Pisano Periods: A Comparison Study

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May 6, 2019

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Abstract

The Pisano period, denoted $\pi(n)$, is the period during which the Fibonacci sequence repeats after reducing the original sequence modulo n. More generally, one can similarly define Pisano periods for any linear recurrence sequence; in this paper we compare the Pisano periods of certain linear recurrence sequences with the Pisano periods of the Fibonacci sequence. We first construct recurrence sequences, defining the initial values as integers from 2 to 1000 and second values as 1. This paper discusses how the constructed sequences are related to the matrix

$$M = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

reduced modulo n. We offer a proof to show that the order of M is equal to the Pisano period of the Fibonacci sequence reduced modulo n. Further, we provide data showing that there are few discrepancies between the order of M and the Pisano periods of the constructed sequences reduced modulo n, for n from 2 to 1000. Finally, we detail progress made in the analysis of the comparison between the Pisano periods of the Fibonacci and Lucas sequences.

Keywords: Pisano period, second order linear recurrence sequence, Fibonacci sequence

1 Introduction and Motivation

One of the most studied sequences in mathematics, now known as the Fibonacci sequence, has been found written on ancient Sanskrit tablets, according to Wall [11]. Since it was introduced to the Western world by the Italian mathematician Fibonacci, other mathematicians have followed suit, discovering the properties of other linear recurrence sequences.

Even given the breadth of research, there are still lingering questions regarding linear recurrence sequences, their properties, and how they are interrelated. This research investigates some properties of second order linear recurrence sequences, of which the definition is found in [2,7].

Definition 1. *Let R be a ring, and let $B, C \in R$ be constants such that $B, C \neq 0$. A second order linear recurrence sequence over R is a sequence $\{A_k\}_{k=0}^{\infty}$ of elements of R for which the following is true:*

$$A_k = Ba_{k-1} + Ca_{k-2}, k \in \mathbb{Z}^+, k \geq 2. \tag{1}$$

Each second order linear recurrence sequence is determined by two initial values, called seed values; altering one or more of these seed values alters the entire sequence. This research considers sequences that each have a second seed value of 1 and for which $B = C = 1$. Therefore, the notation herein identifies sequences based solely on their initial values, with the understanding that their second seed values are 1. We denote linear sequences $A_k(i)$, where k is the sequence index and i is the initial seed value.

The Fibonacci sequence is a second order linear recurrence sequence with initial seed value 0 and second seed value 1. This sequence is denoted $F_k = A_k(0)$, and it is defined as follows:

$$F_{k+2} = F_{k+1} + F_k, \ k \in \mathbb{Z}^+, k \geq 2. \tag{2}$$

Thus, the Fibonacci sequence is defined by adding two prior values to find the next value in the sequence. The first few values are given below:

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots. \tag{3}$$

Every value of the Fibonacci sequence can be generated by the matrix

$$M = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}; \tag{4}$$

more precisely

$$M^k = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^k = \begin{bmatrix} F_{k+1} & F_k \\ F_k & F_{k-1} \end{bmatrix}$$

by [2]. For example,

$$M^2 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^2 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} F_3 & F_2 \\ F_2 & F_1 \end{bmatrix}.$$

The *Pisano period of the Fibonacci sequence modulo n*, denoted $\pi_0(n)$, is the period in which the Fibonacci sequence repeats after reducing the original sequence modulo n. Currently, there is no known general formula for the Pisano periods of the Fibonacci sequence by [3,5,10,11]. An interesting question to pose, however, is how the Pisano periods of other second order linear recurrence sequences compare to those of the Fibonacci sequence; changing only the initial value of the Fibonacci sequence, we compare the resulting Pisano periods in the same modulus. Noting that M is an invertible, square matrix with $\det(M) = -1$, we consider the following definition by [6]:

Definition 2. *Let $n \in \mathbb{Z}^+$. Let A be a $m \times m$ matrix with integer entries, and let A be reduced $\mod n$. Suppose there exists $k \in \mathbb{Z}^+$ such that $A^k = I_m \mod n$, where I_m is the $m \times m$ identity matrix. The order of A is the smallest such k.*

Proposition 1. *If a matrix's determinant is ± 1, the order exists. In particular, for every positive integer n, there exists a positive integer k such that $M^k \equiv I_2 \mod n$.*

Proposition 1 is a consequence of Lagrange's Theorem from group theory. Because the recurrence matrix has a determinant of -1, we are ensured that the sequences related to M^k are finite and cyclic. These sequences reduced $\mod n$ produce strings of repeating sequence values. Thus, we are ensured that strings of repeating sequence values occur at a minimum of j, where j is the order of M mod n.

