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Folding Mathematics: A Mathematical Approach to Origami

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Abstract

From constructing a midpoint on a line to observing specific divisions of a plane, the art form of Origami borrows many mathematical tools in order to create complex, and often symmetrical, patterns in a paper medium known as a fold. For this project, the traditional fold known as the Origami Crane/Swan will be thoroughly examined as it contains the unique property to lie completely flat when complete. This phenomenon occurs because the vertices holding the fold together are not all considered to be flat folds. The different types of vertices interacting with each other create a natural locking mechanism within the medium and make it impossible for the medium to unravel. Using established geometrical and origami theorems, this project intends to deconstruct these locks and investigate the mathematics behind how the construction works.

Keywords: Origami, Geometry, Patterns

1 Introduction

Origami is an art form that has been practiced for centuries as a stylistic expression of geometry. As such, the ties between origami and mathematics are quite apparent. In ancient Japan, it is even suggested that origami was used as a medium to learn geometry. Furthermore, in the early twentieth century, origami was found to be able to be used in solving quadratic and cubic equations.[4] This background of mathematics makes the art produced by it very special. Robert Lang, a physist known for his origami mathematics and artwork, explains that origami is not a trial and error art form, but one that requires mathematical thought and is governed by the limitation of what one can do with a single sheet of paper.[3]

If one were to desire a fold that lay flat when complete, there must be a mathematical or geometrical solution that allows for this to happen. However, flat fold origami is a subset of origami that is very fickle about the solutions it can provide. This project seeks to find an answer as to what causes this behavior by analyzing a well-known flat fold called the orizuru or paper crane.

2 Creation

Building a crane is a relatively simple process. The highlights of it for a beginner in origami is the straightforward nature of the fold and the importance of symmetry it imparts. However, looking at this simple construct through a mathematical lens, things are a lot more interesting. Simple movements and creases hold much more meaning as they completely change the paper's geometry, lines that should have meaning no longer do, and angles perform grand trigonometry in order to conform to their new confines. To appreciate this act of mathematical acrobatics, some rules need to be set in place.

2.1 Background

To begin the investigation into the mystery of origami, the first step is to identify the known concepts and apply an explanation for each. The following axioms, definitions and theorems appear in several articles and books pertaining to origami mathematics.

(Axiom 1) The line connecting two constructible points is a constructible line.[1]

(Axiom 2) The point of coincidence of two constructible lines is a constructible point.[1]

(Axiom 3) The perpendicular bisector of the segment connecting two constructible points is a constructible line.[1]

(Axiom 4) The line bisecting any given constructed angle can be constructed.[1]

(Axiom 5) Given a constructed line l and constructed points P, Q, then whenever possible, the line through Q, which reflects P onto l, can be constructed.[1]

(Axiom 6) Given constructed lines l, m and constructed points P, Q, then whenever possible, any line which simultaneously reflects P onto l and Q onto m, can be constructed.[1]

(Definition 2.1) A *fold* is any folded paper object.[5]

(Definition 2.2) A *Mountain crease* is a convex crease or a crease made that directs toward the maker. These creases will be denoted M and marked - - - on the paper model.[5]

(Definition 2.3) A *Valley crease* is a concave crease or a crease made that directs away from the maker. These creases will be denoted V and marked - . - on the paper model.[5]

(Definition 2.4) A *crease pattern* is the pattern generated on the paper by the construction of the fold when fully unraveled.[5]

(Definition 2.5) A *point* (x, y) is the ordered pair where x and y are the Cartesian coordinates of the point.[2]

(Definition 2.6) A *line* (X, Y) is the set of points (x, y) that satisfy the equation $Xx + Y y + 1 = 0$.[2]

(Theorem 2.1) *Kawasaki theorem*: Let v be a vertex of degree 2n in a single vertex fold and let α_1 ,...,α_{2n} be the consecutive angles between the creases. Then v is a flat vertex fold if and only if $\alpha_1 - \alpha_2 + \alpha_3 - ... - \alpha_{2n} = 0$.[6]

This theorem explains that, for a fold to be flat, the paper must be folded a total of 360^o to return to its flattened state. Furthermore, it shows that the angles between the creases determine whether the fold will stay flat.[8] This theorem is a bit laborious to use when dealing with multiple verticies at once, so the next theorem will be used on primary inspection.

(Theorem 2.2) *Maekawa theorem*:The difference between the number of mountain and valley creases meeting at a vertex in a flat-foldable crease pattern is two. That is, for the creases c_i adjacent to the vertex, we must have $\sum \mu(c_i) = \pm 2$.[7] [1]

The Maekawa theorem is a rather convenient theorem since it simply relies on the count of mountain and valley creases leading to a vertex in order to determine the eligibility of the vertex being flat. This allows for quick assignments of flat fold verticies without having to

[1]For detailed proof of each theorem, see source as each are interesting, but not relevant to the result.

measure every angle in the fold.

