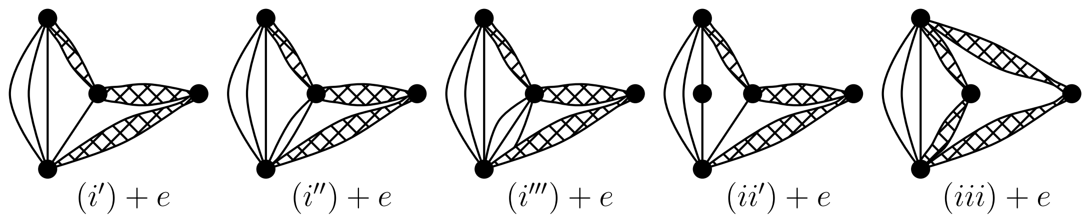


Figure 2.29: $G\backslash f$ for $f \notin E(L)$ for Figure 2.28.

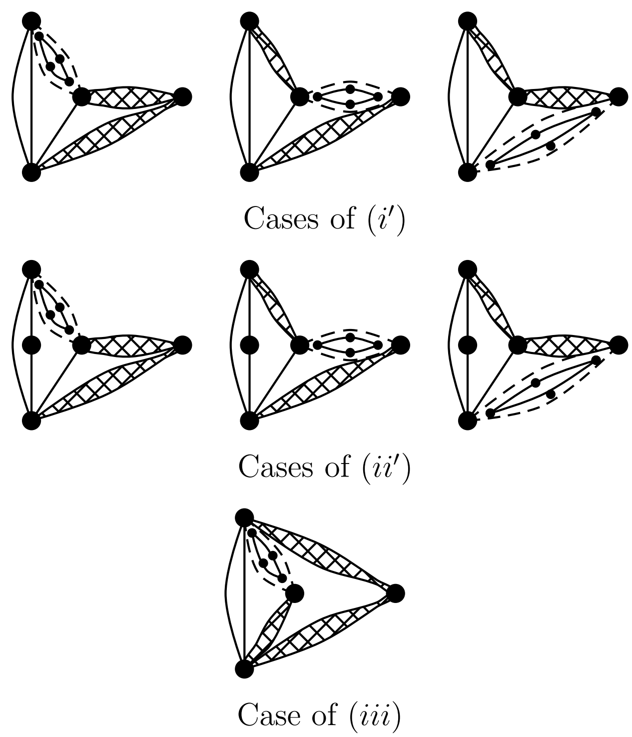


Figure 2.30: Cases of $G\backslash e\backslash f$ from Figure 2.29 with limbs that dominate $K_{2,2}$.

Lemma 2.13. *If a full-K^4 XNOP graph has a limb that dominates $K_{2,2}$ but is not of type LE or ME, then the limb must be edge-separable from a joint that is not a branch vertex of the $K_{2,2}$ subdivision.*

Proof. Suppose that a limb dominates $K_{2,2}$, but is not type LE or ME and is not edge-separable from a joint that is not a branch vertex of the $K_{2,2}$. Then G dominates S_1 or S_2. See Figure 2.31. □

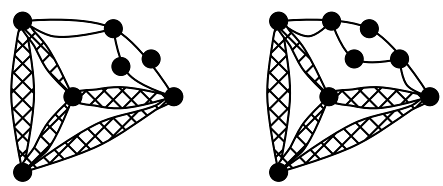


Figure 2.31: Graphs of a limb that is not edge-separable and that dominates $K_{2,2}$.

Lemma 2.14. *If an XNOP graph has an edge-separable limb L that dominates $K_{2,2}$, then L is either type KL or KM, and the non-adjacent limb to L must be type $P2$.*

Proof. Let G be an edge-separable XNOP graph with a limb L that dominates $K_{2,2}$. We know that the limb does not dominate $K_{2,3}$ or K^4, and, by Lemma 2.13, it must be edge-separable from a joint that is not a branch vertex of the $K_{2,2}$. Let e be the separating edge of L. So, $G\backslash e$ is NOP and dominates $K_{2,3}$. By Lemma 2.12, none of the remaining limbs of $G\backslash e$ can dominate $K_{2,2}$. In order for $G\backslash e$ to dominate $K_{2,3}$, the limb that is adjacent to all of the remaining limbs must have an internal vertex. See Figure 2.32. But, then $G \succ KF_{2A}$ or $G \succ KF_{2B}$. Hence, a limb of a full-K^4 XNOP graph that dominates $K_{2,2}$ must be type KL or KM and the non-adjacent limb is type P2.

□

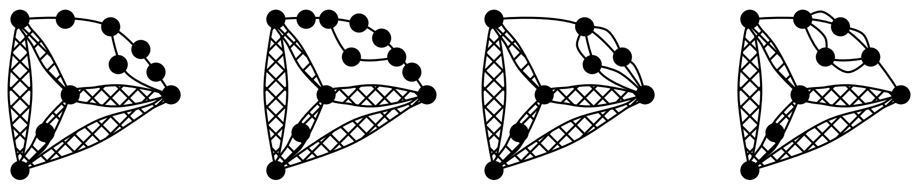


Figure 2.32: Graphs with limbs that strictly dominate limbs of type KL or KM.

Lemma 2.15. *If an XNOP graph has a limb of type DE, then the non-adjacent limb is also type DE.*

Proof. Let L be a limb of type DE and let e be one of the edges of L. Since G is XNOP, it follows that there is an edge f such that $G\backslash e\backslash f$ is OP. Furthermore, since $G\backslash e \succ K^4$, $G\backslash e\backslash f$ must separate a limb of $G\backslash e$. There are three cases of $G\backslash e\backslash f$ as shown in Figure 2.33 $(i) - (iii)$. Since $G\backslash e\backslash f$ is OP, we can further refine the three cases. For cases (i) and (ii), the limb that is adjacent to all of the other limbs must not have an internal vertex. So, it can be of type T or type DE. See Figure 2.34.

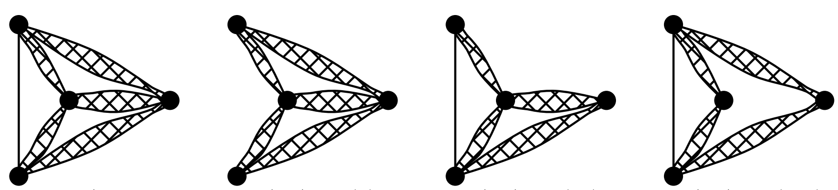


Figure 2.33: An illustration of $G\backslash e$ and the three cases of $G\backslash e\backslash f$.

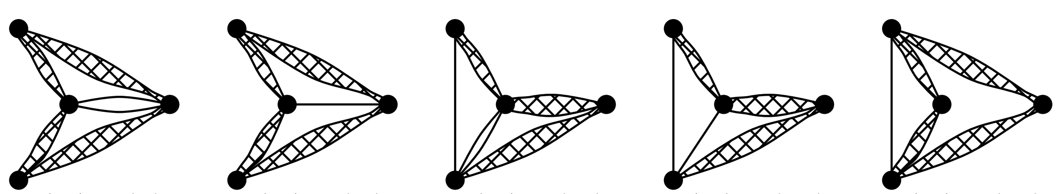


Figure 2.34: Refinement of the three cases of $G\backslash e\backslash f$ from Figure 2.33.

With these refinements, we can look at $G\backslash f$ by adding back the edge e. See Figure 2.35. Since $G\backslash f$ is not OP, we can eliminate $G\backslash f$ $(ii)-(iii)$ since these cases are OP.

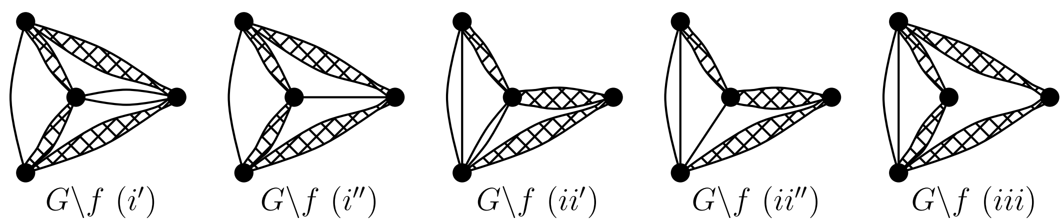


Figure 2.35: $G\backslash f$ from Figure 2.34.

By adding back the edge f to $G\backslash f$ (i), we can refine G as shown in Figure 2.36. It is noteworthy that both cases of G have a limb that is non-adjacent to L that has no internal vertex.

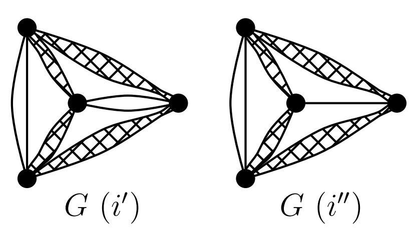


Figure 2.36: G from Figure 2.35.

We can further explore G by looking at what would happen if an edge d of the limb non-adjacent to L is removed. See Figure 2.37. But, $G\backslash d$ (i'') is OP, a contradiction to the assumption that it should not be OP.

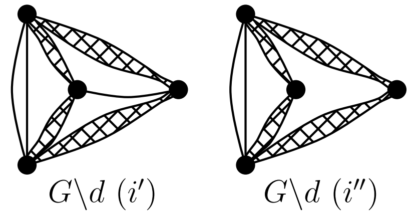


Figure 2.37: G from Figure 2.36.

If we reconstruct G from the only viable case of $G\backslash d$, we find that if a limb is type DE, then the non-adjacent limb is also type DE. See Figure 2.38 □

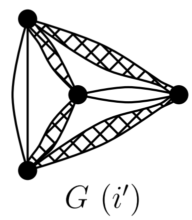


Figure 2.38: G from Figure 2.37.

Lemma 2.16. *If a limb of an XNOP graph is edge-separable, has an internal vertex, and does not dominate $K_{2,2}$, then it is type P2.*

Proof. Suppose M is edge-separable with an internal vertex, does not dominate $K_{2,2}$, and is not type P2. See Figure 2.39 for possibilities and a general depiction of G. Suppose further that G' is a graph such that M is replaced by M', a limb of type DE. It is easy to verify that $G \succ G'$.

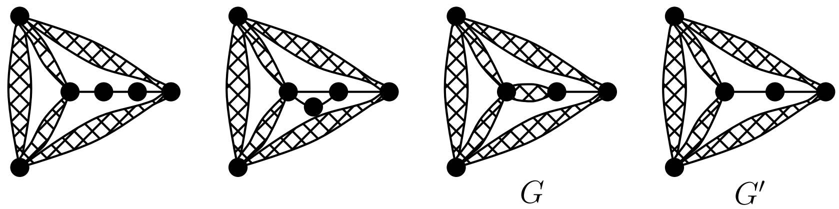


Figure 2.39: M is edge-separable, with internal vertices and does not dominate $K_{2,2}$.

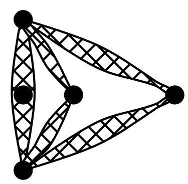


Figure 2.40: $G\backslash e$ for Figure 2.39.

Since M is edge-separable, let e be the edge of M such that $G\backslash e$ separates M from its joint. The limb L that is non-adjacent to M must have an internal vertex as in Figure 2.40, because $G\backslash e$ is not OP.

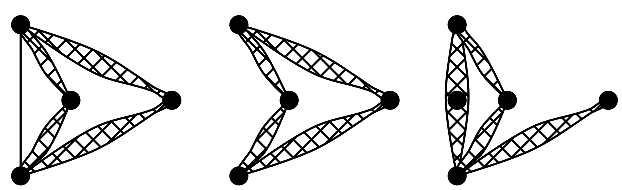


Figure 2.41: $G\backslash e\backslash f$ for Figure 2.40.

Since G is XNOP, there exists f such that $G\backslash e\backslash f$ is OP. The edge $f \notin M\backslash e$, otherwise G is not OP. Furthermore, $G\backslash e\backslash f$ must separate at least one other limb or must separate the bridge with an internal vertex on the non-adjacent limb. See Figure 2.41.

But, if $G \succ G'$ and no edge or vertex of M other than e is deleted or suppressed, then G' must also be XNOP. Hence, G is not minimal, a contradiction. The graph G', as shown in Figure 2.39 is minimal. □

Lemma 2.17. *If a limb of a full-K^4 graph is type $P2$ and its non-adjacent limb is type T, then G is not XNOP.*

Proof. Suppose that G is an XNOP graph with a limb L that is type P2, and a non-adjacent limb M that is type T as in Figure 2.42. Let e be an edge of L. But, $G \backslash e$ is OP unless one limb dominates $K_{2,2}$. See Figure 2.43 (i). Suppose a limb dominates $K_{2,2}$. Let us call that limb N. If N dominates $K_{2,2}$, then the limb can have one bridge or two bridges. If it is two bridges, then by Proposition 2.11, G strictly dominates KF_{1A} or KF_{1B} since one limb of G is type P2. See Figure 2.43 (ii). So, the limb must have one bridge. By Lemma 2.14, the limb non-adjacent to N must have be type P2. But, then G strictly dominates one of KF_{2A} or KF_{2B}, a contradiction. See Figure 2.43 (iii). Hence, $G \backslash e$ is OP and G is not XNOP.

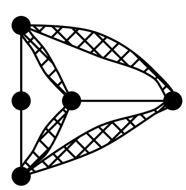


Figure 2.42: L is type T and M is type P2.

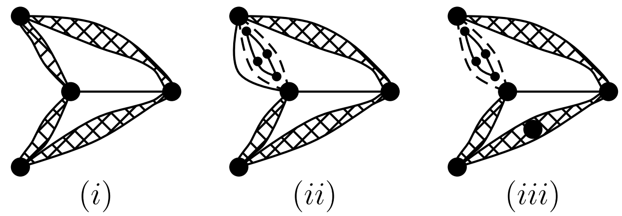


Figure 2.43: $G\backslash e$ for Figure 2.42.

□

Lemma 2.18. *If a limb of a full-K^4 graph G consists of two bridges, one of which has an internal vertex and is not edge-separable, then G is not XNOP.*

Proof. Let M be a limb with a bridge that is not edge-separable and a second trivial bridge as shown in Figure 2.44.

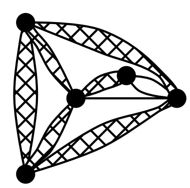
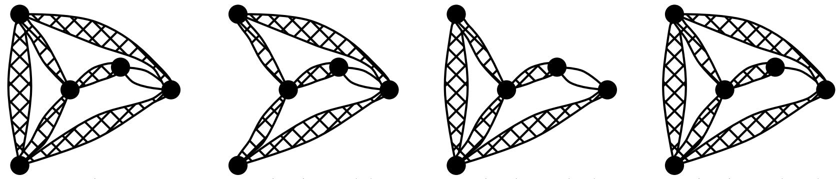


Figure 2.44: G has two bridges, one that is not edge-separable.

Let e be the trivial edge of M. Since G is XNOP, then $G\backslash e$ is not OP. Then, there exists $f \in E(G\backslash f)$ such that $G\backslash e\backslash f$ is OP. The edge f can be in any limb as shown in Figure 2.45.



$G\backslash e$ $G\backslash e\backslash f$ (i) $G\backslash e\backslash f$ (ii) $G\backslash e\backslash f$ (iii)

Figure 2.45: $G\backslash e$ and the three cases of $G\backslash e\backslash f$ for Figure 2.18.

But, only $G\backslash e\backslash f$ (ii) is OP and only if the limb that is adjacent to all of the other limbs does not have an internal vertex. Let us label this limb N. It does not have an internal vertex, so it must be a limb of trivial bridges. It must not be more than two by Proposition 2.10. Since the non-adjacent limb was edge-separable and hence not type DE, the limb N also must not be type DE, by Lemma 2.15. So, N must be a single edge. See Figure 2.46.

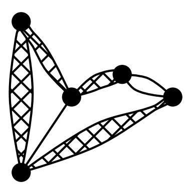


Figure 2.46: $G\backslash e\backslash f$ (ii) is OP.

But, this is a contradiction since $G\backslash f$ should not be OP, but it is, as shown in Figure 2.47.

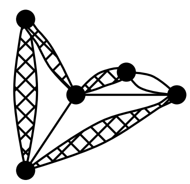


Figure 2.47: $G \backslash f$ (ii) is OP, a contradiction.

□

Lemma 2.19. *If a limb of an XNOP graph consists of two bridges, one of which has an internal vertex, then the limb is type PE and the non-adjacent limb is trivial.*

Proof. Let G be an XNOP graph and M be a limb with a bridge that is edge-separable with an internal vertex and a second trivial bridge as shown in Figure 2.48. Let G' be a graph such that M is replaced by M', a limb of type PE. It is easy to verify that $G \succ G'$. We divide this proof into two parts: (1) prove that M is type PE and (2) prove that the non-adjacent limb is trivial.

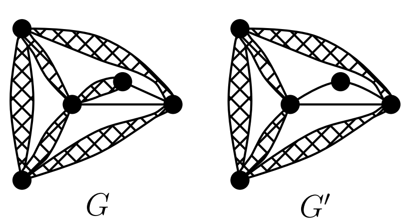


Figure 2.48: G and G' have two bridges, one that is edge-separable and the other trivial.

Let e be the trivial edge of M. Since G is XNOP, $G \backslash e$ is not OP. Then, there exists $f \in E(G \backslash f)$ such that $G \backslash e \backslash f$ is OP. Figure 2.49 shows the cases of $G \backslash e \backslash f$ for an f in each limb of $G \backslash e$.

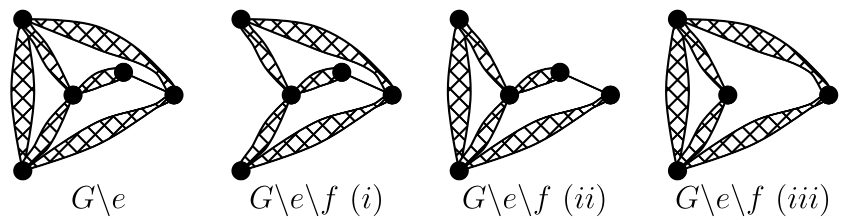


Figure 2.49: $G\backslash e$ and the three cases of $G\backslash e\backslash f$ for Figure 2.19.

But, $G\backslash e\backslash f$ (i) is not OP, and the other cases are OP only if the limb that is adjacent to all of the other limbs in each case does not have an internal vertex. Let us label this limb N. In a manner similar to Lemma 2.18, N must be a single edge. See Figure 2.50.

If $G \succ G'$ and no edge or vertex of M other than e is deleted or suppressed, then G' must also be XNOP. Hence G', not G, is minimal, and M must be type PE.

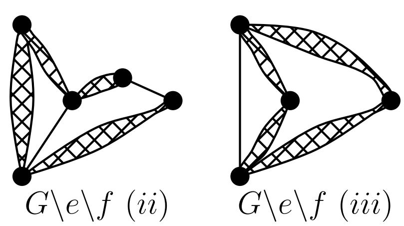


Figure 2.50: The cases of $G\backslash e\backslash f$ that are OP.

By adding back edge e, we can look at $G'\backslash f$ (ii) and (iii) as shown in Figure 2.51. But, $G'\backslash f$ (ii) is a contradiction since it should not be OP. Hence, N must be a trivial limb as shown in Figure 2.52.

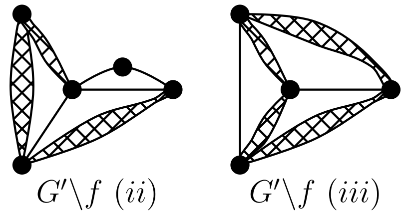


Figure 2.51: $G'\backslash f$ for Figure 2.50.

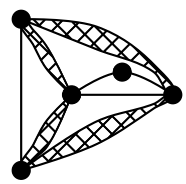


Figure 2.52: G' (ii) for Figure 2.51.

□

Lemma 2.21. *A limb of a full-K^4 XNOP graph G is type $P2$, if and only if its non-adjacent limb is either type $P2$, KL, or KM.*

Proof. Let L be a limb L of type P2, of G and let the edge e be an edge of L. Since G is XNOP, $G\backslash e \succ K_{2,3}$. So, a limb of $G\backslash e$ dominates $K_{2,2}$ or the non-adjacent limb M must have an internal vertex. See Figure 2.53.

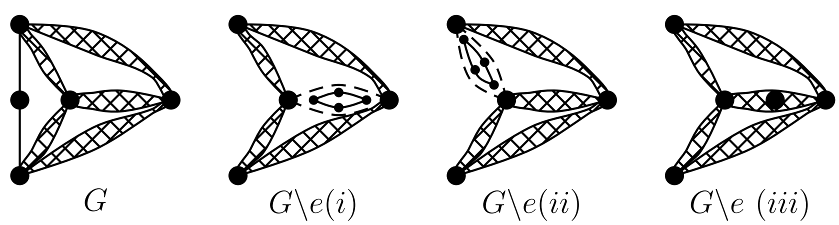


Figure 2.53: G and $G\backslash e$ for L of type P2.

If a limb of $G\backslash e$ dominates $K_{2,2}$, then it can be an adjacent limb or a non-adjacent limb to L as shown in Figure 2.54 case (i) and (ii). If $K_{2,2}$ is a non-adjacent limb, then G properly dominates KF_{1A} or KF_{1B} and is not XNOP, or G is KF_{2A}, or KF_{2B} and M is of type KL or KM.

If the limb that dominates $K_{2,2}$ is an adjacent limb, say N, then G must be as in Figure 2.54 (ii). Since $N \succ K_{2,2}$, it must be of type KL, KM, LE, or ME. But, if it is of type LE or ME, then G properly dominates KF_{1A} or KF_{1B}. If it is of type KL or G, then there exists an edge d such that $G\backslash d$ is not OP. If $G\backslash d$ is not OP, then either another limb of $G\backslash d \succ K_{2,2}$ or the limb adjacent to N has an internal vertex. By Lemma 2.12, if the limb dominates $K_{2,2}$, then G is not XNOP. So, the limb adjacent to N must have an internal vertex as in Figure 2.54. But, then G dominates KF_{2A}, or KF_{2B}, and G is not XNOP.