We consider the cyclic nature of the sequences reduced modulo n and ask whether it is possible for the sequence to repeat with greater frequency than the order of M. One may assume that because changing one seed value alters the entire sequence, the resulting Pisano periods would be changed dramatically. Surprisingly, there are few discrepancies between the Pisano periods of the Fibonacci sequence reduced modulo n and other second order linear recurrence sequences that differ by only the initial seed value. By comparing the Pisano periods of the Fibonacci sequence with those of the Lucas sequence, $A_k(2)$, we conjecture that the discrepancies exist only at moduli n where the 2×1 seed value matrix is an eigenvector of M modulo n.

2 Background and Related Studies

2.1 Closed Form of the Fibonacci Sequence

As noted in the general definition (1), linear recurrence sequences can be written as a combination of terms with constant coefficients. The Fibonacci sequence, (2), is no exception. There is, however, another form for the sequence. The following appears in [1, 2, 4].

Theorem 1. *Let $F_0 = 0$ and $F_1 = 1$. Let F_k be defined recursively in accordance with (2). Then, for all $k \geq 0$,*

$$F_k = \frac{1}{\sqrt{5}}(\phi^k - \bar{\phi}^k) \tag{5}$$

where

$$\phi = \frac{1+\sqrt{5}}{2},\ \bar{\phi} = \frac{1-\sqrt{5}}{2}.$$

Somer and Robinson state in [8, 10] that the closed forms of other linear recurrence sequences can also be found using Binet's formulas for recursion, in conjunction with the seed values of the sequence.

2.2 Eigenvectors and eigenvalues of the recurrence matrix M

The eigenvectors for the recurrence matrix M given in (4) are determined by the usual method:

$$\begin{aligned}\det(M-\lambda I)=0 &\Leftrightarrow \begin{vmatrix}1-\lambda & 1\\ 1 & -\lambda\end{vmatrix}=0\\ &\Leftrightarrow (1-\lambda)(-\lambda)-1=0\\ &\Leftrightarrow \lambda^2-\lambda-1=0.\end{aligned}$$

By the quadratic formula, the roots of the polynomial are ϕ and $\bar{\phi}$. These are, therefore, the eigenvalues of the recurrence matrix M that generates the Fibonacci sequence [7].

Substituting first $\lambda=\phi$, then $\lambda=\bar{\phi}$ into $M-\lambda I$, we solve for eigenvectors in the following way:

$$\begin{aligned}\begin{bmatrix}1-\phi & 1\\ 1 & -\phi\end{bmatrix}\begin{bmatrix}1\\ x\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix} &\Leftrightarrow \begin{bmatrix}1-\phi+x\\ 1-\phi x\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}\\ &\Rightarrow 1-\phi x=0\\ &\Rightarrow x=\frac{1}{\phi}\\ &\Rightarrow x=\frac{2}{1+\sqrt{5}}.\end{aligned}$$

Multiplying by the conjugate of the denominator, we see that

$$\begin{aligned}x&=\frac{2(1-\sqrt{5})}{(1+\sqrt{5})(1-\sqrt{5})}\\ &=\frac{2(1-\sqrt{5})}{-4}\\ &=\frac{1-\sqrt{5}}{-2}\\ &=-\bar{\phi}.\end{aligned}$$

Therefore, our first eigenvector is

$$v_1=\begin{bmatrix}1\\ \dfrac{1-\sqrt{5}}{-2}\end{bmatrix}=\begin{bmatrix}1\\ -\bar{\phi}\end{bmatrix},$$

which has an eigenvalue of ϕ . Likewise,

$$v_2=\begin{bmatrix}1\\ \dfrac{1+\sqrt{5}}{-2}\end{bmatrix}=\begin{bmatrix}1\\ -\phi\end{bmatrix}$$

has an eigenvalue of $\bar{\phi}$.

2.3 Pisano periods of the Fibonacci sequence: Chinese Remainder Theorem

The following definition is found in [2, 10, 11].