With a means to judge a vertex in both it's creases and angles between the creases, the crane's construction can now be fully analyzed from a mathematical perspective. However, first there needs to be a crane and a crease pattern to analyze.

2.2 Finding a Frame

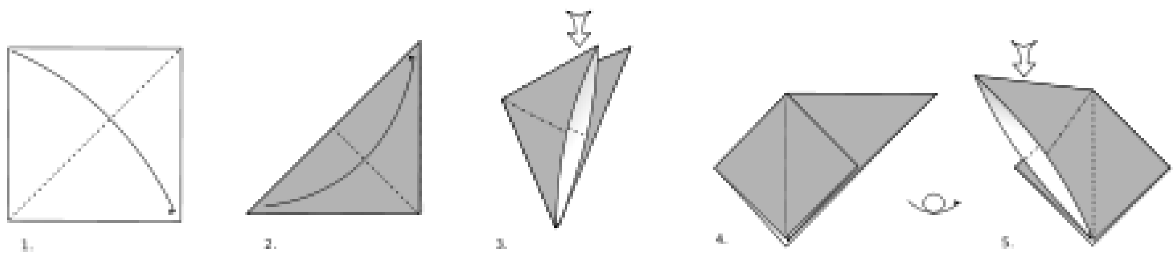

Figure 1: The first steps of folding a Crane.

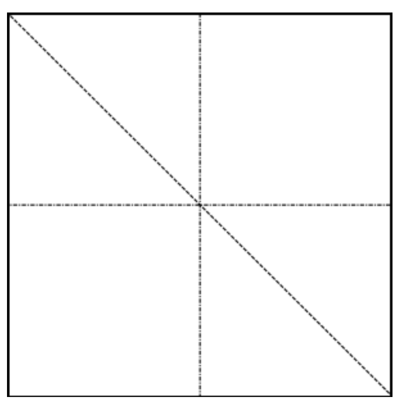

Figure 2: The crease pattern for Figure 1.

Figure 1 details the steps required to build the frame, or the general look that the fold will take. Figure 2 shows a perfect example of a flat vertex fold. Since it has two mountain and four valley creases, the difference between them is two, satisfying the Maekawa theorem. Additionally, $90^o - 45^o + 45^o - 90^o + 45^o - 45^o = 0$ which satisfies the Kawasaki theorem.[8] From here, the wings will take shape and provide an interesting dilemma.

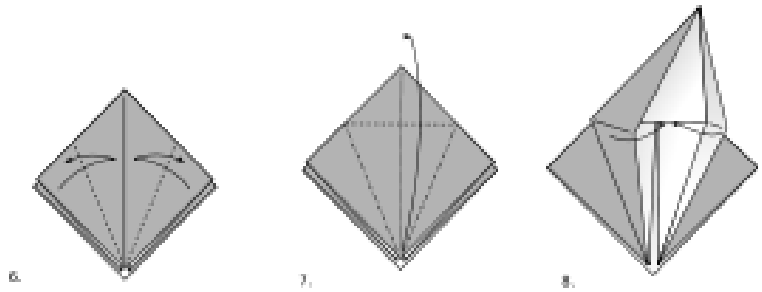

Figure 3: Creating the Wings.

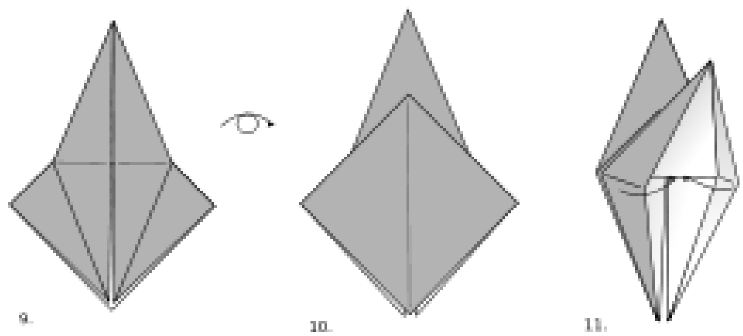

Figure 4: The intermediate steps of folding a Crane.

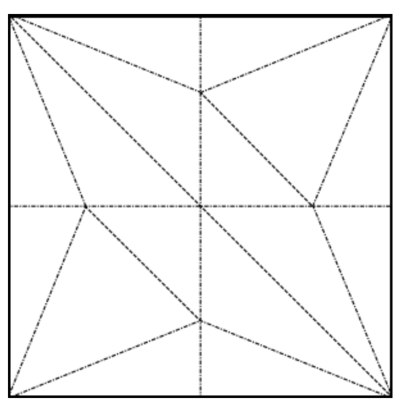

Figure 5: The crease pattern for Figure 3.

2.3 Building a Bird

Figure 4 has the first example of an interesting development in the fold. The four new verticies that have been formed through the wings creasing all have a M to V difference of one when viewed from the crease pattern. This should mean that the fold cannot be flat, yet it can very easily. The observation that explains this is that the line segment between these verticies and the center of each edge no longer exists in the fold. More accurately, these segments are pulled taut to the wing front. With this, the next step is the crane.