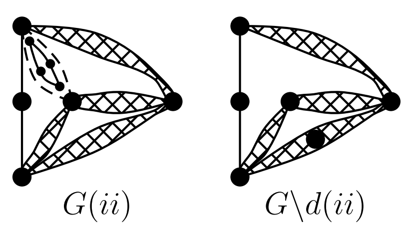


Figure 2.54: G and $G\backslash e$ for L of type P2.

The limb M is edge-separable or it is not. If it is edge-separable, then by Lemma 2.16 and since it must have an internal vertex, it is not type T. So, it must be type P2, KL, or KM.

If M is not edge-separable and has an internal vertex, then it must be type PE or PD. But, by Lemma 2.18 and Lemma 2.19, if M is type PE or PD, L must be type T, a contradiction. Hence, if G has a limb that is type P2, its non-adjacent limb must also be type P2, KL, or KM.

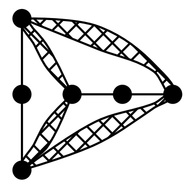


Figure 2.55: Both L and M are type P2.

Conversely, let L be a limb of type P2, KL, or KM, of G. If L is of type P2, then we have already proven that the non-adjacent limb is of type P2. If the limb is of type KL or KM, then let e be an edge of L that separates L from its joints. Since G is XNOP, then $G\backslash e$ must not be OP. So, one of the limbs dominates $K_{2,2}$ or M has an internal vertex. If G is a full-K^4, then by Proposition 2.11 and Lemma 2.12, a second limb must not be of type KL, KM, LE, or ME. So, M must have an internal vertex. The limb M can be edge-separable or not. If M is edge-separable, then it is of type P2. If M is not edge-separable, then it is of type PE or PD. But, by Lemma 2.19 and Lemma 2.20, L must be of type T, a contradiction. Hence, M must be of type P2.

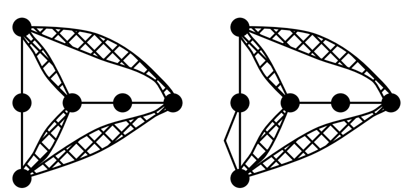


Figure 2.56: Both L and M are type P2.

□

Lemma 2.23. *If a limb of an XNOP graph is type T, then the non-adjacent limb has an internal vertex and is not edge-separable.*

Proof. Suppose that the non-adjacent limb M is type T. See Figure 2.57 for a general depiction of G.

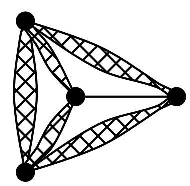


Figure 2.57: M is type T.

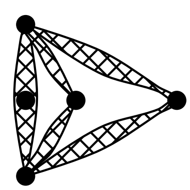


Figure 2.58: $G\backslash e$ for Figure 2.57.

Let e be the single edge of M. The limb L that is non-adjacent to M must have an internal vertex because $G\backslash e$ is not OP as in Figure 2.58.

Since G is XNOP, there exists f such that $G\backslash e\backslash f$ is OP. The graph $G\backslash e\backslash f$ must separate at least one other limb or must separate a bridge with an internal vertex on the non-adjacent limb. See Figure 2.59 for three cases.

We can refine G by adding the edge e back to $G\backslash e\backslash f$ to the three cases as shown in Figure 2.60 and verifying that these cases are not OP per the definition of XNOP. But, case (*iii*) is not possible since it is OP. This is the only case in which removing f results in separating M from its joint. So, removing an edge from M must not separate M from its joint. Hence, M is not edge-separable.

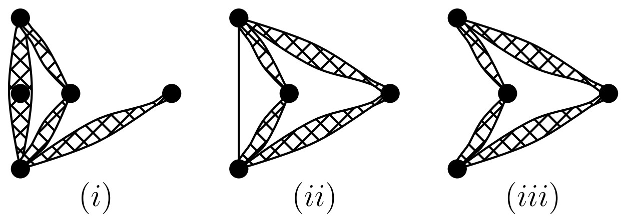


Figure 2.59: Cases of $G\backslash e\backslash f$ for Figure 2.58.

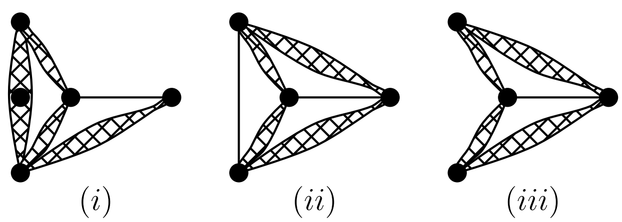


Figure 2.60: Cases of $G\backslash f$ for Figure 2.59.

Of the other two cases of graph $G\backslash f$, case (ii) is not OP, but case (i) is OP unless the limb that is adjacent to all other limbs of $G\backslash f$ has an internal vertex. This gives us two cases for G as shown in Figure 2.61. Case (i) has a limb of type T, and two limbs that have internal vertices, one of them the limb non-adjacent to the one of type T, and case (ii) has a limb of type PE non-adjacent to the one of type T.

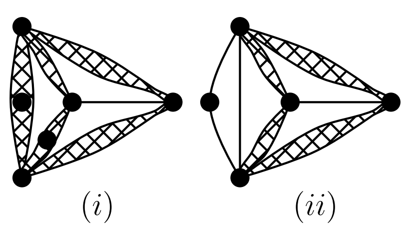


Figure 2.61: G for Figure 2.60.

□

2.3 Algorithms and Results of Full-K^4 XNOP Graphs

In this section, we replace the edges of K^4 with the limbs listed in Figure 2.2, determine if the created graph is XNOP, and verify that the created graph is not isomorphic to one already listed. We could do this by hand, but checking for the XNOP property and also checking for ismorphism would be tedious. We use SageMath to complete this long task and prove the following theorem.

Theorem 2.24. *A 2-connected XNOP graph that is a full-K^4 is one of the following graphs:* KF_{1A}, KF_{1B}, KF_{2A}, KF_{2B}, KF_{3A}, KF_{3B}, KF_{3C}, KF_{3D}, KF_{3E}, KF_{3F}, KF_{3G}, KF_{3H}, KF_{4A}, KF_{4B}, KF_{4C}, KF_{4D}, KF_{4E}, KF_{4F}, KF_{5A}, KF_{5B}, KF_{5C}, KF_{5D}, KF_{5E}, KF_{6A}, KF_{6B}, KF_{6C}, DE_2.

Before we introduce the algorithms for limb permutation, we should remember the definitions of near outer planar (Definition 1.7), excluded outer planar (Definition 1.16) and their associated algorithms.

We use Algorithm 1 to decide whether a graph is NOP by checking to see if a graph is OP. To determine whether a graph is OP, the algorithm uses a built-in function in SageMath called is_circular_planar. This algorithm has a linear time complexity on the number of vertices based on the edge-addition planarity algorithm of [3]. Based on the OP algorithm, we can prove that the NOP algorithm has quadratic time complexity.

Algorithm 1 Determining if a graph is NOP

A graph is NOP if $\exists e \rightarrow G\backslash e$ is OP for $e \in E(G)$
Input: A graph G
Output: True if G is NOP or False if G is not NOP

```
function isNOP(G)
    if G is OP. then
        return True
    end if
    for e ∈ E(G) do
        if e ∉ G.multipleEdges then
            if G\e is OP then
                return True
            end if
        end if
    end for
    return False
end function
```

Proposition 2.25. *Algorithm 1 has a quadratic time complexity.*

Proof. Algorithm 1 has two main parts that contribute to the complexity. By [3], we know that the test for OP has complexity $O(n)$, where $n = |V(G)|$. Hence, the first If-statement and the last If-statement have $O(n)$ complexity. The For-loop executes $m = |E(G)|$ times. Within the For-loop, sits the If-statement with $O(n)$ complexity, so the time complexity for the For-loop can be found by examining mn.

If G is a simple, connected planar graph with $n \geq 3$, then by Euler's formula, $m \leq 3n - 6$. For the purposes of this dissertation, the graphs have at least four vertices and are non-planar. They are not necessarily simple. The graph G must fall into one of three categories - either no edges of G are parallel; some, but not all edges of G are parallel; or all edges of G are parallel. If no edges of G are parallel, then $mn \leq (3n - 6)n = 3n^2 - 6n$. The complexity in this case is $O(n^2)$. If some, but not all edges of G are parallel, then the number of times that the OP algorithm is invoked reduces by the number of sets of parallel edges. Hence, $m \leq 3n - 6 - 1$ and $mn \leq (3n-7)n = 3n^2 - 7n$. The complexity in this case is $O(n^2)$. If all edges of G are parallel, then G is not OP and the algorithm will not invoke the test for OP. Hence, $m \leq 3n - 6$. The complexity in this case is $O(n)$. Therefore, the overall complexity based on the worst case is $O(n^2)$. □

To determine if a graph is XNOP, Algorithm 2 uses Algorithm 1 and two built-in functions from SageMath, one that calculates the neighbors of a vertex and another, that calculates the length of a set.

Algorithm 2 Determining if a graph is XNOP

A graph is XNOP if G is not NOP, but $G \backslash e$ is NOP for every $e \in E(G)$ and a suppression of every vertex of G with exactly two neighbors is NOP.
Input: A graph G
Output: True if G is XNOP or False if G is not XNOP

```
function isXNOP(G)
    if G is NOP. then
        return False
    end if
    for e ∈ E(G) do
        if G\e is not NOP then
            return False
        end if
        for v ∈ V(G) do
            if |neighbors of v| = 2 then
                supress v
                if G supress v is not NOP then
                    return False
                end if
            end if
        end for
    end for
    return True
end function
```

Proposition 2.26. *Algorithm 2 has a cubic time complexity.*

Proof. Algorithm 2 has two main parts that contribute to the complexity. By Proposition 2.25, we know that the test for NOP has complexity $O(n^2)$, where $n = |V(G)|$. Hence, the the three If-statements that test for NOP have $O(n^2)$ complexity. The outside For-loop executes $m = |E(G)|$ times. Within the outside For-loop, there are two loops in series - (1) an If-statement with $O(n^2)$ complexity and (2) another For-loop. We will sum the complexity of each of these, then apply the outside For-loop. Since we know the complexity of (1), we need only analyze the complexity of (2).

The inside For-loop executes $n = |V(G)|$ times. Hence, the inside For-loop has complexity $nO(n^2) = O(n^3)$ complexity. Since the complexity of (2) is greater than that of (1), and the loops are in series, the complexity of the loops within the outside For-loop is $O(n^3)$.

The outside For-loop tests all edges. Hence, the overall complexity can be found by examining mn^3. If G is a simple, connected planar graph with $n \geq 3$, then by Euler's formula, $m \leq 3n - 6$. For the purposes of this dissertation, the graphs have at least four vertices and are non-planar. They are not necessarily simple, but we know from Section 2.1 that the only possible limbs of an XNOP graph are shown in Figure 2.2. No limb in this list has more than two parallel edges. Hence, $m \leq 2(3n-6) = 6n-12$, and the overall complexity is $mO(n^3) \leq (6n-12)n^3 = 6n^4 - 12n^3$. Then, Algorithm 2 has complexity $O(n^4)$. □

With these algorithms for NOP and XNOP, we can look at the programs for permuting the limbs of a full-K^4 XNOP graph G. We will need two algorithms to do this. In Algorithm 3, given two vertices of an edge of K^4 and the specified limb type, we change the edge of K^4 to the limb type. Then, in Algorithm 4, we call the makeLimb function from Algorithm 3 six times, once for each limb, to change all edges of K^4 to the limbs specified.

Algorithm 3 Create graphs with a new limb of type T, P2, DE, PE, PDE, KL1, KL2, KM, KLE1, KLE2, or KME

This function replaces one specified edge of K^4 with a limb type.
Input: K^4, two joints of a limb of K^4, and the limb type.
Output: a full-K^4 graph with one limb of type P2 - KME and five trivial limbs
function MAKELIMB(`graph`, `u`, `v`, `limbType`)
    Given G, with vertices u and v, change the edge from u to v to limbType.
end function

Algorithm 4 Create graphs with six limbs, each of type T, P2, DE, PE, PDE, KL1, KL2, KM, KLE1, KLE2, or KME

This function uses Algorithm 3 six times to replace all of the limbs with a specified limb type.
Input: K^4 and six limb types.
Output: a full-K^4 graph with six limbs of type T - KME.
function SETALLLIMBS(`graph`, `L1`, `L2`, `L3`, `L4`, `L5`, `L6`)
    makeLimb{graph, 0, 1, L1}
    makeLimb{graph, 0, 2, L2}
    makeLimb{graph, 0, 3, L3}
    makeLimb{graph, 1, 2, L4}
    makeLimb{graph, 1, 3, L5}
    makeLimb{graph, 2, 3, L6}
    return graph
end function

Although, it would be easy to program each limb with the eleven types of limbs and check that each graph created is XNOP, it would take a long time to decide whether $11^6 = 1,771,561$ graphs are XNOP, and then to check that the verified XNOP graph was not already on the steadily growing list of XNOP graphs. To reduce this load, we look at symmetries of the graph based on permuting one limb, then two limbs, etc., up to six limbs. We can also eliminate some limbs based on propositions and lemmas from Section 2.1. We begin with one limb.

Let t be the number of nontrivial limbs of G for our permuations of limbs of a full-K^4. In Figure 2.62, we show a full-K^4 with $t = 1$. Since all edges of K^4 are isomorphic, permuting the list of limbs in one limb of G results in the same list of graphs as permuting the list in another limb. Hence, we need only permute the list of limbs once for $t = 1$ as in Algorithm 5. Also, the input to the algorithm is K^4, which is a full-K^4 with all limbs of type T, or (T,T,T,T,T,T). Hence, we need not permute the limb T in subsequent algorithms.

Algorithm 5 List all full-K^4 XNOP graphs with one limb permutation, up to isomorphism.

For $t = 1$, permute all of the limb types T-KME in a full-K^4.
Input: K^4, $L = [T, P2, DE, PE, PDE, KL1, KL2, KM, KLE1, KLE2, KME]$
Output: full-K^4 XNOPList for $t = 1$
for $1 \leq i < 11$ **do**
    G = setAllLimbs(G, $L[i]$, T, T, T, T, T)
    if G is XNOP **then**
        if G is not on XNOPList **then**
            append G to XNOPList
        end if
    end if
end for
return XNOPList

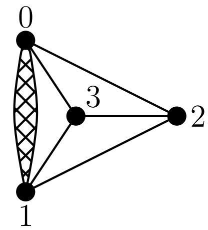


Figure 2.62: G for $t = 1$ in Algorithm 5.

The limbs of type KL and LE are not symmetrical and therefore have two orientations. In Algorithm 3 and Algorithm 4, the limbs KL1, KL2, LE1, and LE2 indicate these orientations.

For $t = 1$, we found two non-isomorphic full-K^4 graphs, (LE1,T,T,T,T,T) and (ME,T,T,T,T,T), which are KF_{1A} and KF_{1B}, respectively.

When $t > 1$, the adjacency of each edge of K^4 determines how many distinct graphs of a full-K^4 can be permuted without repeating the permutation on an isomorphic graph.

For $t = 2$, there are two non-isomorphic graphs as shown in Figure 2.63. We can also reduce the programming load by not creating a graph for every permutation in this case. For instance, (P2,DE,T,T,T,T) is the same graph as (DE,P2,T,T,T,T). Hence, in the algorithm for $t = 2$, combinations can be verified instead of permutations, and the loop for the second limb begins at i and not at 1 as it will for other loops in later algorithms.

The programming load can be further reduced by removing three limb types overall and eliminating some combinations. By Proposition 2.11, a graph with a limb that dominates limb type LE1, LE2 or ME dominates KF_{1A} or KF_{1B}. So, it is redundant to check these limbs in subsequent algorithms. Also, by Lemma 2.12, graphs with two limbs of type KL or KM dominate graphs which are not full-K^4 graphs. Hence, we need not run the algorithm on graphs with two limbs of type KL or KM.

Algorithm 6 List all full-K^4 XNOP graphs with two limb permutations, up to isomorphism.

```
For t = 2, find all permutations of the limb types T-KM in a full-K^4.
Input: K^4, L = [T, P2, DE, PE, PDE, KL1, KL2, KM, KLE1, KLE2, KME]
Output: full-K^4 XNOPList for t = 2
for 1 ≤ i < 8 do
    for i ≤ j < 8 do
        if not (i > 4 and j > 4) then
            G = setAllLimbs(G, L[i], L[j], T, T, T, T)
            if G is XNOP then
                if G is not on XNOPList then
                    append G to XNOPList
                end if
            end if
        end if
    end for
end for
G = K^4
for 1 ≤ i < 8 do
    for i ≤ j < 8 do
        if not (i > 4 and j > 4) then
            G = setAllLimbs(G, L[i], T, T, T, T, L[j])
            if G is XNOP then
                if G is not on XNOPList then
                    append G to XNOPList
                end if
            end if
        end if
    end for
end for
return XNOPList
```

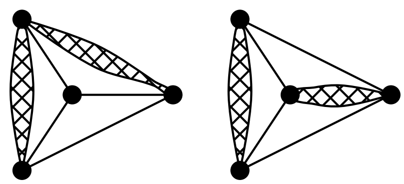


Figure 2.63: G for $t = 2$ in Algorithm 6.

Algorithm 6 resulted in two non-isomorphic full-K^4 XNOP graphs, (P2,T,T,T,T,KL1) and (P2,T,T,T,T,KM), which are KF_{2A} and KF_{2B}, respectively.

For $t = 3$, there are three non-isomorphic graphs as shown in Figure 2.64. For the first two cases in Figure 2.64, we can reduce the programming load by not creating a graph for every permutation. For instance, (P2,DE,PE,T,T,T) is the same graph as (DE,P2,PE,T,T,T). Hence, in the parts of the algorithm for those cases of $t = 3$, combinations can be verified instead of permutations. So, the loop for the second limb begins at i, instead of 1, and the loop for the third limb begins at j.

The third case has symmetries, but not the same as the first two cases. In this case, of the three limbs that are replaced, two are non-adjacent to each other and one is adjacent to the other two limbs. The two limbs that are non-adjacent to each other are symmetric, so duplicates can be eliminated. In the loops, one of these limbs will start at the counter for the other. See Algorithm 7. We can further reduce the programming load by applying Corollary 2.22. Since a full-K^4 XNOP graph with a limb of type KL or KM must be KF_{2A} or KF_{2B}, then no other full-K^4 graphs with limb types KL or KM need be checked, for $t > 2$.

Algorithm 7 List all full-K^4 XNOP graphs with three limb permutations, up to isomorphism.

For $t = 3$, find all permutations of the limb types T-PDE in a full-K^4.
Input: K^4, $L = [T, P2, DE, PE, PDE, KL1, KL2, KM, KLE1, KLE2, KME]$
Output: full-K^4 XNOPList for $t = 3$

```
for 1 ≤ i < 5 do
    for i ≤ j < 5 do
        for j ≤ k < 5 do
            G = setAllLimbs(G, L[i], L[j], L[k], T, T, T)
            if G is XNOP then
                if G is not on XNOPList then
                    append G to XNOPList
                end if
            end if
        end for
    end for
end for
G = K^4
for 1 ≤ i < 5 do
    for i ≤ j < 5 do
        for j ≤ k < 5 do
            G = setAllLimbs(G, L[i], L[j], T, L[k], T, T)
            if G is XNOP then
                if G is not on XNOPList then
                    append G to XNOPList
                end if
            end if
        end for
    end for
end for
G = K^4
for 1 ≤ i < 5 do
    for 1 ≤ j < 5 do
        for j ≤ k < 5 do
            G = setAllLimbs(G, L[i], L[j], T, T, T, L[k])
            if G is XNOP then
                if G is not on XNOPList then
                    append G to XNOPList
                end if
            end if
        end for
    end for
end for
return XNOPList
```

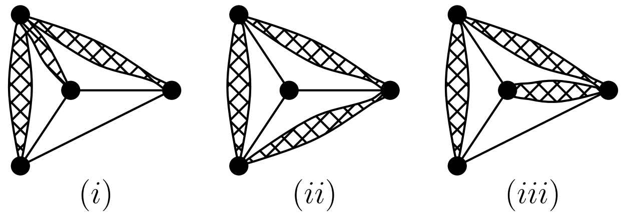


Figure 2.64: G for $t = 3$ in Algorithm 7.

Algorithm 7 resulted in eight non-isomorphic full-K^4 XNOP graphs, (PE,PE,PE,T,T,T), (PE,PE,PD,T,T,T), (PE,PD,PD,T,T,T), (PD,PD,PD,T,T,T), (PE,PE,T,PE,T,T), (PE,PE,T,PD,T,T), (PE,PD,T,PD,T,T), and (PD,PD,T,PD,T,T), which are KF_{3A}, KF_{3B}, KF_{3C}, KF_{3D}, KF_{3E}, KF_{3F}, KF_{3G}, and KF_{3H}, respectively.

For $t = 4$, there are two non-isomorphic graphs as shown in Figure 2.65. We need not check the limbs of the first case, since by Lemma 2.23, if a limb is of type T, then the non-adjacent limb cannot also be type T.

For the second case in Figure 2.65, we can reduce the programming load in a manner similar to cases of $t = 3$ by noting the symmetry of two limbs of this case. In the loops, one of these limbs will start at the counter for the other. See Algorithm 8.

Algorithm 8 List all full-K^4 XNOP graphs with four limb permutations, up to isomorphism.