Definition 3. *Let $i \in \mathbb{Z}$ and let $A_k(i)$ be a second order linear recurrence sequence. When $A_k(i)$ is reduced $\bmod n$, the Pisano period, denoted $\pi_i(n)$, is the period with which the sequence repeats.*

This research considers only second order linear recurrence sequences with second seed values of 1; we therefore use notation that denotes the initial seed value and the modulus, $A_k(i, n)$ where k is the index, i is the initial value, and n is the modulus. For instance, when the Fibonacci sequence is reduced modulo 2, denoted as $F_k(0, 2)$, the following is produced:

$$F_k(0) = 0, 1, 1, 2, 3, 5, 8, 13, 21, \ldots \Rightarrow F_k(0, 2) = 0, 1, 1, 0, 1, 1, 0, 1, 1, \ldots$$

After the reduction, a repeating string of the Fibonacci sequence **0,1,1** is evident. Because the string length is 3, the Pisano period of the Fibonacci sequence reduced modulo 2 is 3. That is, $\pi_0(2) = 3$. Currently, there is no general formula for the Pisano periods of the Fibonacci sequence or any other second order linear recurrence sequence.

Wall and Somer detail that the Chinese Remainder Theorem is applicable for the determination of Pisano periods [10, 11].

Theorem 2 (Chinese Remainder theorem for Pisano periods)**.** *If m and n are co-prime, then $\pi_i(mn)$ is equal to the least common multiple of $\pi_i(m)$ and $\pi_i(n)$.*

Therefore, Pisano periods of linear recurrence sequences can be determined by first decomposing the modulus into prime powers, then finding the least common multiple of those powers' Pisano periods. Interestingly, the Pisano periods of prime powers of the Fibonacci sequence do not follow a general trend (see [5]).

2.4 Anomalous primes 2 and 5

When studying the Pisano periods of the Fibonacci sequence, the Chinese Remainder Theorem tells us that determining the periods of the prime factors of the modulus is important. However, there are two primes that are anomalous with regard to Pisano periods. First, we consider the closed form of the Fibonacci sequence from (5). We see that the form has a factor of $\frac{1}{\sqrt{5}}$. Likewise, the closed form contains a 2 in the denominators of both ϕ and $\bar{\phi}$. This suggests that calculating the Pisano periods for prime factors of moduli 2 and 5 may produce anomalous results.

3 Methods

This section describes the methods used to investigate the properties of linear recurrence sequences reduced modulo n. In particular, it details how the sequences are constructed, how the are related to the recurrence matrix M, and how their properties are compared in this research.

3.1 Constructing recurrence sequences

This research focuses on comparing the Pisano periods of the Fibonacci sequence with those of linear recurrence sequences with one dissimilar seed value; the second seed value for each sequence will be 1. Therefore, we include in our study sequences that begin with integers $i \in \{2, 3, \ldots, 1000\}$. The sequences are defined as follows:

$$A_{k+2}(i) = A_{k+1}(i) + A_k(i), k \in \mathbb{Z}^+.$$

For instance, the Lucas sequence, with initial value of 2, is denoted $A_k(2)$, with the understanding that the second seed value is 1.

3.2 Relating constructed sequences to the recurrence matrix M

As detailed in (4), the matrix M generates the Fibonacci sequence when it is raised to powers k. When M^k is multiplied by a 2×1 matrix that contains only the seed values of a linear recurrence sequence, the entire sequence can be generated. In this research, the initial values range from 2 to 1000, and the second seed value of each constructed sequence is 1. Therefore, for the linear recurrence sequences considered in this research, the following holds:

$$M^k \begin{bmatrix} 1 \\ i \end{bmatrix} = \begin{bmatrix} A_{k+1}(i) \\ A_k(i) \end{bmatrix},$$

where $A_0(i) = i$. Let us consider, for instance, the Lucas sequence with initial value 2:

$$A_k(2) = M^k \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

To find the next two sequence values, we set $k = 2$ and write:

$$M^2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} A_3(2) \\ A_2(2) \end{bmatrix}.$$

Thus, the recurrence matrix M is related to all linear recurrence sequences that we will study.

3.3 Proof: the order of M is equal to the Pisano period of the Fibonacci sequence

We show that the order of recurrence matrix M is equal to the Pisano period of the Fibonacci sequence, thereby eliminating the need to raise matrices to powers and reduce by a modulus.