2.4 Completing the Crane

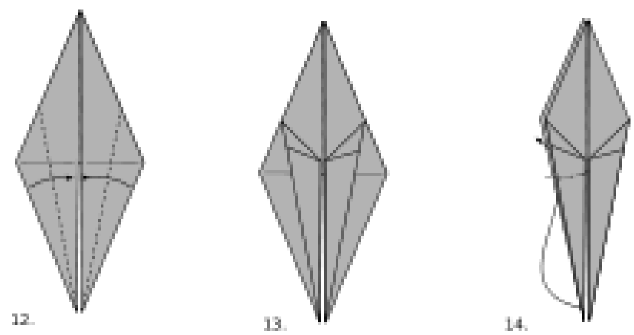

Figure 6: Forming the head and tail.

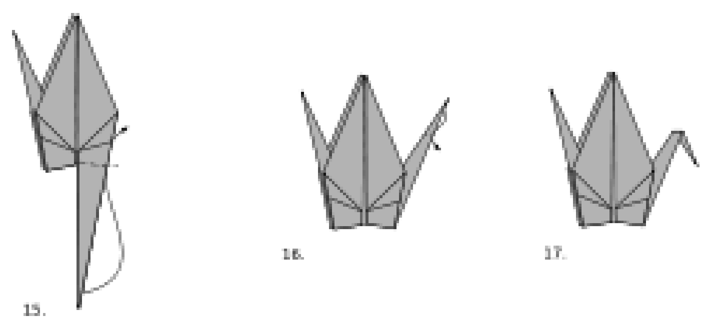

Figure 7: The final steps of folding a Crane.

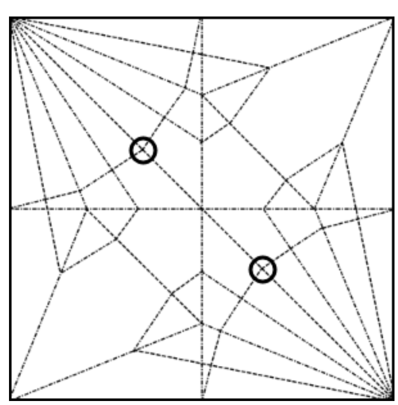

Figure 8: The crease pattern for Figure 5.

The crane has made quite a few new verticies as it finishes, but only two are of interest. The marked verticies contain four mountain and zero valley creases. This violates the Maekawa theorem with no exceptions, but the fold ends flat. In fact, it is decidedly bound to the fold pattern with no movement possible until these two verticies are undone. They would seem to act as locking mechanisms encouraging the fold to stay flat.

3 Conclusion

Flat fold origami is an oddity in both it's creation and exceptions. The creases that disappear in order to lay flat and the vice-like lock are only a small selection of the various ways the piece of paper will conform to lay flat. That said, for the trouble it can cause when determining whether a fold is flat, there is a very practical application for these kinds of folds. This is done with paper using no tearing nor reattaching as per the rules of traditional origami. However, The medium can be substituted for metal with the same locking consistency. Dr. Demaine identifies that folding industrial material would be applicable to a wide range of field from architecture to robotics.[9] Even further in the technological future, origami may be used for self re-programmable machines.[10] It is unclear if the particular locking crease presented in the paper crane is appropriate for such lofty goals, but one thing is certainly clear. The sky is the limit for the modest art form.

References

[1] Roger Alperin. A mathematical theory of origami constructions and numbers. 03 2000.

[2] Roger Alperin and Robert J. Lang. One, two, and multi-fold origami axioms. *Origami4: 4th International Meeting of Origami Science, Mathematics and Education, 4OSME 2006*, 01 2009.

[3] Vanessa Gould. *Between the Folds.* Green Fuse Films, 2008.

[4] Thomas Hull. A note on impossible paper folding. *American Mathematical Monthly*, 103, 03 1996.

[5] Thomas Hull. Counting mountain-valley assignments for flat folds. *Ars Comb.*, 67, 04 2003.

[6] Thomas Hull. The combinatorics of flat folds: a survey. *Origami: Proceedings of the 3rd International Meeting of Origami Science, Math, and Education3*, 07 2013.

[7] Thomas Hull. Coloring connections with counting mountain-valley assignments. 01 2016.

[8] Thomas Hull and Eric Chang. *Flat Vertex Fold Sequences*, pages 599–607. 06 2011.

[9] Jason Ku and Erik D. Demaine. Folding flat crease patterns with thick materials. *Journal of Mechanisms and Robotics*, 8, 11 2015.

[10] Jesse L Silverberg, Arthur Evans, Lauren McLeod, Ryan C Hayward, Thomas Hull, Christian Santangelo, and Itai Cohen. Applied origami using origami design principles to fold reprogrammable mechanical metamaterials. *Science (New York, N.Y.)*, 345:647–650, 08 2014.