For $t = 4$, find all permutations of the limb types T-PDE in a full-K^4.
Input: K^4, $L = [T, P2, DE, PE, PDE, KL1, KL2, KM, KLE1, KLE2, KME]$
Output: full-K^4 XNOPList for $t = 4$
for $1 \leq i < 5$ **do**
  for $1 \leq j < 5$ **do**
    for $1 \leq k < 5$ **do**
      for $k \leq l < 5$ **do**
        G = setAllLimbs(G, $L[k]$, $L[l]$, $L[i]$, $L[j]$, T, T)
        if G is XNOP **then**
          if G is not on XNOPList **then**
            append G to XNOPList
          end if
        end if
      end for
    end for
  end for
end for
return XNOPList

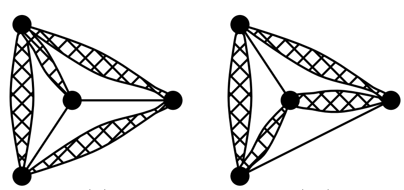


Figure 2.65: G for $t = 4$ in Algorithm 8.

Algorithm 8 resulted in six non-isomorphic full-K^4 XNOP graphs, (PE,PE,P2,P2,T,T), (PE,PD,P2,P2,T,T), (PD,PD,P2,P2,T,T), (PE,PE,DE,DE,T,T), (PE,PD,DE,DE,T,T), and (PD,PD,DE,DE,T,T), which are KF_{4A}, KF_{4B}, KF_{4C}, KF_{4D}, KF_{4E}, and KF_{4F}, respectively.

For $t = 5$, there is one non-isomorphic graph as shown in Figure 2.66. Since we are permuting five limbs, there are two sets of symmetric limbs, and one limb that does not share symmetry. The symmetries are in non-adjacent limbs. See Algorithm 9.

Algorithm 9 List all full-K^4 XNOP graphs with five limb permutations, up to isomorphism.

For $t = 5$, find all permutations of the limb types T-PDE in a full-K^4.
Input: K^4, $L = [T, P2, DE, PE, PDE, KL1, KL2, KM, KLE1, KLE2, KME]$
Output: full-K^4 XNOPList for $t = 5$
for $1 \leq i < 5$ **do**
    for $1 \leq j < 5$ **do**
        for $j \leq k < 5$ **do**
            for $1 \leq l < 5$ **do**
                for $l \leq m < 5$ **do**
                    G = setAllLimbs(G, $L[i]$, $L[j]$, $L[k]$, $L[m]$, $L[l]$, T)
                    if G is XNOP **then**
                        if G is not on XNOPList **then**
                            append G to XNOPList
                        end if
                    end if
                end for
            end for
        end for
    end for
end for
return XNOPList

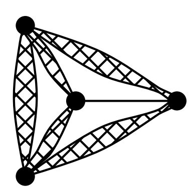


Figure 2.66: G for $t = 5$ in Algorithm 9.

Algorithm 9 resulted in five non-isomorphic full-K^4 XNOP graphs, (PE,P2,P2,P2,P2,T), (PD,P2,P2,P2,P2,T), (PE,P2,DE,DE,P2,T), (PD,P2,DE,DE,P2,T), and (PE,DE,DE,DE,DE,T), which are KF_{5A}, KF_{5B}, KF_{5C}, KF_{5D}, and KF_{5E}, respectively.

For $t = 6$, there is one non-isomorphic graph as shown in Figure 2.67. Two symmetries are programmed to reduce the load. One based on the first three limbs, and the other based on the last three limbs. See Algorithm 10.

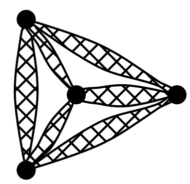


Figure 2.67: G for $t = 6$ in Algorithm 10.

Algorithm 10 resulted in four non-isomorphic full-K^4 XNOP graphs, (P2,P2,P2,P2,P2,P2), (P2,P2,DE,DE,P2,P2), (P2,DE,DE,DE,DE,P2), and (DE,DE,DE,DE,DE,DE), which are KF_{6A}, KF_{6B}, KF_{6C}, and DE_2, respectively.

Algorithm 10 List all full-K^4 XNOP graphs with six limb permutations, up to isomorphism.

For $t = 6$, find all permutations of the limb types T-PDE in a full-K^4.
Input: K^4, $L = [T, P2, DE, PE, PDE, KL1, KL2, KM, KLE1, KLE2, KME]$
Output: full-K^4 XNOPList for $t = 6$
for $1 \leq i < 5$ **do**
  for $i \leq j < 5$ **do**
    for $j \leq k < 5$ **do**
      for $1 \leq l < 5$ **do**
        for $l \leq m < 5$ **do**
          for $m \leq n < 5$ **do**
            G = setAllLimbs(G, $L[i]$, $L[j]$, $L[k]$, $L[n]$, $L[m]$, $L[l]$)
            if G is XNOP **then**
              if G is not on XNOPList **then**
                append G to XNOPList
              end if
            end if
          end for
        end for
      end for
    end for
  end for
end for
return XNOPList

This concludes the permutation of all of the limbs of a full-K^4. The list of 27 full-K^4 graphs in Theorem 2.24 have been verified by programming, to be XNOP and complete.

In the next chapter, we revisit the idea of skeletons to find the next set of skeletons that are one edge away from K^4.

CHAPTER 3

K^4 PLUS ONE EDGE

In the previous chapter, we found all XNOP graphs with K^4 as a skeleton. To find other XNOP graphs or prove that we have a complete list, we must look at other possible skeletons of XNOP graphs, then use the skeletons found along with the limbs found in Chapter 2 to find all of the XNOP graphs with skeletons of K^4 plus one edge. We do this with five sections. The first one gives us the usable skeletons of K^4 plus an edge, which are W_4 and 3-prism. The second and third examine the limbs of XNOP graphs of W_4 to find the complete list of full W_4 XNOP graphs. Lastly, the fourth and fifth sections prove the complete list of full 3-prism XNOP graphs in a similar manner to Sections 3.2 and 3.3.

3.1 Skeletons of K^4 Plus One Edge

In this section, we examine each of the possible BG-operations and determine if these are feasible constructions of skeletons for an XNOP graph with K^4 plus one edge.

By symmetry, every resulting graph from BG-operations (1)-(3) to K^4 is isomorphic to one of the graphs shown in Figure 3.1.

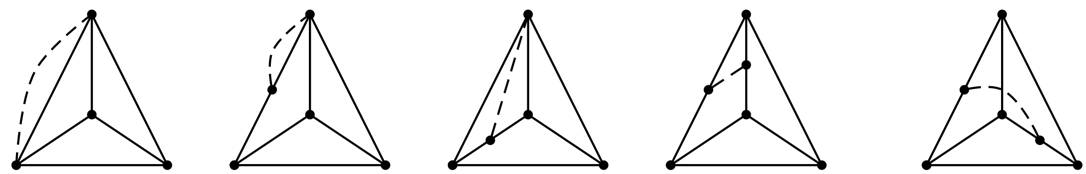


Figure 3.1: Examples of graphs of K^4 plus an edge.

Hence, the addition of a single edge to K^4 gives us two new types of skeletons of XNOP graphs, W_4, and 3-prism. We can rule out $K_{3,3}$ since G would be nonplanar. In the next two sections, we explore the limbs of W^4 and 3-prism to find the XNOP graphs with those graphs as skeletons and prove that those lists are finite.

3.2 Overview of All Possible Full-W_4 XNOP Graphs

In this section, we examine the full-W_4 graphs that are XNOP. For brevity, we only outline the theorems, lemmas and corollaries necessary to prove the complete list of full-W_4 XNOP graphs in this section. We save the detailed proofs of theorems, etc. for Section 3.3. Although we can use programming to confirm our results, the list of full-W_4 XNOP graphs is small enough to prove without programming.

In Chapter 2, we found all possible limbs of a full-K^4. Since $W_4 \succ K^4$ as shown in Figure 3.1, then the same limb list proved for K^4 can be a starting point for the list of possible limbs of a full-W_4.

Also, since $W_4 \succ K^4$, by Proposition 2.11, we need not consider limb types LE and ME as possible limbs for a full-W_4 XNOP graph. Furthermore, we can eliminate limb types KL and KM since the replacement of an edge of W_4 with either limb type KL or KM results in a graph that properly dominates KF_{2A} or KF_{2B}, by Corollary 2.22.

To prove that we have the complete list of full-W_4 XNOP graphs, we use the following theorem, which can be proved with five lemmas and one corollary. To aid the reader in understanding the long proof to the main theorem, many of the proofs to the lemmas and corollary are in Section 3.3. Since W_4 is a wheel, it is helpful to recall the definition of a wheel (Definition 1.5). In particular, we refer to the *rim* and *spokes* of W_4.

Theorem 3.1. *A full-W_4 XNOP graph is one of the following graphs: WF_1, WF_2, WF_3.*

Proof. Let G be a full-W_4 XNOP graph that is not listed above. In Chapter 2, we proved many propositions, lemmas and corollaries about the limbs of a full-K^4. In this chapter, we focus on the spokes of a full-W_4 to prove that the list of full-W_4 XNOP graphs is complete. The first lemma gives us an upper bound on the limbs of the spoke of a full-W_4.

Lemma 3.2. *A full-W_4 graph that has a spoke limb that is not edge-separable is not XNOP.*

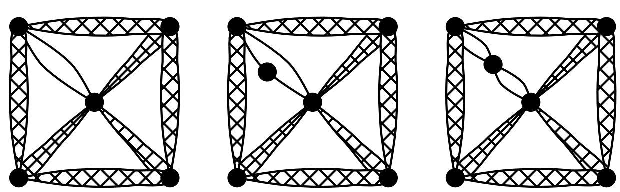


Figure 3.2: Lemma 3.2 - Full-W_4 graphs with spoke limbs that are not edge-separable are not XNOP.

The proof of Lemma 3.2 is in Section 3.3. The following corollary to Lemma 3.2 is easy to verify.

Corollary 3.3. *A limb that is a spoke of a full-W_4 XNOP graph is of type T or $P2$.*

We now have six basic graphs with spoke limbs of type T or P2 as shown in Figure 3.3. Three of the graphs dominate WF_1. The following lemma is easy to verify.

Lemma 3.4. *If a full-W_4 graph G has two non-adjacent spoke limbs of type $P2$, then $G \succ WF_1$.*

This gives us three cases to check for XNOP graphs. These are the only possibilities for full-W_4 XNOP cases that do not dominate WF_1. We start with the graph with two spokes of type P2, Figure 3.3, case (i) and the following lemma, whose proof is in Section 3.3.

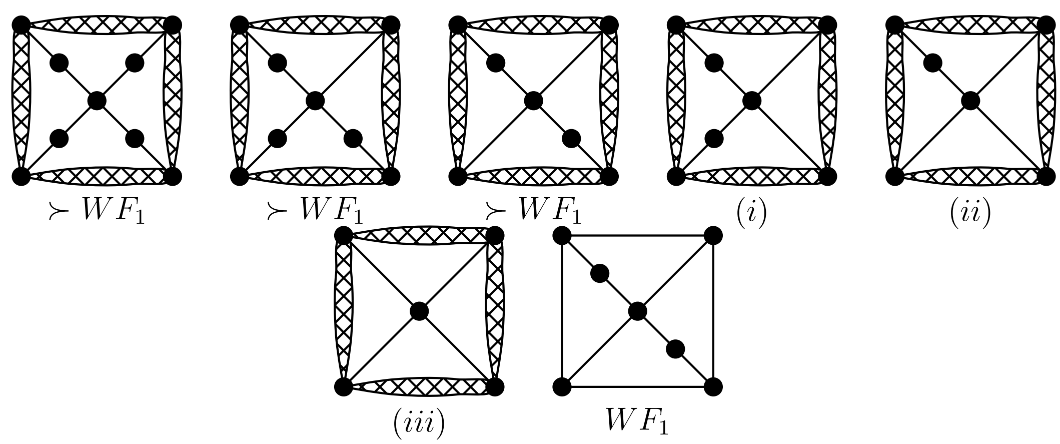


Figure 3.3: There are six full-W_4 graphs with spoke limbs of type T or P2.

Lemma 3.5. *A full-W_4 XNOP graph with two adjacent spoke limbs of type $P2$ dominates S_1.*

Hence, G must have one or no spoke limbs of type P2. If G has exactly one spoke limb of type P2, as in case (ii), then the following lemma applies. See Section 3.3 for the details of the proof.

Lemma 3.6. *A full-W_4 XNOP graph with one spoke limb of type $P2$ dominates WF_2.*

Hence, G must have no spoke limbs of type P2, as in case (iii) and the following lemma applies.

Lemma 3.7. *A full-W_4 XNOP graph with all spoke limbs of type T dominates WF_3.*

See Section 3.3 for the details of the proof.

By Corollary 3.3, the spoke limbs must be of type T or P2. Of the six cases of these type, we have found that a full-W_4 XNOP graph is WF_1, WF_2, or WF_3.

□

3.3 Proofs of All Possible Full-W_4 XNOP Graphs

Theorem 3.2. *A full-W_4 graph that has a spoke limb that is not edge-separable is not XNOP.*

Proof. Let G be a full-W_4 XNOP graph with a limb L that is a spoke and that is not edge-separable. So, L must be of type DE, PE, or PD as in Figure 3.4

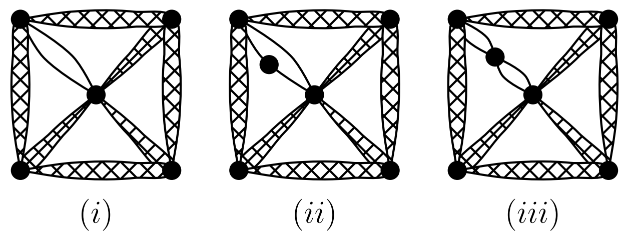


Figure 3.4: A full-W_4 with a limb that is a spoke and that is not edge-separable.

Let e be an edge of L. Since G is XNOP, then $G\backslash e$ is not OP, and there exists an edge f such that $G\backslash e\backslash f$ is OP. The edge f must separate a limb on the outer rim of G, and in cases (ii) and (iii), f must be of a limb adjacent to the L. Otherwise, $G\backslash e\backslash f$ is not OP as shown in Figure 3.5.

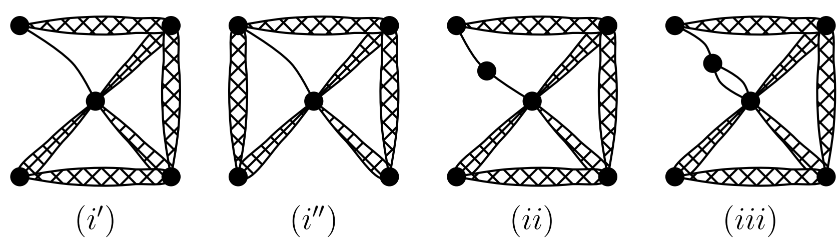


Figure 3.5: The edge f must be an edge of the rim.

The graph $G\backslash e\backslash f$ is OP, so the two remaining inner spoke limbs of $G\backslash e\backslash f$ must not have internal vertices as in Figure 3.6.

Since G is XNOP, then $G\backslash f$ is not OP. But, in each case, it is OP, as shown in Figure 3.7, a contradiction. Hence, a limb that is a spoke of a full-W_4 XNOP graph is edge-separable.

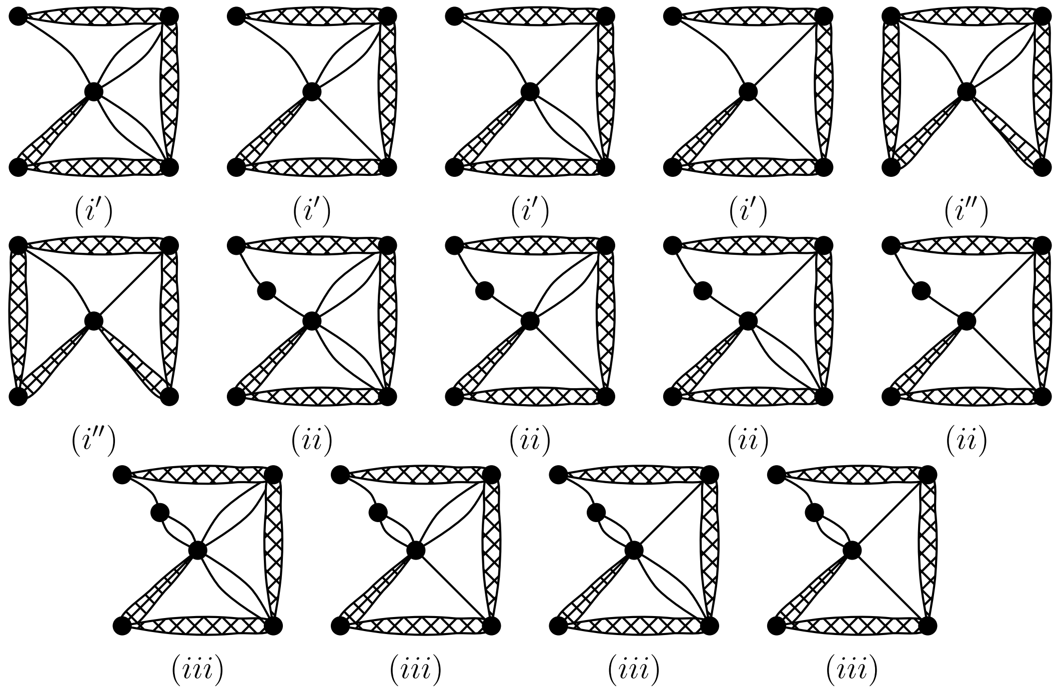


Figure 3.6: Refinement of $G\backslash e\backslash f$.

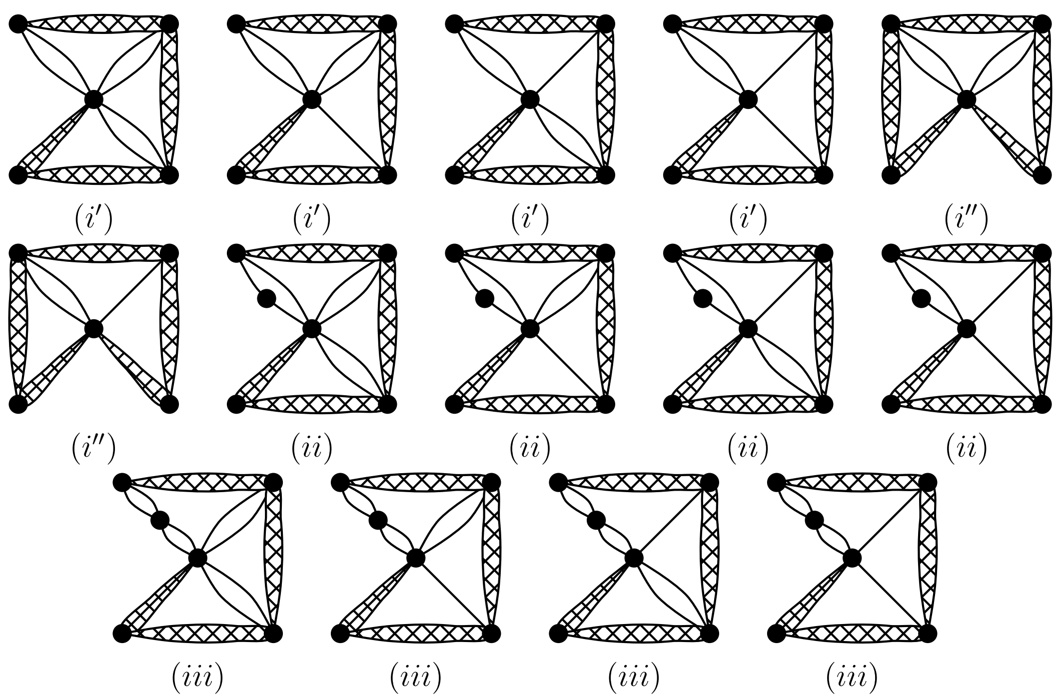


Figure 3.7: $G\backslash f$ is OP, a contradiction.

$\square$

Lemma 3.5. *A full-W_4 XNOP graph with two adjacent spoke limbs of type P2 dominates S_1.*

Proof. Let G be a full-W_4 XNOP graph with two spoke limbs of type P2 and let L be the limb on the rim of G that is adjacent to the two spoke limbs of type P2. The limb L is not edge-separable, otherwise, G is NOP and not XNOP as shown in Figure 3.8. But, if L is not edge-separable, then $G \succ S_1$.

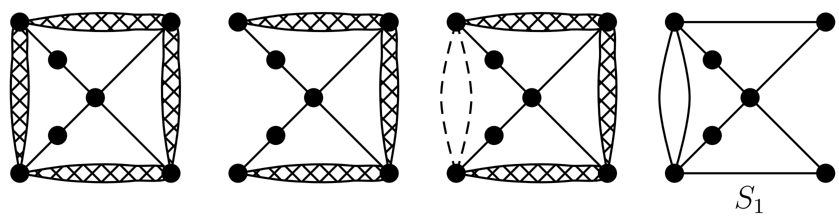


Figure 3.8: Full-W_4 graphs with two non-adjacent spoke limbs of type P2 dominates S_1.

□

Lemma 3.6. *A full-W_4 XNOP graph with one spoke limb of type P2 dominates WF_2.*

Proof. Let G be a full-W_4 XNOP graph with exactly one limb of type P2. The two limbs on the rim of G that are adjacent to the spoke limb of type P2 must not be edge-separable. Otherwise, G is NOP. But, then $G \succ WF_2$.

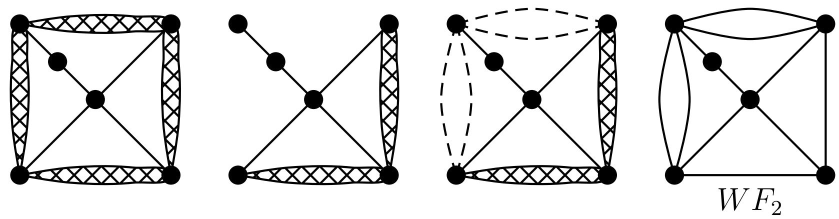


Figure 3.9: Full-W_4 graphs with one spoke limb of type P2 dominates WF_2.