Theorem 3. *The order of M reduced modulo n is equal to the Pisano period of the Fibonacci sequence reduced modulo n.*

Proof. Let j be order of M; that is, let j be the minimal positive integer with $M^j = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$,and let l be the Pisano period of the Fibonacci sequence reduced modulo n. Then,

$$M^{l+1} = \begin{bmatrix} F_{l+2} & F_{l+1} \\ F_{l+1} & F_l \end{bmatrix} = \begin{bmatrix} F_2 & F_1 \\ F_1 & F_0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \equiv M \mod n.$$

$$\Rightarrow M^{l+1} \equiv M \mod n$$
$$\Rightarrow M^{l+1}M^{-1} \equiv MM^{-1} \mod n$$
$$\Rightarrow M^l \cong I_2 \mod n$$

Therefore, $l \geq j$.

Now consider $M^{j+k} = M^j \cdot M^k = I_2 \cdot M^k = M^k$. Therefore,

$$\begin{bmatrix} F_{j+k+2} & F_{j+k+1}l \\ F_{j+k+1} & F_{j+k} \end{bmatrix} = \begin{bmatrix} F_{k+2} & F_{k+1} \\ F_{k+1} & F_k \end{bmatrix}, \text{ so } F_{j+k} = F_k. \text{ Therefore, } l \leq j$$

Since we have both $j \leq l$ and $l \leq j$, by anti-symmetry, we conclude that $j = l$. Hence, the order of M reduced modulo n is equal to the Pisano period of the Fibonacci sequence reduced modulo n.

□

4 Data

Based on Theorem 3, we can compare the Pisano period of the Fibonacci sequence to the Pisano period of a constructed sequence to determine whether the sequence repeats at an index k which is less than the order j of the matrix. To make this comparison, we calculate the Pisano periods of the constructed sequences with initial values between 2 and 1000 reduced by moduli between 2 and 1000.

All necessary calculations are made via Sage. Code is first written to generate the Fibonacci sequence and its Pisano periods reduced modulo n, $2 \leq n \leq 1000$. Then, additional code

Table 1: Sample Data Set

Initial Value i	Domain	Anomalous Moduli n	Ratio: $\pi_i(n)/\pi_0(n)$
$i = 5x + 2$	$0 \leq x \leq 199$	$5n, 1 \leq n \leq 200$	5
$i = 3 + 11x$	$0 \leq x \leq 90$	11, 22	2
$i = 4 + 19x$	$0 \leq x \leq 52$	19, 38	2
$i = 8 + 71x$	$0 \leq x \leq 13$	71, 142	2
$i = 11 + 131x$	$0 \leq x \leq 7$	131, 262	2
$i = 13 + 181x$	$0 \leq x \leq 5$	181, 362	2
$i = 15 + 239x$	$0 \leq x \leq 4$	239, 478	2
$i = 16 + 271\text{x}$	$0 \leq x \leq 3$	271, 542	2
$i = 18 + 31x$	$0 \leq x \leq 31$	31, 62	2
$i = 19 + 379x$	$0 \leq x \leq 2$	379, 758	2
$i = 23 + 29x$	$0 \leq x \leq 33$	29, 58	2
$i = 24 + 599x$	$0 \leq x \leq 1$	599	2
$i = 23$		551	2
$i = 25$		649	2
$i = 36$		121, 242	2
$i = 42$		361, 722	2
$i = 80$		209, 418, 341, 682	2

generates the Pisano periods of other second order linear recurrence sequences with initial values $i \in \mathbb{Z}$, $2 \leq i \leq 1000$, reduced modulo n, $2 \leq n \leq 1000$. Finally, we use Sage to generate the ratios of the Pisano periods of the Fibonacci sequence to those of the other linear recurrence sequences in the same moduli. The code used to generate the data is detailed in Appendix A. The data that was generated with this code was cross-referenced by hand and with reference tables in [7].

Table 1 details a sample data set that is representative of the data collected in this research. The first column indicates the initial value i of the sequence described. The second column, called "Anomalous Moduli n," lists moduli for which the sequence $A_k(i)$ has a different Pisano period than the Pisano period of the Fibonacci sequence for the same modulus. The third column displays the following ratio: Pisano period for $A_k(i) \mod n$ divided by the Pisano period of the Fibonacci sequence $\mod n$.