□

Lemma 3.7. *A full-W_4 XNOP graph with all spoke limbs of type T dominates WF_3.*

Proof. Let G be a full-W_4 XNOP graph with all spoke limbs of type T. All of the rims of G must not be edge-separable. Otherwise, G is NOP. But, then $G \succ WF_3$, a contradiction.

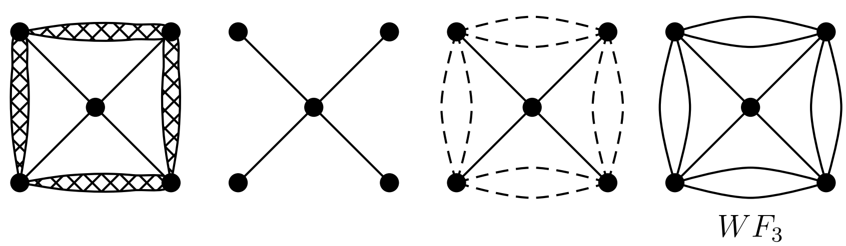


Figure 3.10: Full-W_4 graphs with all spoke limbs of type T dominates WF_3.

□

In the next two sections, we look at the only other skeleton that is one edge away from K^4, a 3-prism.

3.4 Overview of All Possible Full-3-Prism XNOP Graphs

In the next two sections, we examine the full-3-prism graphs, or full-TP graphs, that are XNOP. In this section, we provide a main theorem and an outline of its lemmas and corollaries, but with no proofs of the lemmas and corollaries. We save the proofs for Section 3.5, as there are many parts to the main theorem and the proofs of the lemmas and corollaries can be long. Although we can use programming to confirm our results, the list of full-3-prism XNOP graphs is small enough to prove without programming.

In Chapter 2, we found all possible limbs of a full-K^4. Since 3-prism $\succ K^4$ as shown in Figure 3.11, then the same limb list proved for K^4 can be a starting point for the list of possible limbs of a full-TP.

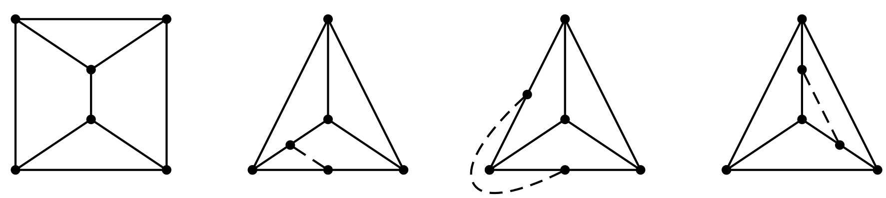


Figure 3.11: 3-prism $\succ K^4$.

Since 3-prism $\succ K^4$, by Proposition 2.11, we need not consider limb types LE and ME as possible limbs for a full-TP XNOP graph. Furthermore, we can eliminate limb types KL and KM since the replacement of an edge of 3-prism with either limb type KL or KM results in a graph that properly dominates KF_{2A} or KF_{2B}, by Corollary 2.22.

To prove that we have the complete list of full-TP XNOP graphs, we use the following theorem, which can be proved with seven lemmas and one corollary. For brevity, the proofs to the lemmas and corollary are in Section 3.5. In the following theorem and proofs, it is also helpful to recall the definition of a prism (Definition 1.6). In particular, we will refer to the *cycles* and *spurs* of a 3-prism.

Theorem 3.8. *A full-3-prism XNOP graph is one of the following graphs:* TP_1, TP_2, TP_3, TP_4, TP_5, TP_6.

Proof. Let G be a full-TP XNOP graph that is not listed above. We will prove an important lemma that gives us an upper bound on the limbs of the cycles of G, then focus on the limbs of the spurs of G.

Lemma 3.9. *If a limb of a cycle of a full-TP graph* G *is not edge-separable, then* G *is not XNOP.*

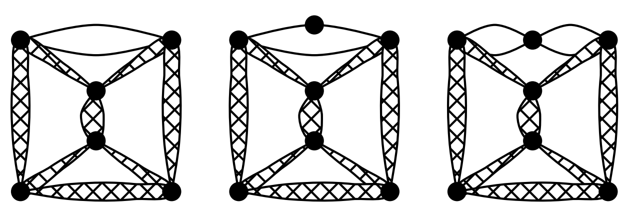


Figure 3.12: Lemma 3.9 - Full-TP graphs with cycle limbs that are not edge-separable are not XNOP.

This corollary follows.

Corollary 3.10. *A limb of a cycle of a full-TP XNOP graph is of type T or P2.*

Since we know that the cycle limbs must be of type T or P2, we can find all combinations of the basic full-TP graphs with limbs of type T or P2. This gives us thirteen non-isomorphic cases for cycle limbs of type T or P2 as shown in Figure 3.13. We can classify the graphs by locating the limbs of type P2 in relation to the cycle that it is located and in relation to the face that it is located.

A number of these dominate TP_1 or TP_2, which gives us the two following lemmas that are easy to verify.

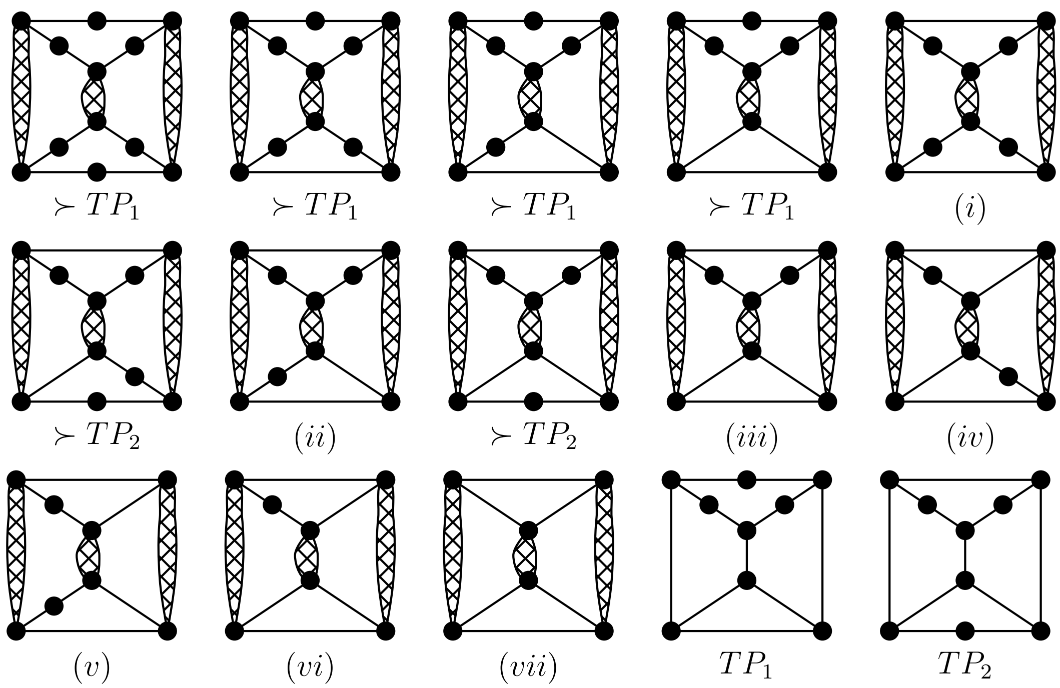


Figure 3.13: Full-TP graphs with cycle limbs of type T or P2.

Lemma 3.11. *A full-TP graph with three limbs of type P2 all in one cycle dominates* TP_1.

Lemma 3.12. *A full-TP graph with three limbs of type P2, two of which are in one cycle and one that does not share a face with another limb of type P2, dominates* TP_2.

We can now examine cases (i)-(iii) in the following lemma.

Lemma 3.13. *A full-TP graph that is not NOP, with two limbs, L and M, of type P2 in one cycle and a limb N of type T in the opposite cycle such that N does not share a face with L or M, dominates* TP_3.

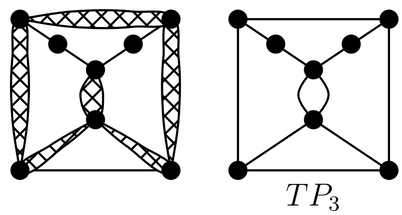


Figure 3.14: Lemma 3.13 - Full-TP graphs that are not NOP with two limbs of type P2 in one cycle dominate TP_3.

For case (iv), we have the following lemma.

Lemma 3.14. *A full-TP graph that is not NOP, with two cycle limbs, L and M, of type P2 such that L and M do not share a face, dominates* TP_4.

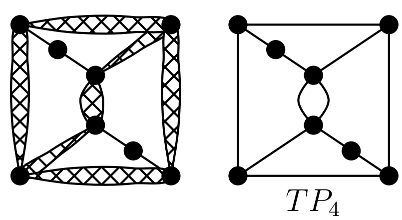


Figure 3.15: Lemma 3.14 - Full-TP graphs that are not NOP with two limbs of type P2 such that L and M do not share a face dominate TP_4.

The lemma for cases (v) and (vi) is the following.

Lemma 3.15. *A full-TP graph that is not NOP, with one cycle with exactly one limb L of type P2 and the other cycle such that the limbs that do not share a face with L are of type T, dominates* TP_5.

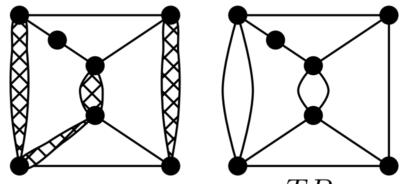


Figure 3.16: Lemma 3.15 - Full-TP graphs that are not NOP dominate TP_5.

For case (vii), we have the following lemma.

Lemma 3.16. *A full-TP graph that is not NOP, with no cycle limbs of type $P2$ dominates TP_6.*

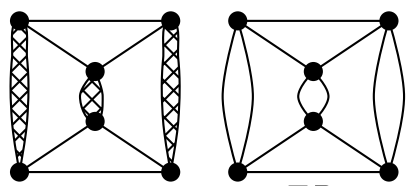


Figure 3.17: Lemma 3.16 - Full-TP graphs that are not NOP and have no cycle limbs of type P2 dominate TP_6.

□

3.5 Proofs of All Possible Full-3-Prism XNOP Graphs

Lemma 3.9. *If a limb of a cycle of a full-TP graph G is not edge-separable, then G is not XNOP.*

Proof. Let G be a full-TP XNOP graph with a cycle limb L that is not edge-separable as in Figure 3.18. Let e be an edge of L. Since G is XNOP, $G \backslash e$ is not OP, and there exists an edge $f \in E(G \backslash e)$ such that $G \backslash e \backslash f$ is OP. It is easy to verify that the edge f must be an edge of one of the spur limbs of G. See Figure 3.19.

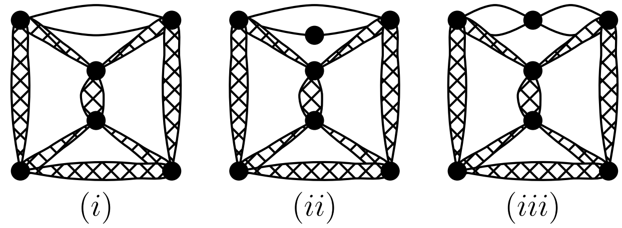


Figure 3.18: Lemma 3.9 - Full-TP graphs with cycle limbs that are not edge-separable are not XNOP.

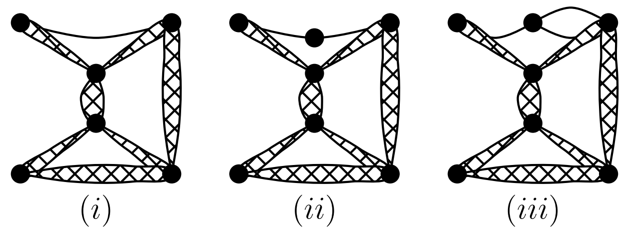


Figure 3.19: $G\backslash e\backslash f$ for Figure 3.18.

The graph $G\backslash e\backslash f$ is OP, so the two cycle limbs that are adjacent to both remaining spur limbs of $G\backslash e\backslash f$ must not have internal vertices as in Figure 3.20.

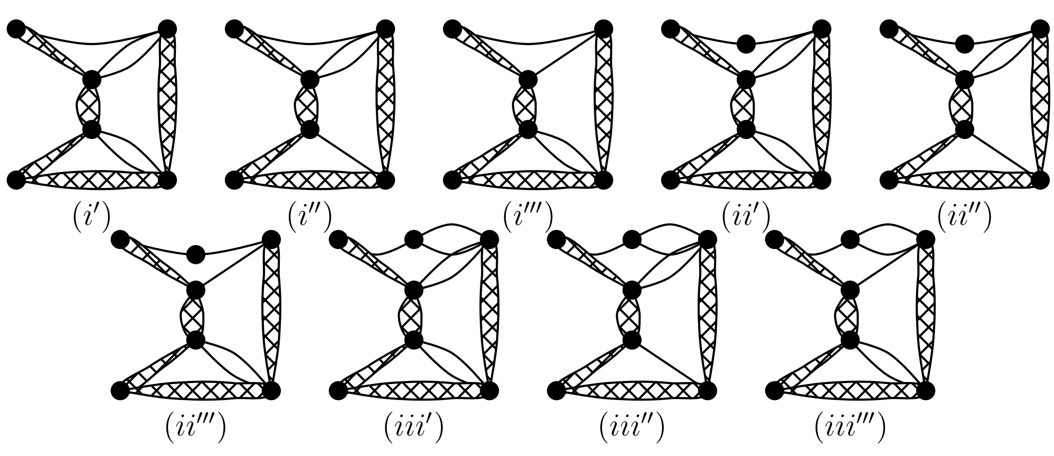


Figure 3.20: Refinement of $G\backslash e\backslash f$ for Figure 3.19.

Since G is XNOP, then $G\backslash f$ is not OP. But, in each case, it is OP, as shown in Figure 3.21, a contradiction. Hence, a cycle limb of a full-TP XNOP graph is edge-separable.

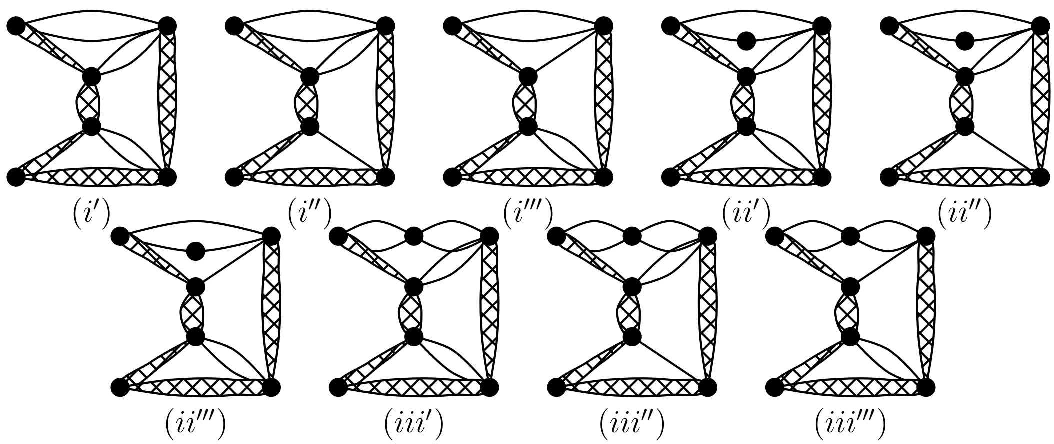


Figure 3.21: $G\backslash f$ for Figure 3.20.

□

Lemma 3.13. *A full-TP graph that is not NOP, with two limbs, L and M, of type $P2$ in one cycle and a limb N of type T in the opposite cycle such that N does not share a face with L or M, dominates TP_3.*

Proof. Let G be a full-TP XNOP graph with two limbs, L and M, of type P2 in one cycle and a limb N of type T in the opposite cycle such that N does not share a face with L or M as shown in Figure 3.22 (a). The cycle limb that is adjacent to L and M must be of type T, otherwise $G \succ TP_1$. See Figure 3.22 (b). The spur limb that is adjacent to both L and M must not be edge-separable. Otherwise, G is NOP as shown in Figure 3.22 (c). But, then $G \succ TP_3$. See Figure 3.22 (d).

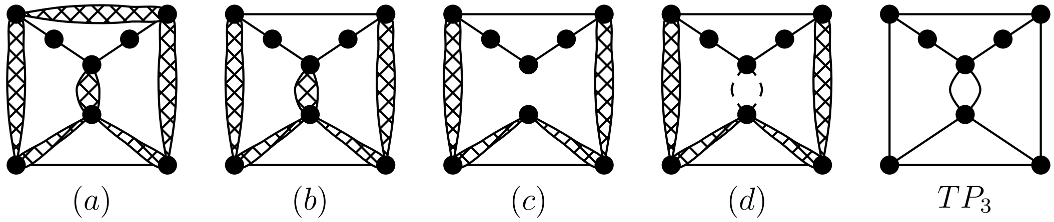


Figure 3.22: Lemma 3.13 - Full-TP graphs that are not NOP dominate TP_3.

□

Lemma 3.14. *A full-TP graph that is not NOP, with two cycle limbs, L and M, of type $P2$ such that L and M do not share a face, dominates TP_4.*

Proof. Let G be a full-TP XNOP graph with two limbs, L and M, of type P2 such that L and M do not share a face as in Figure 3.22 (a). All other cycle limbs must be of type T, otherwise, G dominates TP_1, TP_2, or TP_3. See Figure 3.22 (b). The spur limb that is adjacent to both L and M must not be edge-separable as shown in 3.22 (c). Otherwise, G is NOP. But, then $G \succ TP_4$. See Figure 3.22 (d).

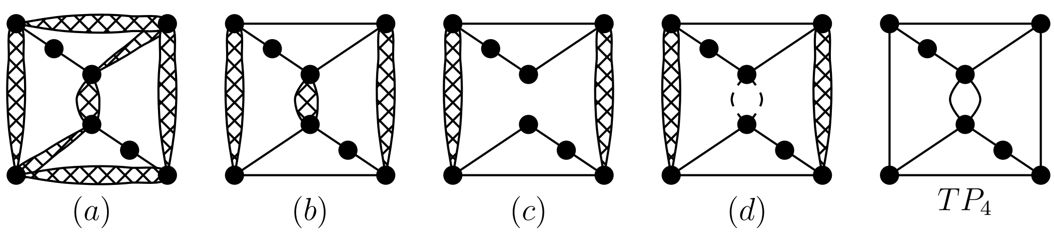


Figure 3.23: Lemma 3.14 - Full-TP graphs that are not NOP and have exactly one limb of type P2 dominate TP_4.

□

Lemma 3.15. *A full-TP graph that is not NOP, with one cycle with exactly one limb L of type P2 and the other cycle such that the limbs that do not share a face with L are of type T, dominates TP_5.*

Proof. Let G be a full-TP XNOP graph with one cycle with exactly one limb L of type P2 and the other cycle such that the limbs do not share a face with L are of type T as in Figure 3.24 (a). The two spur limbs that are adjacent to L must not be edge-separable as shown in 3.24 (b). Otherwise, G is NOP. But, then $G \succ TP_5$. See Figure 3.24 (c).

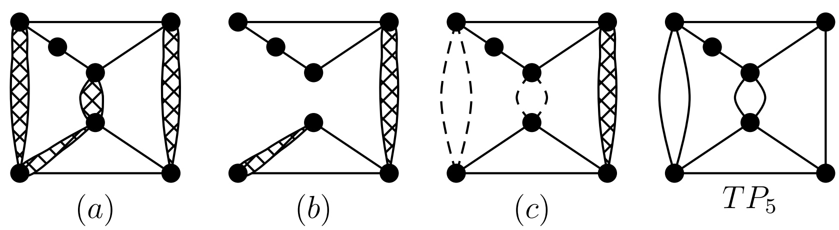


Figure 3.24: Lemma 3.15 - Full-TP graphs that are not NOP and have no cycle limbs of type P2 dominate TP_5.

□

Lemma 3.16. *A full-TP graph that is not NOP, with no cycle limbs of type P2 dominates TP_6*

Proof. Let G be a full-TP XNOP graph with no cycle limbs of type P2 as shown in Figure 3.25 (a). All three spur limbs must not be edge-separable as shown in 3.25 (b). Otherwise, G is NOP. But, then $G \succ TP_6$. See Figure 3.25 (c).

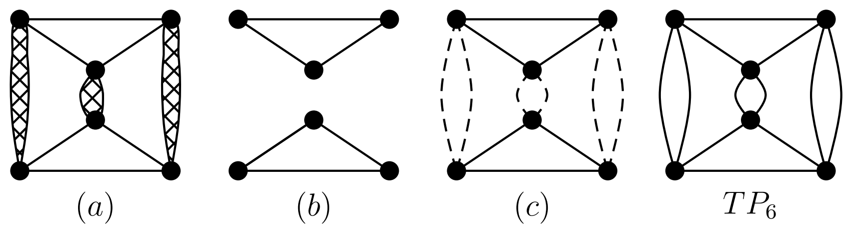


Figure 3.25: Lemma 3.16 - Full-TP graphs that are not NOP and have no cycle limbs of type P2 dominate TP_6.

□

CHAPTER 4

W_4 PLUS ONE EDGE AND 3-PRISM PLUS ONE EDGE

In Chapter 3, we found all XNOP graphs with W_4 and 3-prism as a skeleton. To find other XNOP graphs or to prove that we have a complete list, we must look at other possible skeletons of XNOP graphs, then use the skeletons found, along with the limbs found in Chapter 2 to find all of the XNOP graphs with skeletons of W_4 plus one edge and 3-prism plus one edge. We do this with four sections. In the first two sections, we examine the skeletons of W_4 plus an edge and 3-prism plus an edge, respectively. In the third section, we look at the double hub, one of the skeletons found in the two previous sections, and proof the complete list of full double hub XNOP graphs. Lastly, we look at the 1-eared wheel and 1-eared prism and prove that no XNOP graphs exist with these as a skeleton.

4.1 Skeletons of W_4 Plus an Edge

We begin with W_4. By symmetry, every resulting graph from BG-operations (1)-(3) to W_4 is isomorphic to one of the graphs shown in Figure 4.1, Figure 4.2, or Figure 4.3.

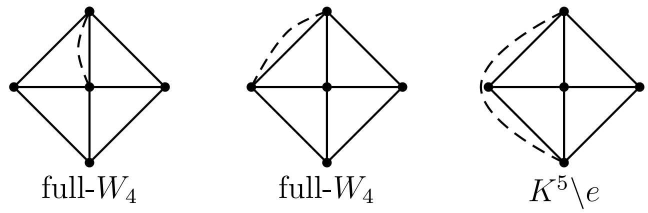


Figure 4.1: Examples of graphs of $W_4 + f$, where f is of type (1).

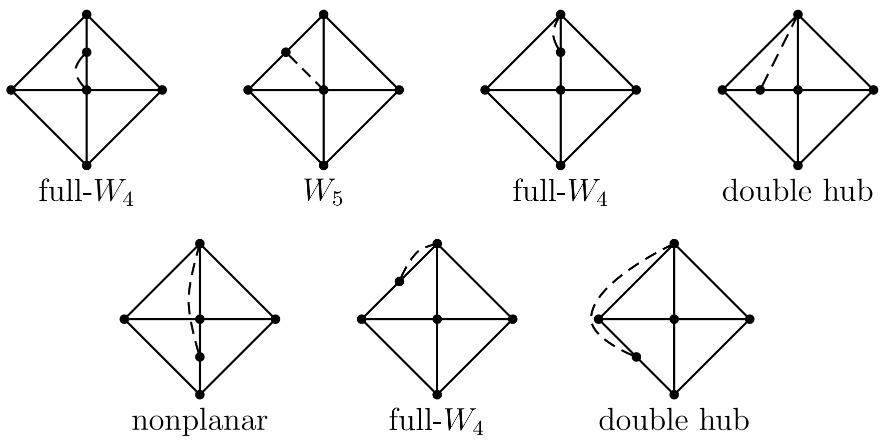


Figure 4.2: Examples of graphs of $W_4 + f$, where f is of type (2).

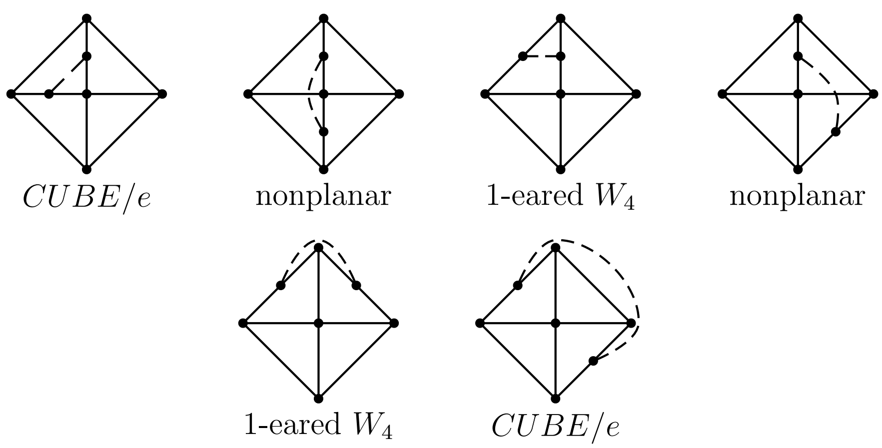


Figure 4.3: Examples of graphs of $W_4 + f$, where f is of type (3).

Hence, the addition of a single edge to W_4 gives us five new types of skeletons of XNOP graphs, $K^5\backslash e$, W_5, the double hub, the 1-eared W_4, and $CUBE/e$. Two of these, $K^5\backslash e$ and $CUBE/e$ are XNOP. We will explore the double hub and 1-eared W_4 as skeletons of XNOP graphs in Sections 4.3 and 4.4, respectively. Although W_5 should be considered as a skeleton in the same manner as the double hub and 1-eared W_4, no XNOP graph dominates W_5, as proved in [10].

4.2 Skeletons of 3-Prism Plus an Edge

Now we explore skeletons that are constructed by adding an edge f to 3-prism.

By symmetry, every resulting graph from BG-operations (1)–(3) to 3-prism is isomorphic to one of the graphs shown in Figure 4.4, Figure 4.5, or Figure 4.6.

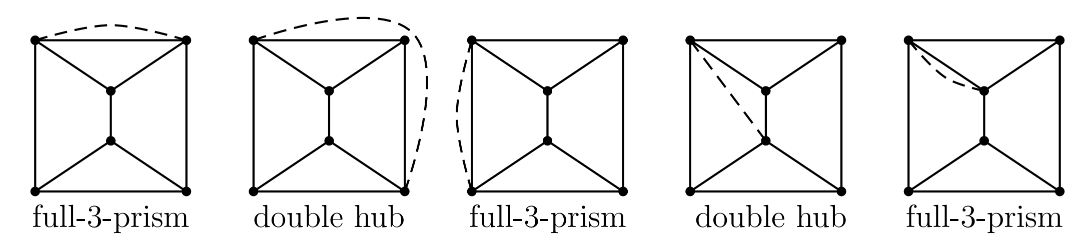


Figure 4.4: Examples of graphs of 3-prism+f, where f is of type (1).

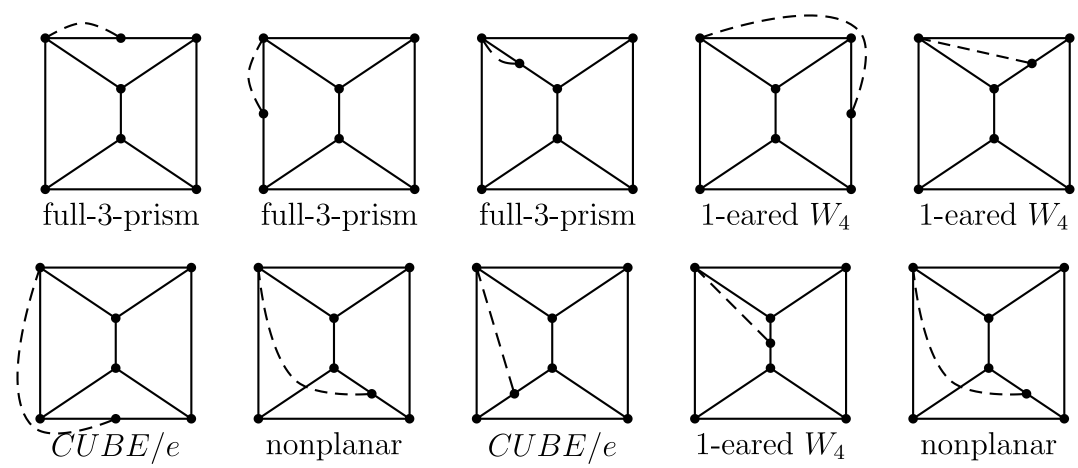


Figure 4.5: Examples of graphs of 3-prism+f, where f is of type (2).

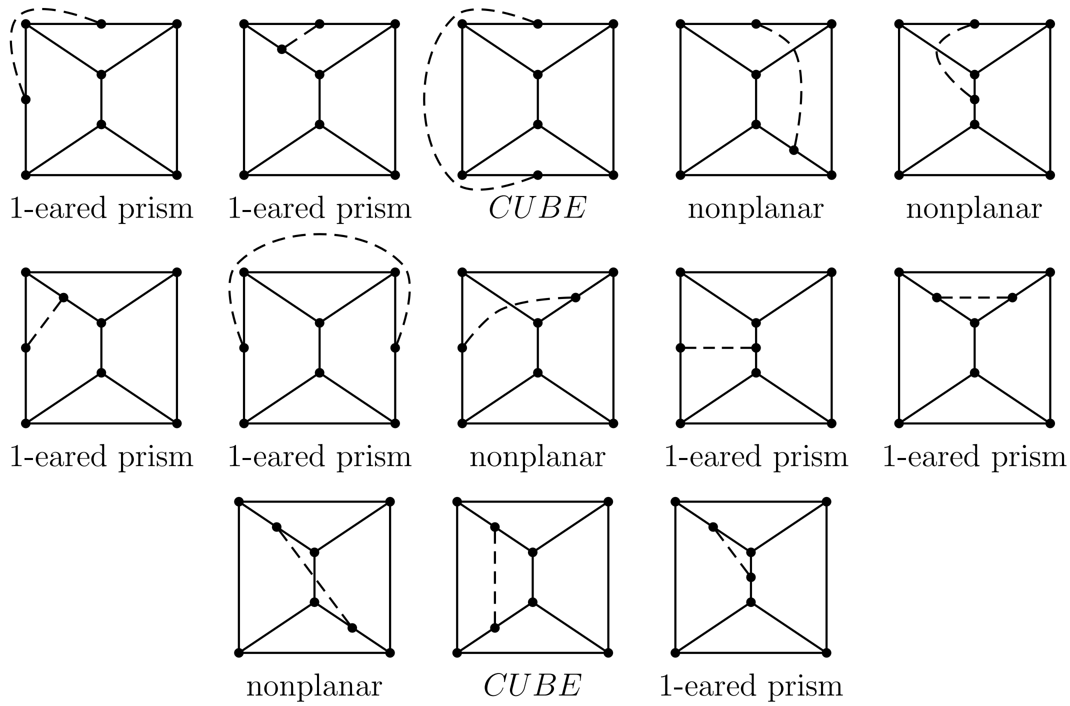


Figure 4.6: Examples of graphs of 3-prism+f, where f is of type (3).

Hence, the addition of a single edge to 3-prism gives us two new types of skeletons of XNOP graphs, the 1-eared prism, and $CUBE$. The $CUBE$ is an XNOP graph and needs no further exploration. We explore the 1-eared prism as a skeleton of an XNOP graph in Section 4.4.

4.3 Full-Double-Hub XNOP Graphs

In this section, we examine the full-double hub graphs, or full-DH graphs, that are XNOP. We use lemmas to prove the main theorem below.

The following definition is helpful in this chapter.

Definition 4.1. A *vulnerable edge* of a graph G is an edge g of G such that $G\backslash g$ is an outer-planar graph. An edge whose deletion does not result in an outer-planar graph is called a *non-vulnerable edge.*

The following main theorem uses the definition in one of the two lemmas below.

Theorem 4.2. *A full-DH XNOP graph is* DH_1.

Proof. The double-hub has exactly one vulnerable edge. We focus on this edge g as a limb $L = \sigma(g)$ of a full-DH and use this to assess the properties of the other limbs. See Figure 4.7 for an illustration of the vulnerable limb of a full-DH.

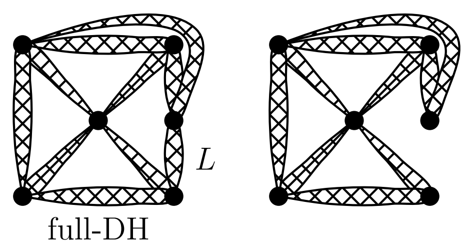


Figure 4.7: G and $G\backslash e$ for an edge-separable L.

Lemma 4.3. *The non-vulnerable limbs of a full-DH XNOP graph are edge-separable.*

Proof. Let G be a full-DH XNOP graph, and let M be a non-vulnerable limb of G. It is easy to verify that if M is not edge-separable, then $G\backslash e$ for $e \in E(M)$ is not OP. Then if $G\backslash e\backslash f$ is OP, the edge f is an edge of the vulnerable limb L and the inner limbs of $G\backslash e\backslash f$ have no internal vertices. But, then $G\backslash f$ is also OP, a contradiction. Figure 4.8 shows an example of one of the nine non-vulnerable limbs of G. The other cases are easy to verify.

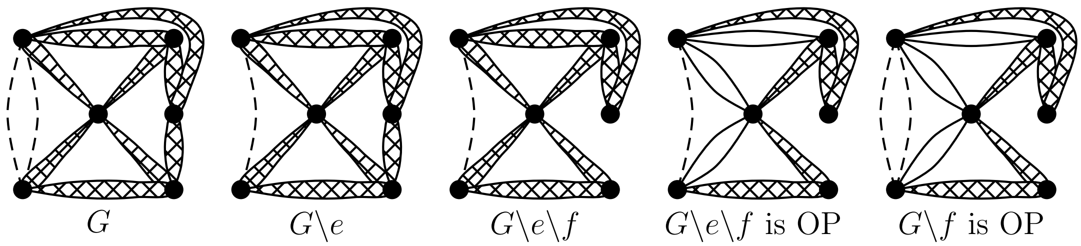


Figure 4.8: A non-vulnerable limb of G is edge-separable.

□

Hence, all limbs of G are edge-separable except the vulnerable limb of G. The vulnerable limb can be edge-separable or not. The following lemma proves that it is not.

Lemma 4.4. *If a full-DH graph G has a vulnerable limb that is edge-separable, then G dominates WF_1 or $K_{2,4}$.*

Proof. Let G be a full-DH XNOP graph with a vulnerable limb L that is edge-separable. Then $G\backslash e$ is not OP for some $e \in E(L)$. So, one or more of three limbs must contain an internal vertex as shown in Figure 4.9.

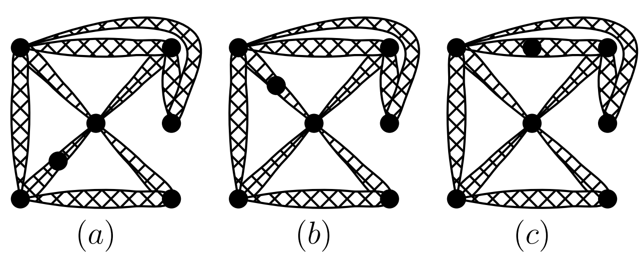


Figure 4.9: $G\backslash e$ is not OP.

But, if either of the limbs in Figure 4.9 (a) or (c) have an internal vertex, then $G \succ WF_1$. See Figure 4.10. So, the internal vertex must be as in Figure 4.9 (b). But, then $G \succ K_{2,4}$, see Figure 4.10, a contradiction.

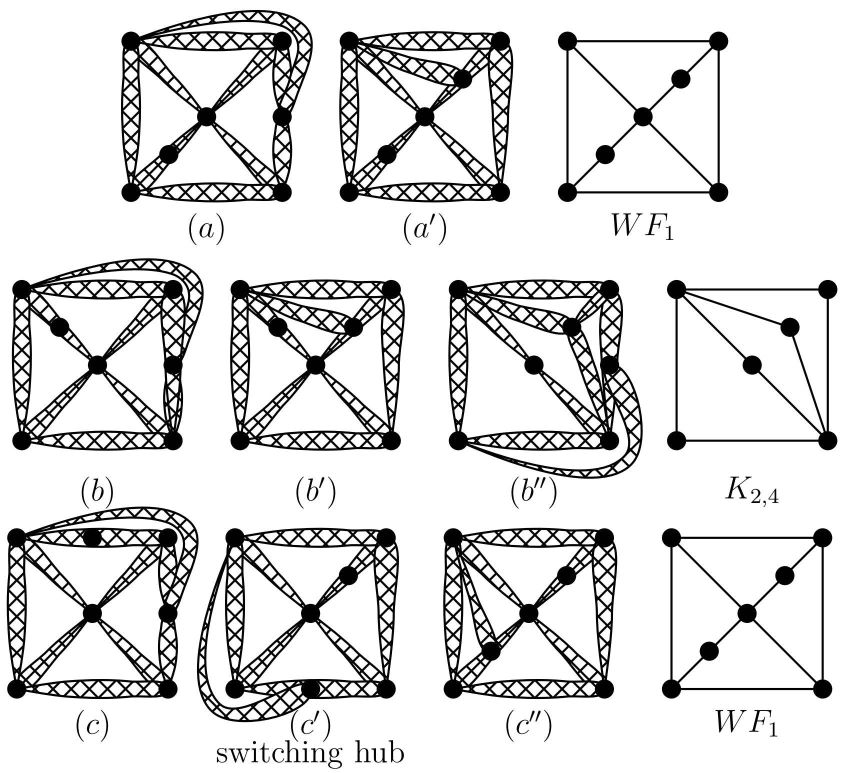


Figure 4.10: Refinement of $G\backslash e$ shows that G dominates an XNOP graph.

□

This corollary follows.

Corollary 4.5. *The vulnerable limb of a full-DH XNOP graph is not edge-separable.*

Hence, all non-vulnerable limbs of G are edge-separable and the vulnerable limb is not edge-separable. But, then $G \succ DH_1$.

□

4.4 Full-1-Eared-Wheel or Full-1-Eared-Prism XNOP Graphs

In this section, we use an approach similar to the one in Section 4.3. We look at the vulnerable edges and the non-vulnerable edges to proof the following main theorem.

Theorem 4.6. *No XNOP graph is a full-1-eared-wheel or a full-1-eared-prism.*

Proof. We divide this theorem into two sections, one on the full-1-eared-wheel, or full-EW, and one on the full-1-eared-prism, or full-EP. We start with the full-EW.

As in our proof of the full-DH XNOP graphs, we look first at the vulnerable edges of a 1-eared-wheel. There are two vulnerable edges, which are symmetric as shown in Figure 4.11

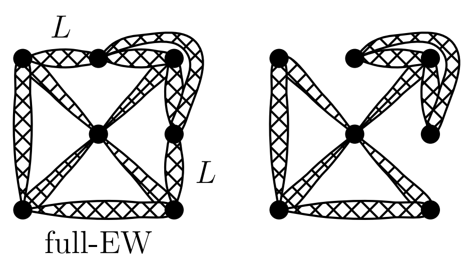


Figure 4.11: G and $G\backslash g$ for an edge-separable $L = \sigma(g)$ for the vulnerable edge g.

This lemma gives us a starting point for the non-vulnerable limbs of a full-EW.

Lemma 4.7. *The non-vulnerable limbs of a full-EW are edge-separable.*

Proof. Let G be a full-EW XNOP graph. Let L and M be the two vulnerable limbs of G. It is easy to verify that if a non-vulnerable limb N of G is not edge-separable, then $G\backslash e$ for $e \in E(N)$ is not OP. Then if $G\backslash e\backslash f$ is OP, the edge f is an edge of L or M and the inner limbs of $G\backslash e\backslash f$ have no internal vertices. But, then $G\backslash f$ is also

OP, a contradiction. Figure 4.12 shows an example of one of the nine non-vulnerable limbs of G. The other cases are easy to verify.

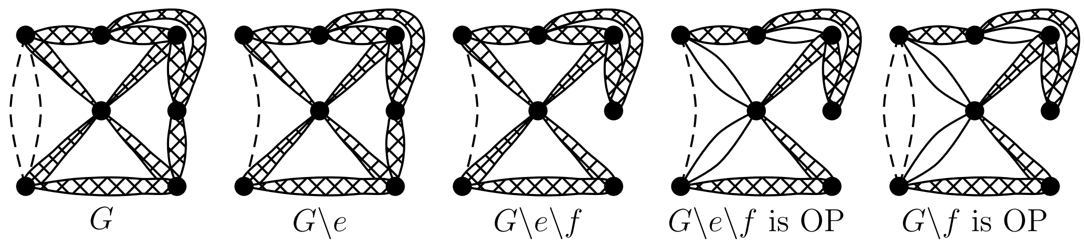


Figure 4.12: A non-vulnerable limb of G is edge-separable.

□

Hence, all limbs of G are edge-separable except the two vulnerable limbs of G. The vulnerable limbs can be edge-separable or not. The following lemma proves that if a full-EW is XNOP, then the vulnerable limbs are not edge-separable.

Lemma 4.8. *If a full-EW graph G has a vulnerable limb that is edge-separable, then G dominates WF_1, TP_3, or TP_4.*

Proof. Let G be a full-EW XNOP graph with a vulnerable limb L that is edge-separable. Then $G\backslash e$ is not OP for some $e \in E(L)$. So, one or more of three limbs must contain an internal vertex as shown in Figure 4.13.

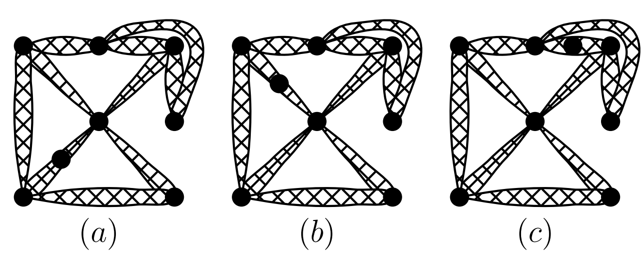


Figure 4.13: $G\backslash e$ is not OP.

But, if the limb in Figure 4.13 (a) has an internal vertex, then $G \succ WF_1$. See Figure 4.14. So, the internal vertex must be as in Figure 4.13 (b) or (c). But, in these cases, if the internal limbs have internal vertices, then the second vulnerable limb M must not be edge separable, otherwise $G \backslash e$ is OP for some edge $e \in E(M)$. See Figure 4.14.