The data collected can be divided into three classifications:(initial values and moduli that produce a ratio of 5), (initial values and prime moduli that produce a ratio of 2), and (initial values and composite moduli that produce a ratio of 2).

5 Results Overview

The sample data above illustrates that when there are discrepancies between the Pisano period of the Fibonacci sequence and the Pisano period of the constructed sequences, the ratio between the two is 5 or 2. Upon examining the data generated in Sage, we see that every initial value $i \in \{2, 7, 12, 17, 22, 27, \ldots\}$ has anomalous moduli $5n$, for all $1 \leq n \leq 200$. We also observe that moduli that have a ratio of 2 as compared to the Pisano periods of the Fibonacci sequence tend to appear in pairs and fall into one of two categories: n ending in 1 (hence $2n$ ending in 2), or n ending in 9 (hence $2n$ ending in 8).

5.1 Analysis of ratios: Fibonacci to Lucas Pisano periods

To better understand the circumstances that produce anomalous moduli, we compare the resulting Pisano periods of the Lucas sequence to those of the Fibonacci sequence. First, we note that the Lucas sequence, $A_k(2)$, has anomalous moduli at $5n$. The ratio at these moduli is 5; this implies that the sequence repeats 5 times as often as the order of M demands. Therefore, the matrix $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ must become an eigenvector of the matrix $M^k \mod 5n$, for some $k \geq 1$ such that $M^k \neq I_2$. To determine at which moduli this is possible, we consider:

$$M \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Since the eigenvalues of M are ϕ and $\bar{\phi}$, to be an eigenvector for $M \mod n$,

$$\begin{bmatrix} 3 \\ 1 \end{bmatrix} \equiv \phi \begin{bmatrix} 1 \\ 2 \end{bmatrix} \equiv \begin{bmatrix} \phi \\ 2\phi \end{bmatrix} \quad \mod n$$

or

$$\begin{bmatrix} 3 \\ 1 \end{bmatrix} \equiv \bar{\phi} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \equiv \begin{bmatrix} \bar{\phi} \\ 2\bar{\phi} \end{bmatrix} \quad \mod n.$$

It suffices to consider when $1 \equiv 2\phi \mod n$

$$\begin{aligned} 1 &\equiv 2\phi \quad \mod n \\ \Rightarrow 1 &\equiv 2\left(\frac{\sqrt{5}+1}{2}\right) \quad \mod n \end{aligned}$$

This expression is true only when $\sqrt{5}$ is 0 when reduced mod 5. Therefore, the Lucas sequence's seed value matrix becomes an eigenvector with eigenvalue 1 at $n = 5$.

5.2 Analysis of ratio 2: Quadratic Reciprocity

As noted earlier, the moduli with ratio 2 tend to occur in pairs, with n ending in 1 or 9 and $2n$ ending in 2 or 8. When we examine the circumstances necessary for this to be true in moduli p, where p is a prime number, we see that in anomalous moduli, $\begin{bmatrix} 1 \\ i \end{bmatrix}$ is an eigenvector for M mod p. This means that $i \equiv -\phi \mod p$ or $i \equiv -\bar{\phi} \mod p$. Based on the eigenvalues for M, this would imply that $\sqrt{5}$ exists mod p. The following appears in [9]:

Definition 4 (Legendre symbol)**.** *Let p be a prime and a be an integer not divisible by p. The Legendre symbol is given by:*

$$\left(\frac{a}{p}\right) = \begin{cases} 1 & \textit{if } a \textit{ is a quadratic residue modulo } p \\ -1 & \textit{if } a \textit{ is a nonresidue modulo } p \end{cases}$$

where a *quadratic residue modulo* p is a nonzero number that is congruent to a square modulo p. We use the Law of Quadratic Reciprocity to determine in which moduli p 5 is a quadratic residue, since 5 is a quadratic residue mod p if and only if 5 is a square mod p. In Legendre symbol notation, we write:

$$\left(\frac{5}{p}\right) = 1 \tag{6}$$

In [9], Silverman defines the *Law of Quadratic Reciprocity* as follows:

Theorem 4 (Law of Quadratic Reciprocity)**.** *Let p and q be distinct odd primes.*

$$\left(\frac{-1}{p}\right) = \begin{cases} 1 & \textit{if } p \equiv 1 \mod 4 \\ -1 & \textit{if } p \equiv 3 \mod 4 \end{cases}$$
$$\left(\frac{2}{p}\right) = \begin{cases} 1 & \textit{if } p \equiv 1, 7 \mod 8 \\ -1 & \textit{if } p \equiv 3, 5 \mod 8 \end{cases}$$
$$\left(\frac{q}{p}\right) = \begin{cases} \left(\frac{p}{q}\right) & \textit{if } p \equiv 1 \mod 4 \textbf{\textit{ or }} q \equiv 1 \mod 4 \\ -\left(\frac{p}{q}\right) & \textit{if } p \equiv 3 \mod 4 \textbf{\textit{ and }} q \equiv 3 \mod 4 \end{cases}$$

By the Law of Quadratic Reciprocity, we can compute (6) in the following manner:

$$\left(\frac{5}{p}\right) = 1 \Leftrightarrow \left(\frac{p}{5}\right) = 1.$$

Therefore, the number 5 has a square root if and only if $p = 2, p = 5$, or $p \equiv 1, 4 \mod 5$. Thus, in these moduli, we will find discrepancies between the Pisano period of a constructed sequence and the Pisano period of the Fibonacci sequence.

6 Conclusion and future studies

Second order linear recurrence sequences modulo n are finite and cyclic, but it is possible for the sequences to repeat with more frequency than the order guarantees. Because the order of the recurrence matrix M reduced modulo n is equal to the Pisano period of the Fibonacci sequence reduced modulo n, comparing the Pisano periods of constructed sequences with those of the Fibonacci sequence in the same moduli allows us to determine the moduli and initial values for which this is possible. Surprisingly, the results of this project show that altering the initial seed value of the Fibonacci sequence produces few instances of change in the Pisano periods, modulo n. Within the search parameters, the ratio between the two is always 1, 2, or 5. When the ratio is 1, the sequence does not repeat more frequently than the order of the associated recurrence matrix M mod n. We consider the anomalous moduli that produce a ratio of 5 or 2. When the ratio is 5, this occurs at moduli $5n$. The comparison study between the Lucas sequence and the Fibonacci sequence suggests that this occurs when the seed value matrix becomes an eigenvector with eigenvalue 1, forcing the sequence to repeat 5 times as often as the order demands. The eigenvalues of the Fibonacci sequence allows us to determine that this is possible when $n = 5$. When the ratios between the Pisano periods of the Fibonacci sequence and the constructed sequence for the same modulus are 2, they appear in pairs, n and $2n$. This suggests that the seed value matrix for the sequence becomes an eigenvector for $M \mod n$. This is only possible for integers n such that 5 is a square. We can use the Law of Quadratic Reciprocity to determine that this is possible for prime values of n. That is, a ratio of 2 will be produced for prime moduli n, if n is equivalent to 1 mod 5 or 4 mod 5.

In the future, studies should be conducted to compare the closed forms of second order linear recurrence sequences to that sequence's anomalous moduli, in an effort to generalize the occurrence of anomalies. For moduli beyond the scope of this research, studies need to be conducted to determine if anomalous moduli with ratio 2 always appear in pairs.

A Sage Code

The following Sage code was used to collect data for second order linear recurrence sequences with initial seed values from 2 to 1000 and second seed values of 1, reduced modulo n for $2 \leq n \leq 1000$. The code first generates the Pisano periods of the Fibonacci sequence, then the Pisano periods of the other linear recurrence sequences. Lastly, the code produces a ratio of the Pisano period for the sequence to the Pisano period for the Fibonacci sequence in the same modulus.

```
start=0
list = [];
for n in range(2,1000):
    k=0
    a=start
    b=1
    while s[:k]!=s[k:]or k<2:
        k=len(s)//2;
        a,b=b,a+b
```

```
list+=[k];
for startx in range(2,1000):
listx = [];
for n in range (2,1000):
     sx=[]
     k=list[n-2]
     kx=0
     ax=startx
     bx=1
     while sx[:kx]!=sx[kx:]or kx<2:
          sx+=[ax%n];
          kx=len(sx)//2;
          ax,bx=bx,ax+bx
listx+=[kx];
print s,n,kx/k;
```

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