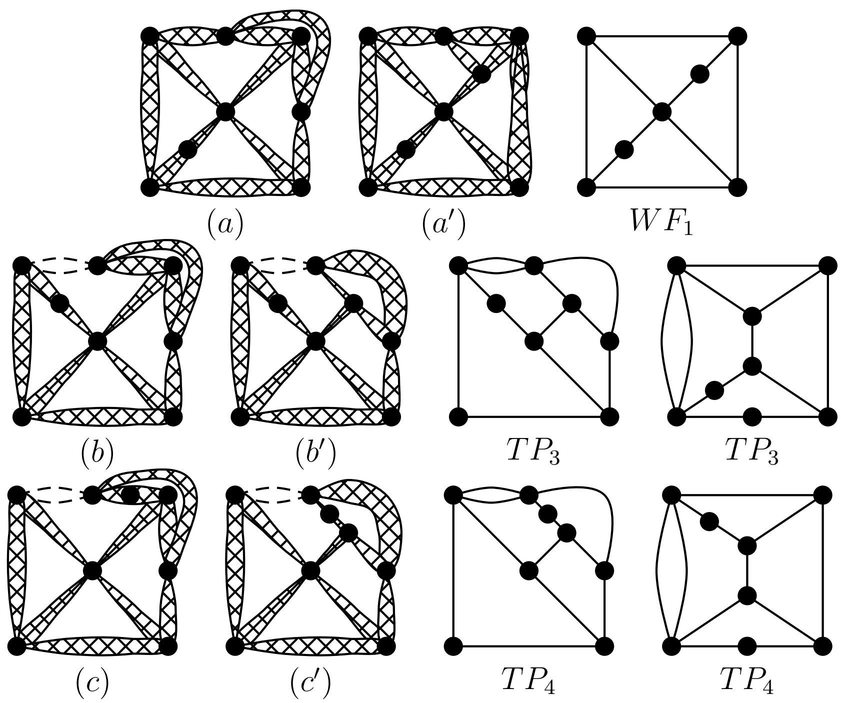


Figure 4.14: Refinement of $G\backslash e$ shows that G dominates an XNOP graph.

In the case of Figure 4.14 (b), if M is not edge-separable, then $G \succ TP_3$. So, G must be as in case (c), but then $G \succ TP_4$. See Figure 4.14.

□

This corollary follows.

Corollary 4.9. *The vulnerable limbs of a full-EW XNOP graph are not edge-separable.*

Hence, all non-vulnerable limbs of G are edge-separable and the vulnerable limbs are not edge-separable. But, then G properly dominates TP_5. Since we have addressed all limbs of a full-EW and found no full-EW XNOP graphs, we have proved the first part of the proof.

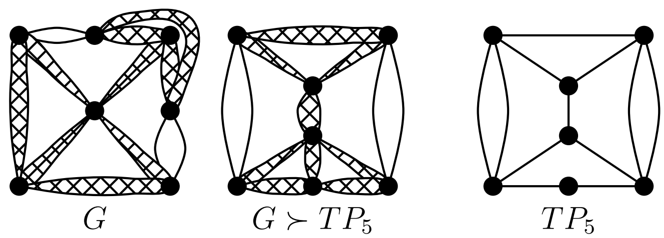


Figure 4.15: $G \succ TP_5$.

We now focus on graphs that are full-1-eared-prisms. As in our proof of the full-EW XNOP graphs, we look first at the vulnerable edges of a 1-eared-prism. There is one vulnerable edge as shown in Figure 4.16.

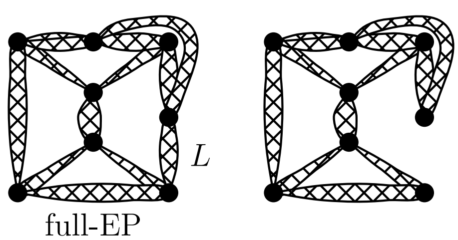


Figure 4.16: G and $G\backslash e$ for an edge-separable L.

The following lemma on the non-vulnerable limbs is our starting point.

Lemma 4.10. *The non-vulnerable limbs of a full-EP are edge-separable.*

Proof. Let G be a full-EP XNOP graph. Let L be the vulnerable limb of G. It is easy to verify that if a non-vulnerable limb M of G is not edge-separable, then $G\backslash e$ for

$e \in E(M)$ is not OP. Then if $G \backslash e \backslash f$ is OP, the edge f is an edge of L and the inner limbs of $G \backslash e \backslash f$ have no internal vertices. But, then $G \backslash f$ is also OP, a contradiction. Figure 4.17 shows an example of one of the eleven non-vulnerable limbs of G. The other cases are easy to verify.

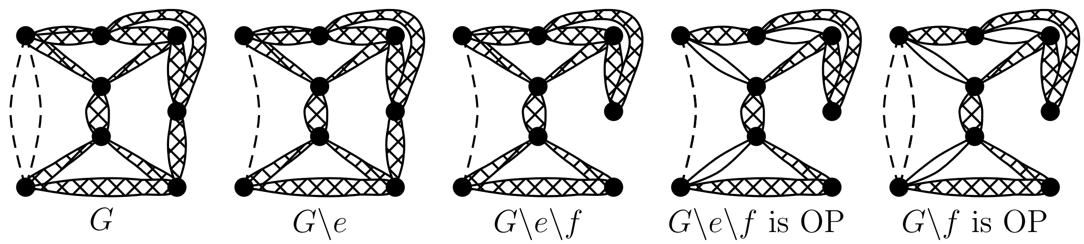


Figure 4.17: A non-vulnerable limb of G is edge-separable.

□

Hence, all limbs of G are edge-separable except the vulnerable limb of G. The vulnerable limb can be edge-separable or not. The following lemma proves that if a full-EP is XNOP, then the vulnerable limb is not edge-separable.

Lemma 4.11. *If a full-EP graph G has a vulnerable limb that is edge-separable, then G dominates TP_1 or TP_2.*

Proof. Let G be a full-EP XNOP graph with a vulnerable limb L that is edge-separable. Then $G \backslash e$ is not OP for some $e \in E(L)$. So, one or more of three limbs must contain an internal vertex as shown in Figure 4.18.

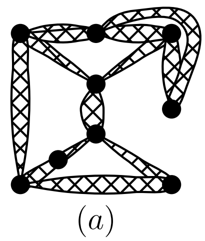


Figure 4.18: $G\backslash e$ is not OP.

But, if the limbs in Figure 4.18 (a) and (c) have an internal vertex, then $G \succ TP_2$. See Figure 4.19. So, the internal vertex must be as in Figure 4.18 (b). But, then $G \succ TP_1$, a contradiction. See Figure 4.19.

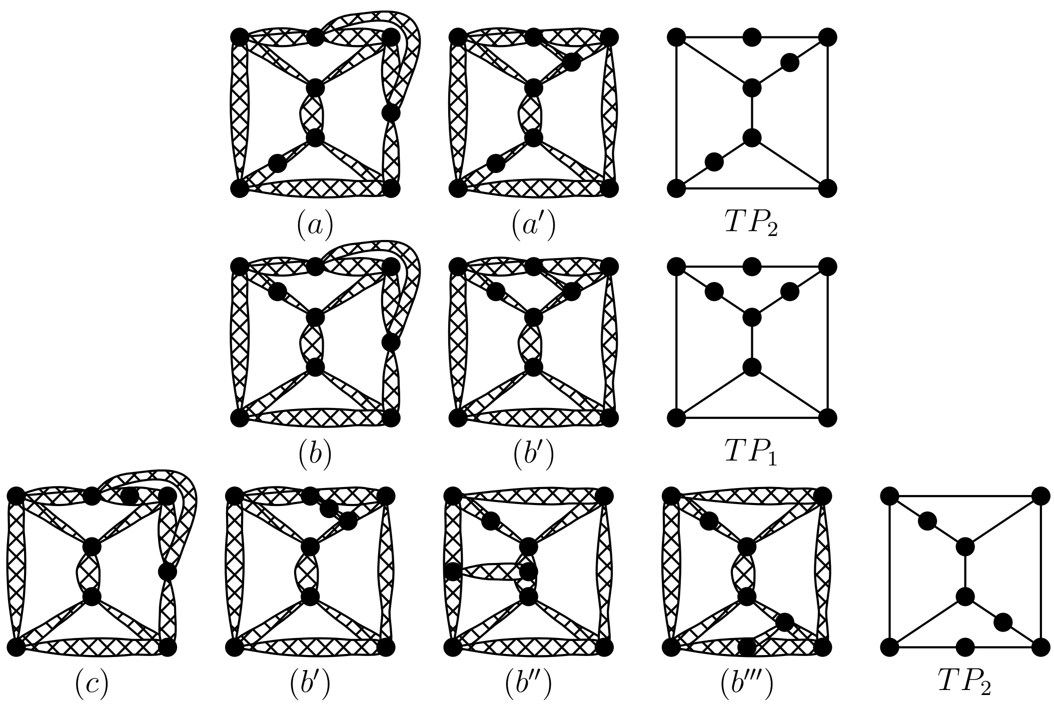


Figure 4.19: $G\backslash e$ is not OP.

□

This corollary follows.

Corollary 4.12. *The vulnerable limb of a full-EP XNOP graph is not edge-separable.*

Hence, all non-vulnerable limbs of G are edge-separable and the vulnerable limb is not edge-separable. But, then $G \succ TP_3$. We have addressed all limbs of a full-EP graph and found no full-EP XNOP graphs.

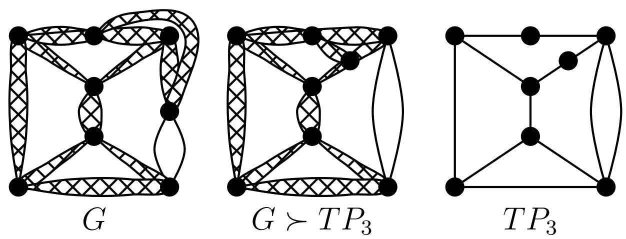


Figure 4.20: $G \succ TP_3$.

We have examined both the full-1-eared-wheel and the full-1-eared-prism and found no XNOP graphs. Hence, there is no need to search for XNOP graphs with skeletons that have the 1-eared-wheel or the 1-eared-prism as a minor.

□

CHAPTER 5

DOUBLE-HUB PLUS ONE EDGE

In Chapter 4, we found all XNOP graphs with double hub plus an edge and proved that no XNOP graph has a 1-eared wheel or a 1-eared prism as a skeleton. To find other XNOP graphs or to prove that we have a complete list, we must look at other possible skeletons of XNOP graphs, then use the skeletons found, along with the limbs found in Chapter 2 to find all of the XNOP graphs with skeletons of double hub plus one edge. We do this with two sections. In the first section, we examine the skeletons of the double hub plus an edge. Lastly, we look at the heptahedral and octahedral, the skeletons found in the previous section, and proof that no XNOP graphs exist with these as a skeleton.

5.1 Skeletons of Double-Hub Plus One Edge

Theorem 5.1. *No skeleton of an XNOP graph properly dominates the double-hub.*

The addition of two edges to K^4 results in four new skeletons of XNOP graphs: $K^5\backslash e$, the double-hub, $CUBE/e$, and the $CUBE$. The graphs $K^5\backslash e$, $CUBE/e$, and the $CUBE$ are XNOP. Therefore, they need no further exploration. The double-hub is a confirmed skeleton, as there is one XNOP graph with the double-hub as a skeleton. So, S must be a graph that can be constructed from K^4 with at least three edges.

We use the double-hub, along with the BG-operations to find this skeleton of XNOP graphs.

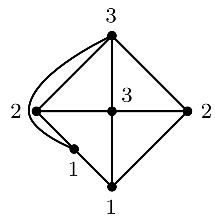


Figure 5.1: Two drawings of the double-hub with labeled symmetries.

The double-hub can be drawn as in Figure 5.1 to observe its symmetries. Every resulting graph from BG-operations (1)-(3) to the double-hub is isomorphic to one of the graphs shown in Figure 5.2, Figure 5.3, or Figure 5.4.

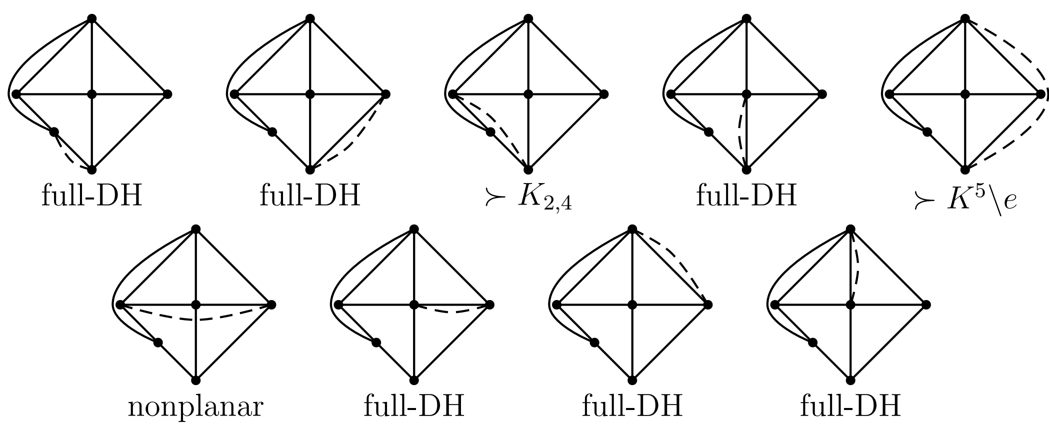


Figure 5.2: Graphs of DH+f, where f is of type (1).

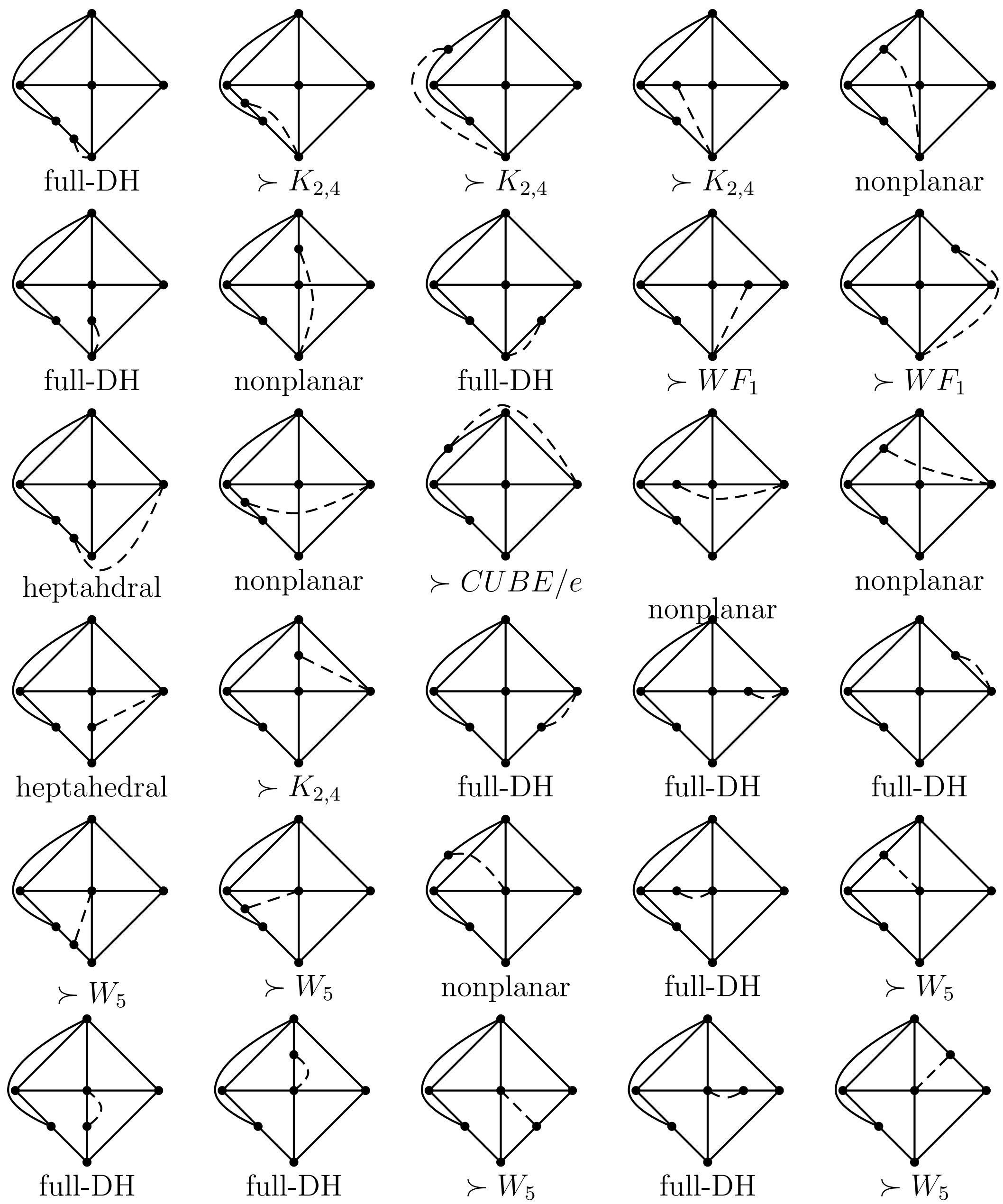


Figure 5.3: Graphs of DH+f, where f is of type (2).

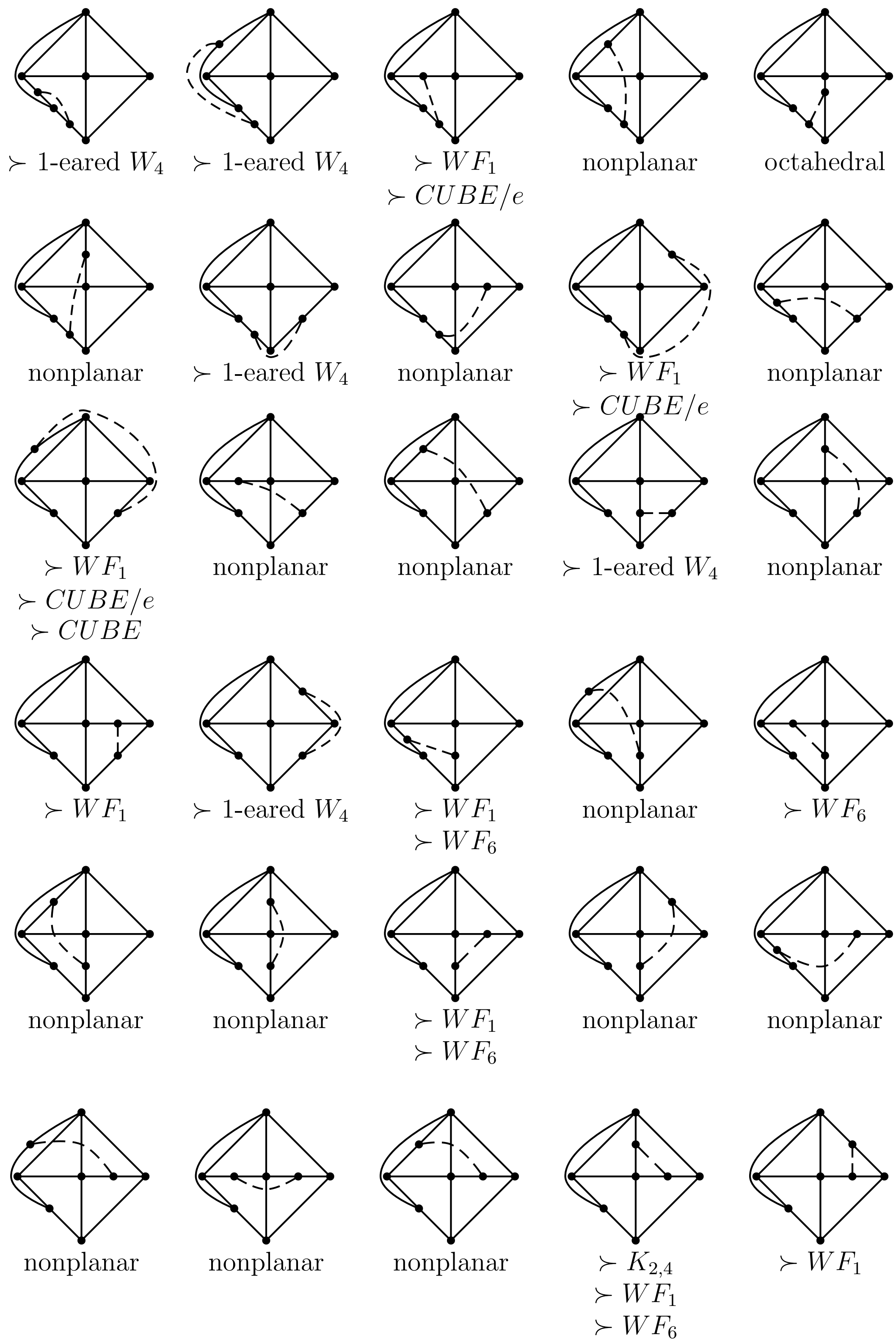


Figure 5.4: Graphs of DH+f, where f is of type (3).

Hence, the addition of a single edge to the DH gives us two new types of skeletons of XNOP graphs, the heptahedral, the octahedral. We explore the heptahedral and the octahedral as skeletons of an XNOP graph in the next section.

5.2 Full-Heptahedral or Full-Octahedral XNOP Graphs

In this section, we use an approach similar to the ones in Sections 4.3 and 4.4. We look at the vulnerable edges and the non-vulnerable edges to prove the following main theorem.

Theorem 5.2. *No XNOP graph is a full-heptahedral or a full-octahedral.*

Proof. We divide this theorem into two sections, one on the full-heptahedral, or full-HH, and one on the full-octahedral, or full-OH. We start with the full-HH.

We look first at the vulnerable edge of the heptahedral as shown in Figure 5.5

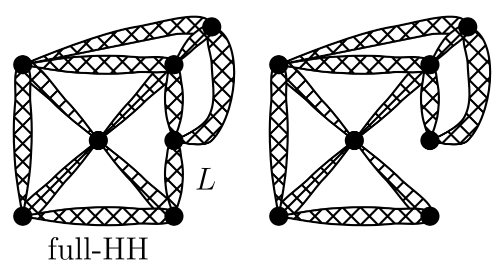


Figure 5.5: G and $G\backslash e$ for an edge-separable L.

This lemma gives us a starting point for the non-vulnerable limbs of a full-HH.

Lemma 5.3. *The non-vulnerable limbs of a full-HH are edge-separable.*

Proof. Let G be a full-HH XNOP graph. Let L be the vulnerable limb of G. It is easy to verify that if a non-vulnerable limb M of G is not edge-separable, then $G\backslash e$

for $e \in E(N)$ is not OP. Then if $G\backslash e\backslash f$ is OP, the edge f is an edge of L and the inner limbs of $G\backslash e\backslash f$ have no internal vertices. But, then $G\backslash f$ is also OP, a contradiction. Figure 5.6 shows an example of one of the twelve non-vulnerable limbs of G. The other cases are easy to verify.

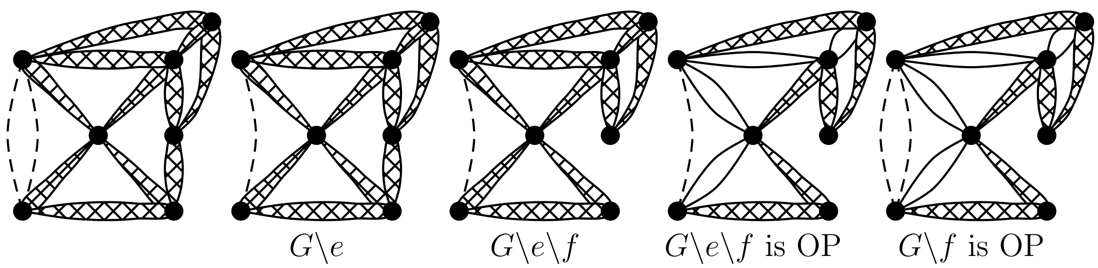


Figure 5.6: A non-vulnerable limb of G is edge-separable.

□

Hence, all limbs of G are edge-separable except the vulnerable limbs of G. The vulnerable limb is edge-separable or not. The following lemma proves that if a full-HH is XNOP, then the vulnerable limb is not edge-separable.

Lemma 5.4. *If a full-HH graph G has a vulnerable limb that is edge-separable, then G dominates $K_{2,4}$ or WF_1.*

Proof. Let G be a full-HH XNOP graph with a vulnerable limb L that is edge-separable. Then $G\backslash e$ is not OP for some $e \in E(L)$. So, one or more of four limbs must contain an internal vertex as shown in Figure 5.7.

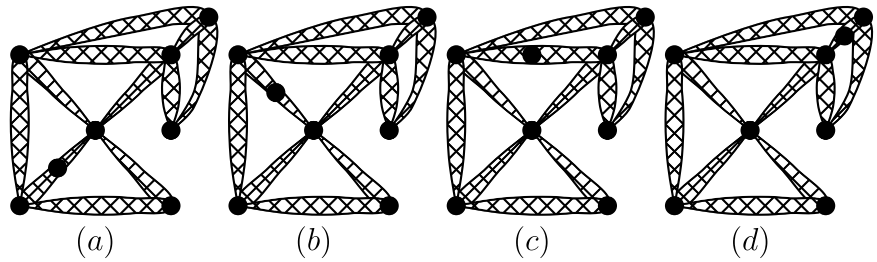


Figure 5.7: $G\backslash e$ is not OP.

But, if the limbs in Figure 5.7 (a), (c), or (d) have an internal vertex, then $G \succ WF_1$. So, the internal vertex must be as in Figure 5.7 (b). But, then $G \succ K_{2,4}$.

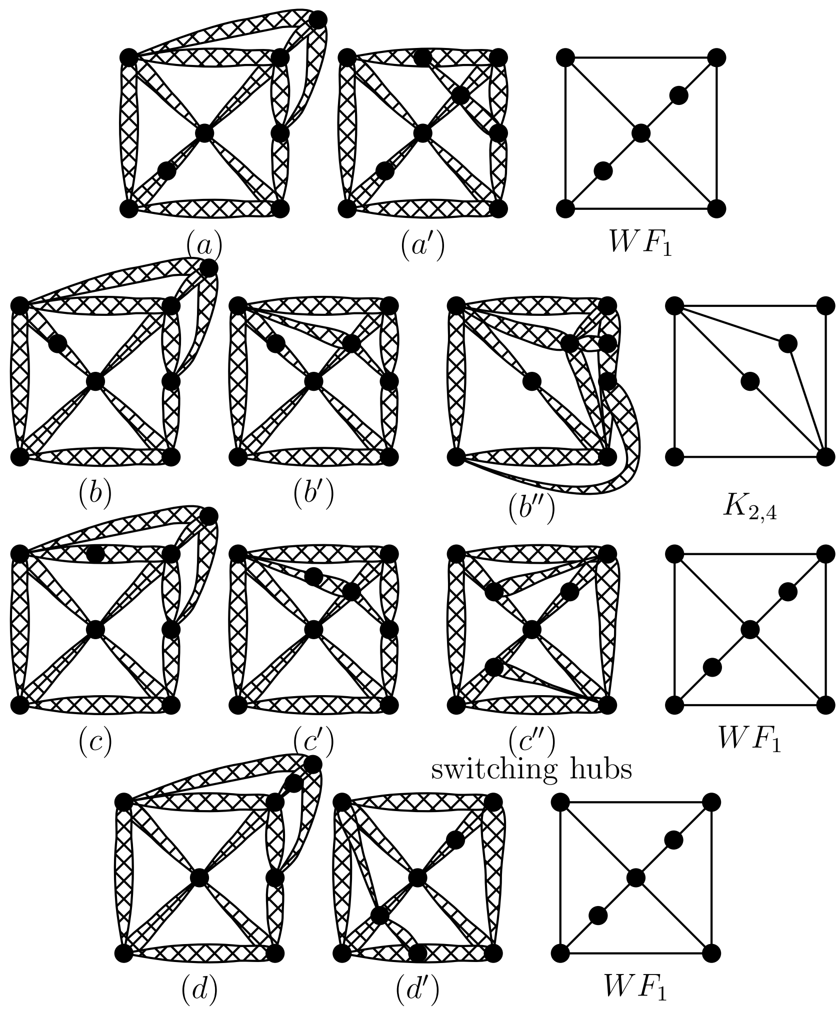


Figure 5.8: $G\backslash e$ shows that G dominates an XNOP graph.

□

This corollary follows and is easy to verify.

Corollary 5.5. *The vulnerable limb of a full-HH XNOP graph is not edge-separable.*

We have proved that all non-vulnerable limbs of G are edge-separable and the vulnerable limb is not edge-separable. But, then G properly dominates S_1 as shown in Figure 5.9.

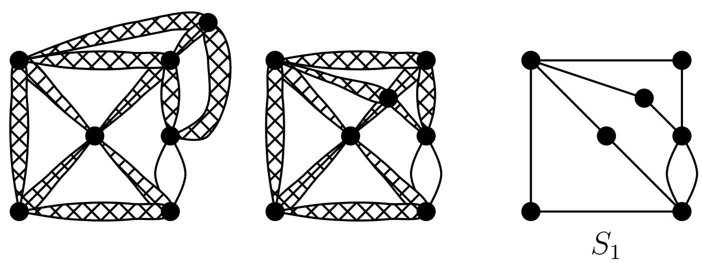


Figure 5.9: $G \succ S_1$.

Since we have addressed all limbs of a full-HH and found no full-HH XNOP graphs, we have proved the first part of the proof.

We now focus on graphs that are full-octahedrals. As in our proof of the full-HH XNOP graphs, we look first at the vulnerable edges of a octahedral. There is one vulnerable edge as shown in Figure 5.10

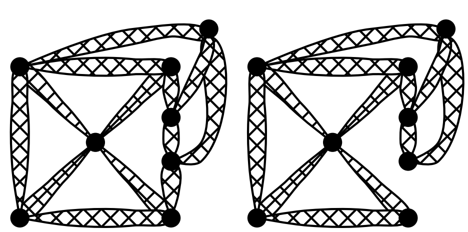


Figure 5.10: G and $G \backslash e$ for an edge-separable L.

The following lemma on the non-vulnerable limbs is our starting point.

Lemma 5.6. *The non-vulnerable limbs of a full-OH are edge-separable.*

Proof. Let G be a full-OH XNOP graph. Let L be the vulnerable limb of G. It is easy to verify that if a non-vulnerable limb M of G is not edge-separable, then $G\backslash e$ for $e \in E(M)$ is not OP. Then if $G\backslash e\backslash f$ is OP, the edge f is an edge of L and the inner limbs of $G\backslash e\backslash f$ have no internal vertices. But, then $G\backslash f$ is also OP, a contradiction. Figure 5.11 shows an example of one of the twelve non-vulnerable limbs of G. The other cases are easy to verify.

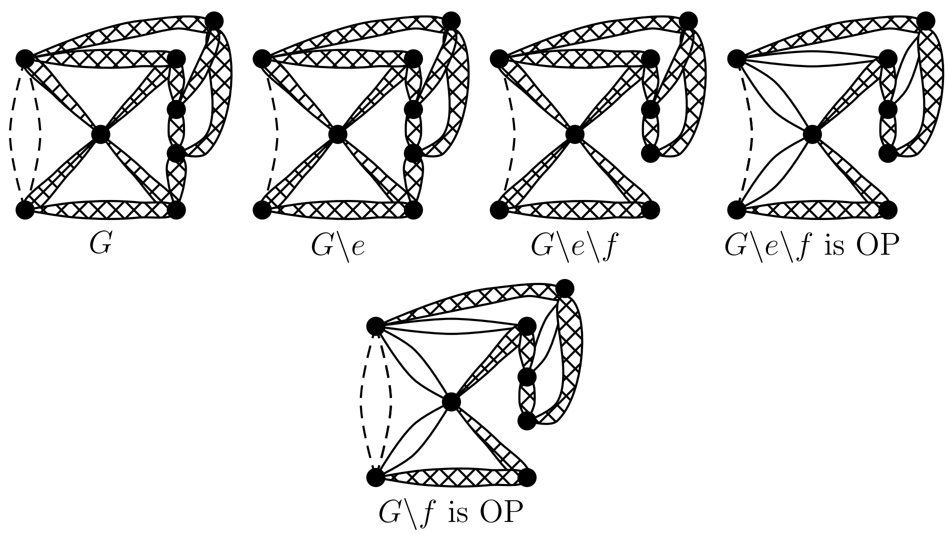


Figure 5.11: A non-vulnerable limb of G is edge-separable.

□

Hence, all limbs of G are edge-separable except the vulnerable limb of G. The vulnerable limb can be edge-separable or not. The following lemma proves that if a full-OH is XNOP, then the vulnerable limb is not edge-separable.

Lemma 5.7. *If a full-OH graph G has a vulnerable limb that is edge-separable, then G dominates $K_{2,4}$, S_5, or WF_1.*

Proof. Let G be a full-OH XNOP graph with a vulnerable limb L that is edge-separable. Then $G\backslash e$ is not OP for some $e \in E(L)$. So, one or more of three limbs must contain an internal vertex as shown in Figure 5.12.

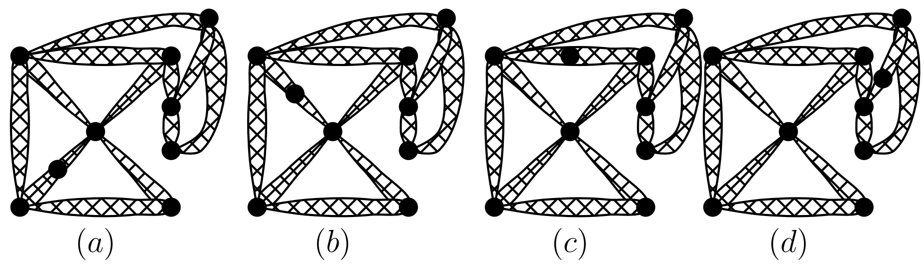


Figure 5.12: $G\backslash e$ is not OP.

But, if the limbs in Figure 5.12 (a) and (c) have an internal vertex, then $G \succ WF_1$. So, the internal vertex must be as in Figure 5.12 (b) or(d). If it is as in case (b), then $G \succ K_{2,4}$. And, if it is as in case (d), then $G \succ S_5$. Hence, G is not a full-OH XNOP graph.

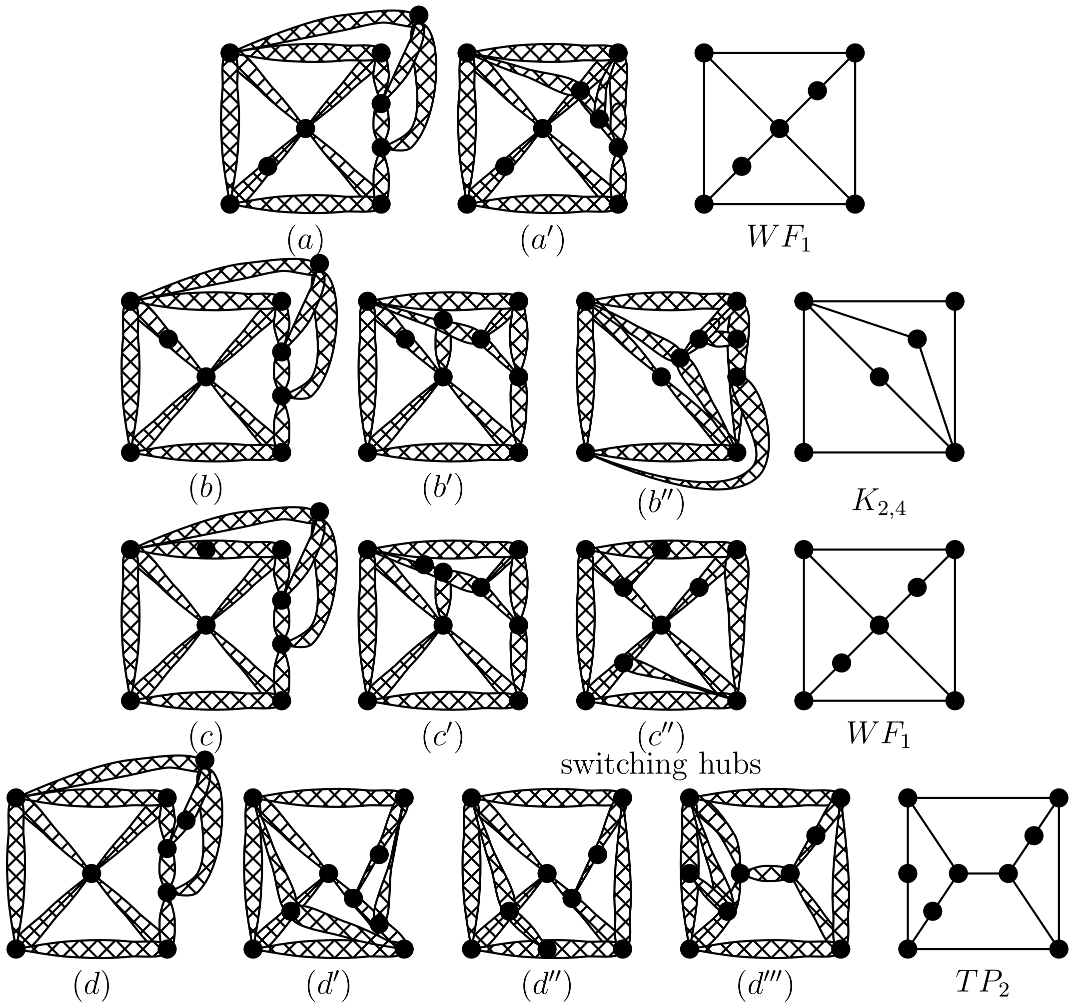


Figure 5.13: $G\backslash e$ is not OP.

□

This corollary follows and is easy to verify.

Corollary 5.8. *The vulnerable limb of a full-OH XNOP graph is not edge-separable.*

We have proved that all non-vulnerable limbs of G are edge-separable and the vulnerable limb is not edge-separable. See Figure 5.14. But, then $G \succ S_1$.

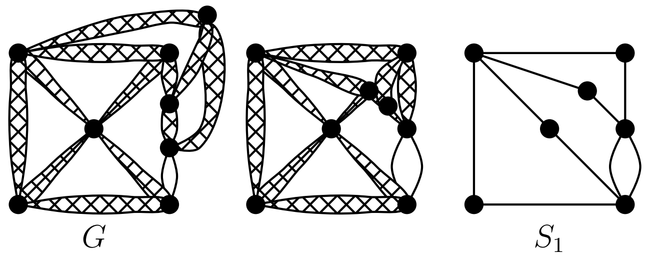


Figure 5.14: $G \succ S_1$.

In conclusion, we have addressed all limbs of a full-OH graph and found no full-OH XNOP graphs. We have also examined both the full-heptahedral and the full-octahedral and found no XNOP graphs. Hence, there is no need to search for XNOP graphs with skeletons that have the heptahedral or the octahedral as a minor.

□

CHAPTER 6

CONCLUSIONS

6.1 Summary

This dissertation supplies another link in the chain of excluded or forbidden graph characterization. By using the domination relation, we described a finite list of minimal XNOP graphs. To do this, we introduced the concept of skeletons, limbs, and joints. We also developed algorithms for testing whether or not a graph is NOP or XNOP. With these algorithms, we replaced edges of skeletons with limbs and proved the complete list of full-K^4 XNOP graphs. Finally, we introduced the concept of vulnerable edges to prove that the list of 58 XNOP graphs is complete.

6.2 Future Work

From Theorem 2.6 we found that excluding K^4 yields a series parallel graph. A natural extension of the work with near outer-planar graphs and series parallel graphs would be to find an excluded list for near series parallel graphs. From preliminary work, the excluded near series parallel graph (XNSP) list is not minor-closed and not finite under topological minors. It is the author's belief that near series parallel graphs are closed under domination and that the XNSP list is finite. Also, since $K_{2,3}$ is allowed, no graphs that are XNSP have subdivided edges, and the skeleton ideas developed in this dissertation can be used, but with different specifications. Algorithms for testing whether a graph is series parallel have been proved at linear complexity. We can adapt these to SageMath, if necessary. It is the author's conjecture that the number of XNSP graphs will be fewer than the number of XNOP graphs.

Another idea for future work was found while generating the list of possible skeletons of XNOP graphs. We found an infinite list of 3-connected NOP graphs, which we call *bubble graphs.* These graphs have a couple of unique features. Let G be a bubble graph. There exists an edge e on a face with the largest number of vertices, such that $G\backslash e$ is OP. The edge e, a vulnerable edge, lies on two faces, F_1 and F_2. The *length of a face* of a plane graph G is the length of the walk in G that bounds it. For the graphs that we have listed, we observe that the faces of G that are not F_1 or F_2 are of length three or four. There may be a relationship between these graphs and fan graphs, which are a key element of maximal outer-planar graphs in [1]. Figure 6.1 shows the first fifteen bubble graphs with one vulnerable edge depicted with dashed lines. The author would like to characterize these graphs in the future.

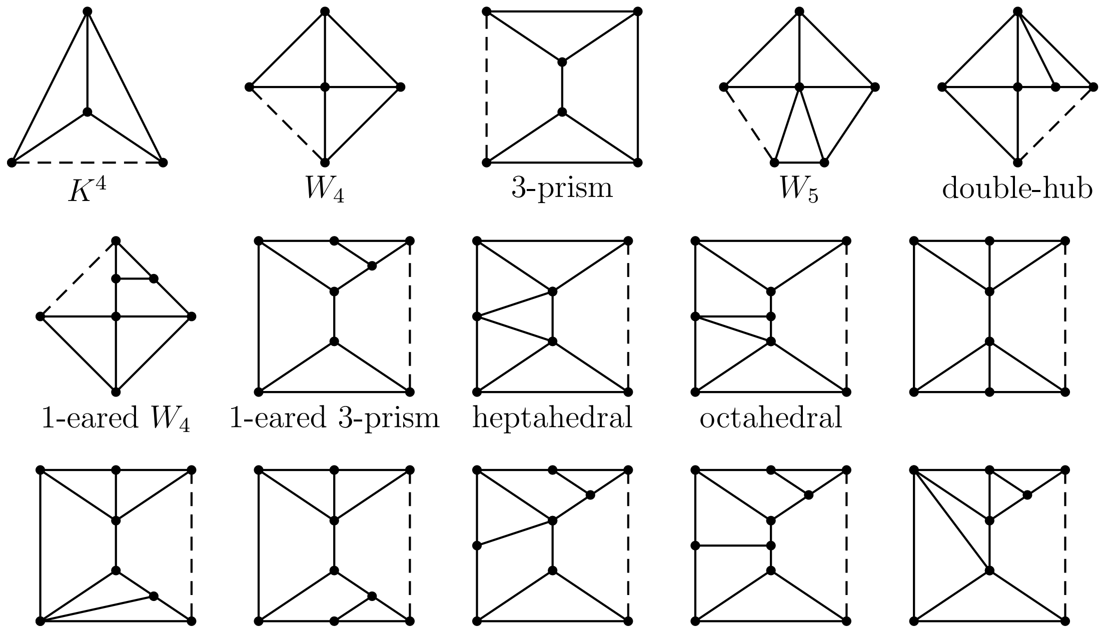
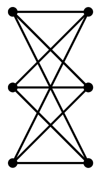


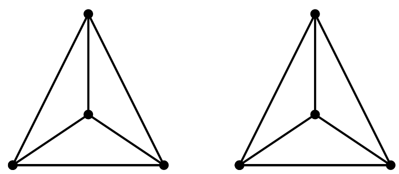
Figure 6.1: The first fifteen bubble graphs, or skeleton graphs that are NOP, and a vulnerable edge.

APPENDIX A

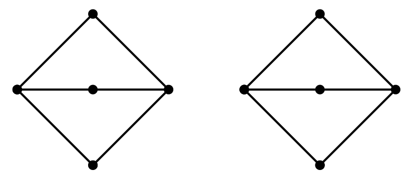
LIST OF XNOP GRAPHS



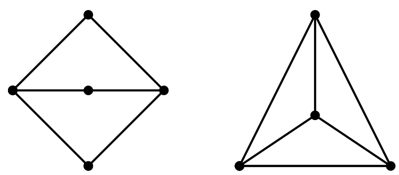
$K_{3,3}$



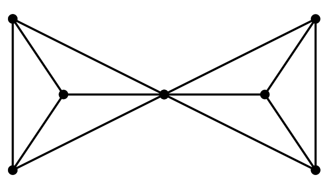
D_1



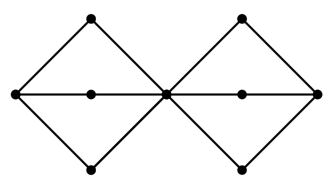
D_2



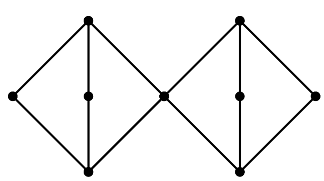
D_3



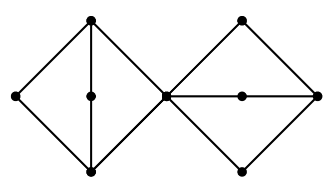
CV_1



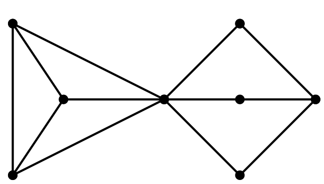
CV_2



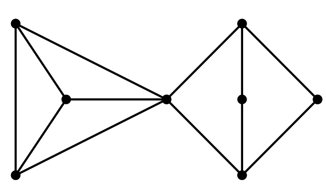
CV_3



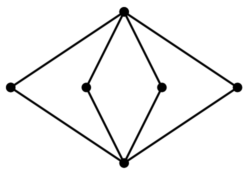
CV_4

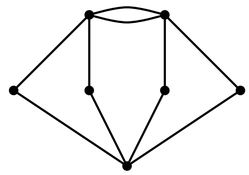


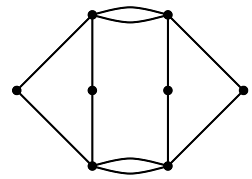
CV_5

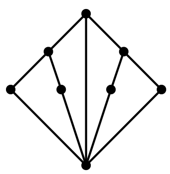


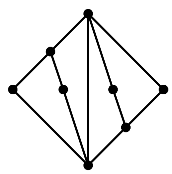
CV_6

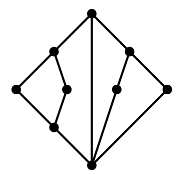

$K_{2,4}$

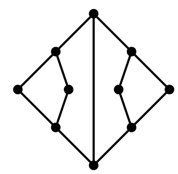

S_1

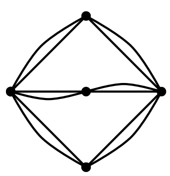

S_2

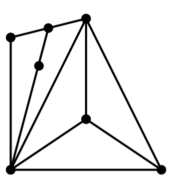

S_3

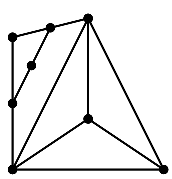

S_4

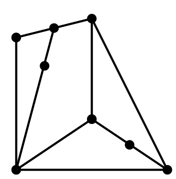

S_5

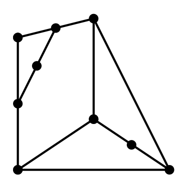

S_6

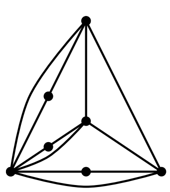

DE_1

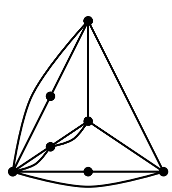

KF_{1A} (WT_{17})

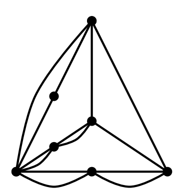

KF_{1B} (WT_{16})

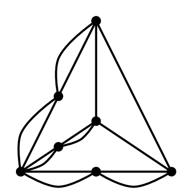

KF_{2A} (WT_{15})

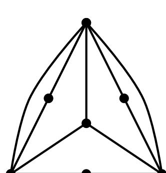

KF_{2B} (WT_{14})

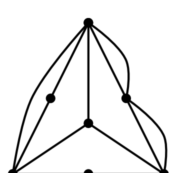

KF_{3A} (WT_{12b})

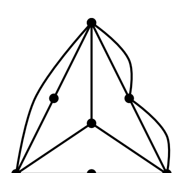

KF_{3B} (WT_{12c})

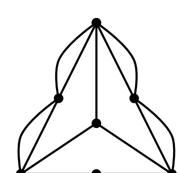

KF_{3C} (WT_{12d})

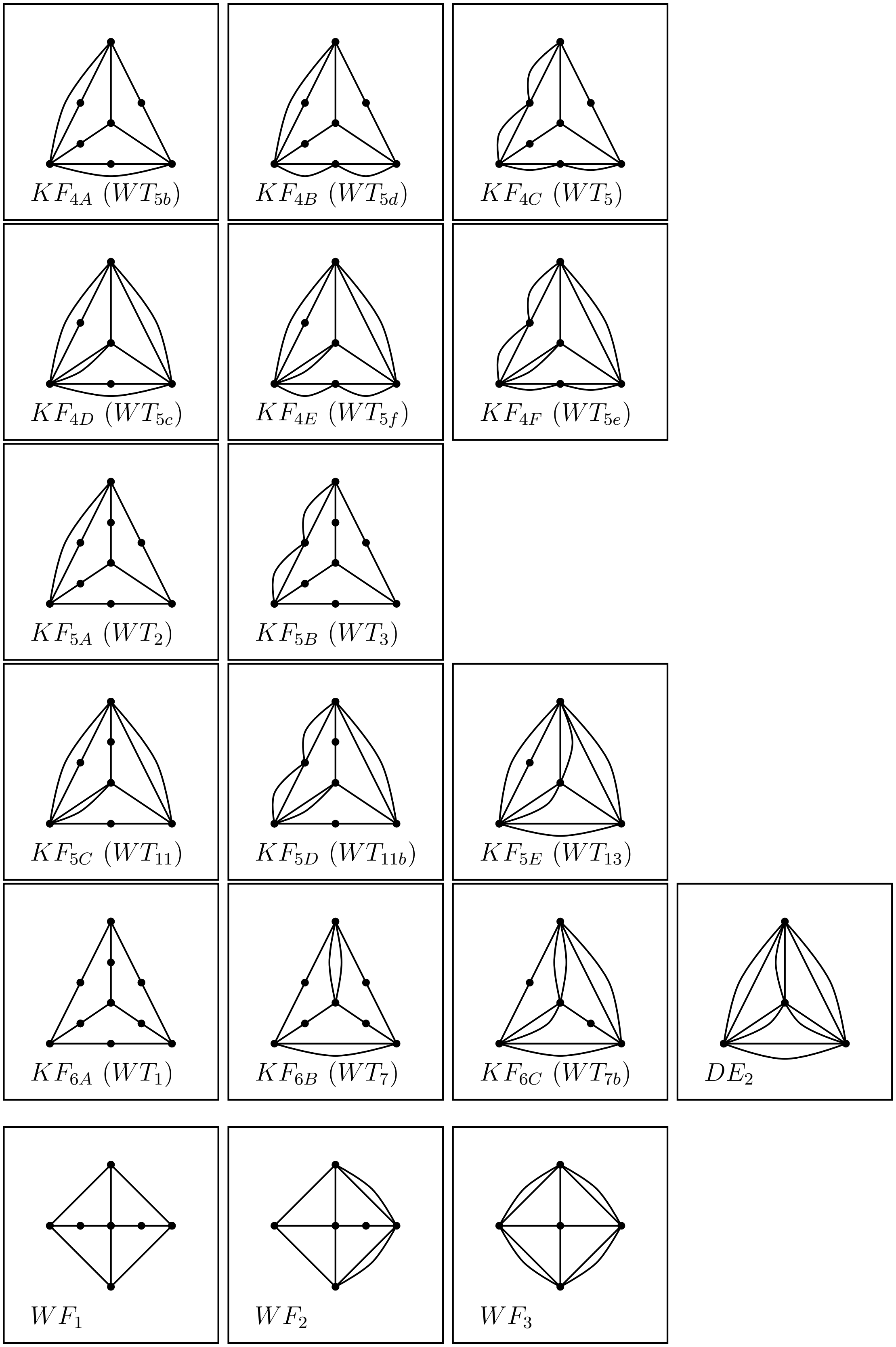

KF_{3D} (WT_{12e})


KF_{3E} (WT_{9b})

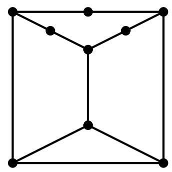

KF_{3F} (WT_{9d})


KF_{3G} (WT_{9c})

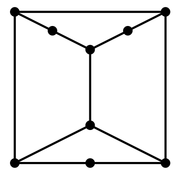

KF_{3H} (WT_{9})



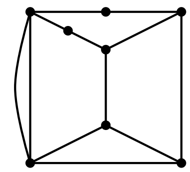
KF_{4A} (WT_{5b})
KF_{4B} (WT_{5d})
KF_{4C} (WT_{5})
KF_{4D} (WT_{5c})
KF_{4E} (WT_{5f})
KF_{4F} (WT_{5e})
KF_{5A} (WT_{2})
KF_{5B} (WT_{3})
KF_{5C} (WT_{11})
KF_{5D} (WT_{11b})
KF_{5E} (WT_{13})
KF_{6A} (WT_{1})
KF_{6B} (WT_{7})
KF_{6C} (WT_{7b})
DE_{2}
WF_{1}
WF_{2}
WF_{3}



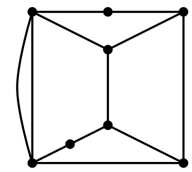
TP_1 (WT_4)



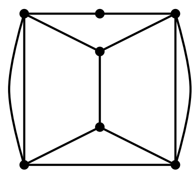
TP_2 (WT_8)



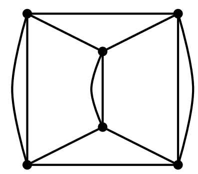
TP_3 (WT_6)



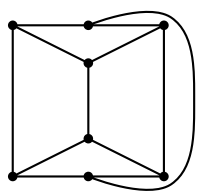
TP_4



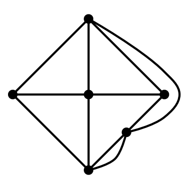
TP_5 (WT_{10})



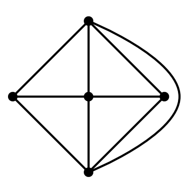
TP_6 (WT_{12})



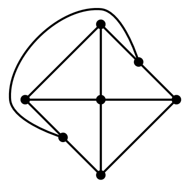
$CUBE$



DH_1 (WF_4)



$K_5 \backslash e$



$CUBE/e\ (WF_6)$

APPENDIX B

PLANAR EXAMPLES OF GRAPHS WHOSE LIMBS DOMINATE $K_{2,3}$ OR K^4

Planar Examples of Graphs of Theorem 2.4.

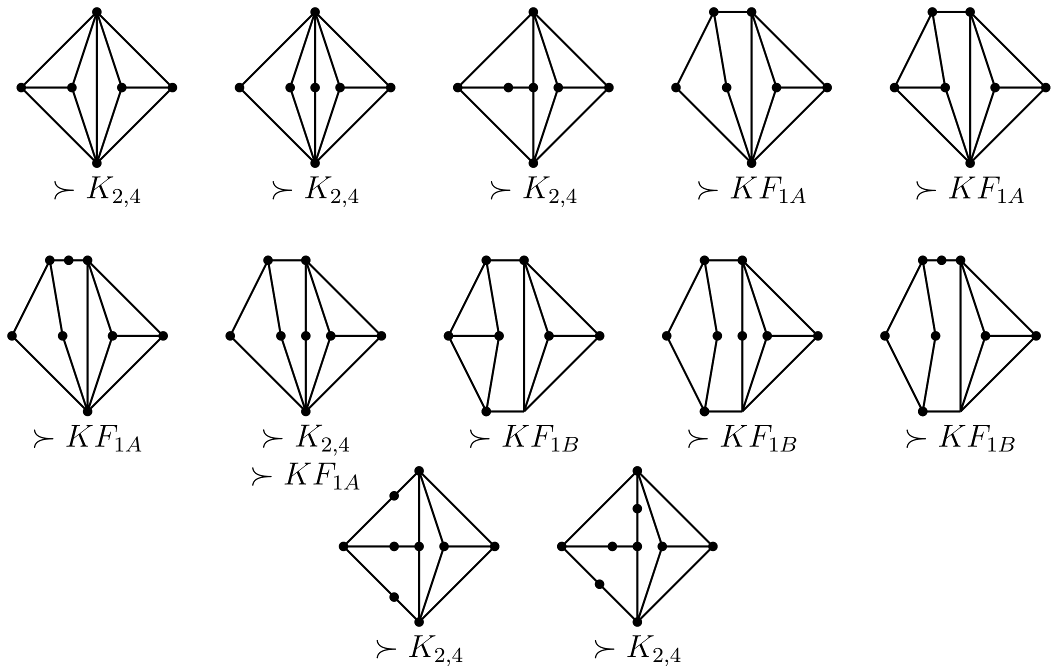


Figure B.1: Planar, full-K^4 graphs with limbs of $K_{2,3}$ or K^4, of case (i), in the graph $L \cup H$.

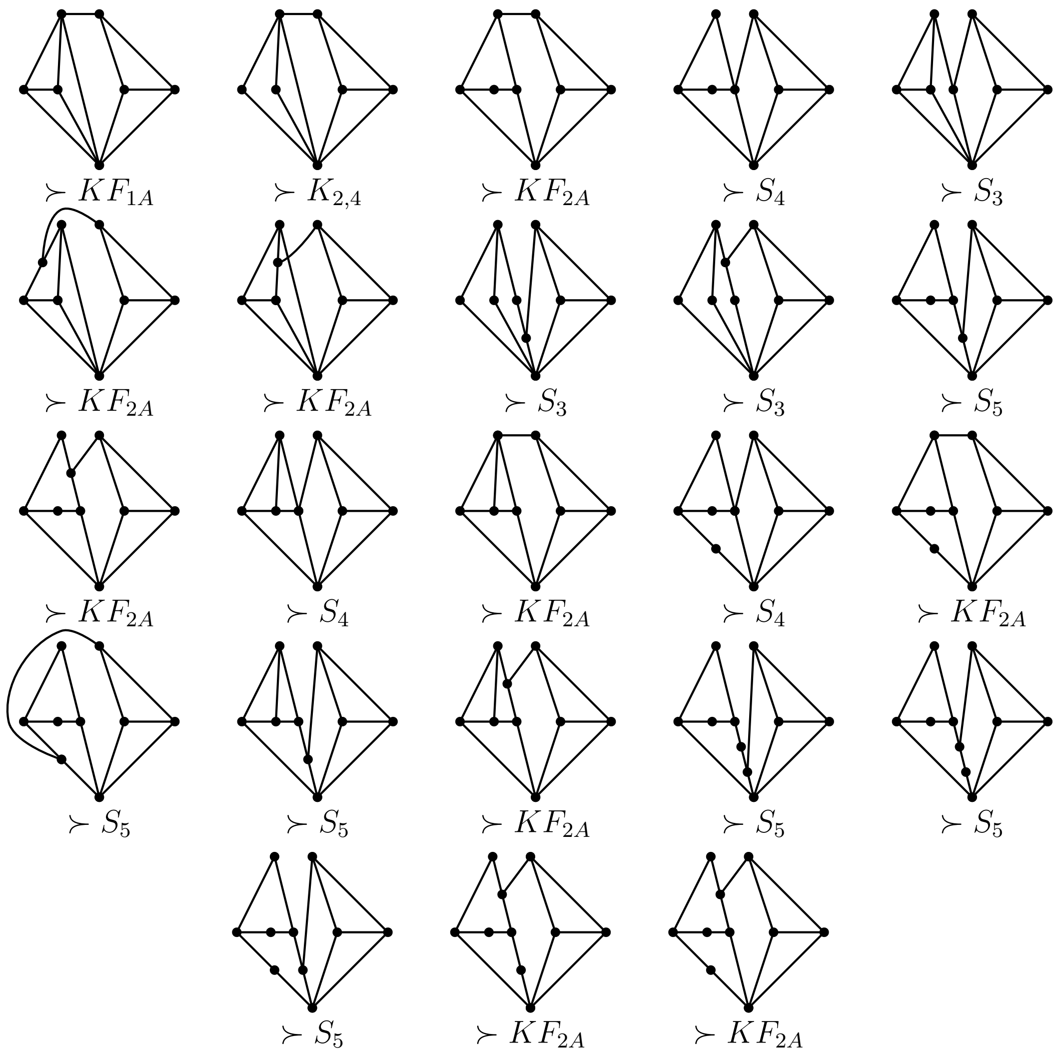


Figure B.2: Planar, full-K^4 graphs with limbs of $K_{2,3}$ or K^4, of case (ii), in the graph $L \cup H$.

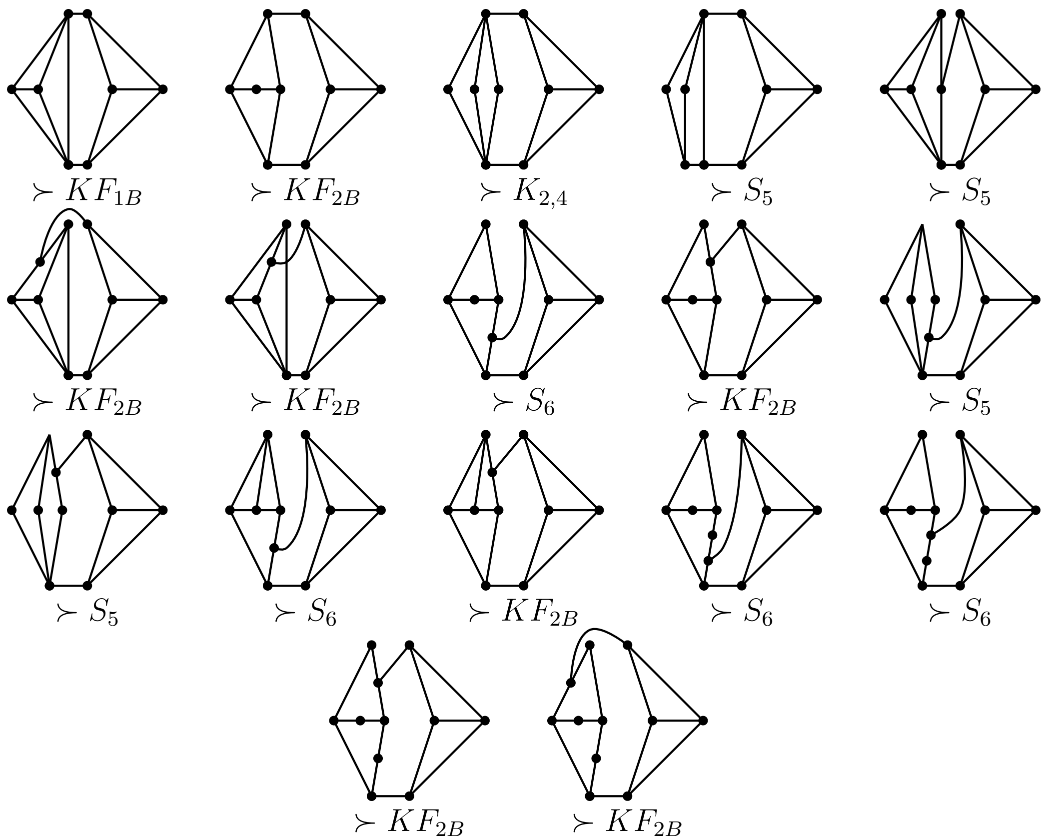


Figure B.3: Planar, full-K^4 graphs with limbs of $K_{2,3}$ or K^4, of case (iii), in the graph $L \cup H$.

BIBLIOGRAPHY

[1] Allgeier, B., Structure and properties of maximal outerplanar graphs, (2009), Dissertation, University of Louisville.

[2] Barnette, D.W. and Grünbaum, B., On Steinitz's theorem concerning convex 3-polytopes and on some properties of 3-connected planar graphs, *Many Facets of Graph Theory* (1969), 27—40

[3] Boyer, J. and Myrvold, W., On the Cutting Edge: Simplified O(n) Planarity by Edge Addition. *Journal of Graph Algorithms and Applications* (2004), 8, (3): 241–273

[4] Chartrand, G. and Harary, F., Planar permutation graphs, *Annales de l'Institut Henri Poincaré* **B** (1967), 3 (4): 433—438

[5] Diestel, R., Graph Theory, 3rd Edition, Graduate Texts in Mathematics, Springer, Verlag, Heidelberg 2006

[6] Dirac, G., A property of 4-chromatic graphs and remarks on critical graphs, *J. London Math. Soc.* (1952), 27: 85—92.

[7] Duffin, R.J., Topology of series-parallel networks, *J. Math. Anal. Appl.* (1965), 10: 303—318.

[8] Dziobiak, S., Excluded minor characterization of apex-outer-planar graphs, (2011). Dissertation, LSU.

[9] Kuratowski, K., Sur le problème des courbes gauches en topologie, *Fund. Math.* (1930), 15: 271—283.

[10] Lueder genannt Luehr, T.A., A characterization of near outer-planar graphs (2010). Masters thesis, LSU.

[11] Liu, C., Graph Structures and Well-Quasi-Ordering (2014). PhD thesis, Georgia Tech.

[12] Robertson, N. and Seymour, P.D., Graph Minors. XX. Wagner's Conjecture, *J. Combin. Theory*, Ser. B **92** (2004):325–357

[13] Wagner, K., Über eine Eigenschaft der ebenen Komplexe, *Math. Ann.* (1937), 114: 570-–590

[14] West, D., Introduction to Graph Theory, 2nd Edition, Pearson Education Inc, New Jersey, 2000

Index